Addendum: Baire Category

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These notes follow closely the notes of A. Sokal, available under www.ucl.ac.uk/∼ucahad0/3103_handout_7.pdf.

1 Nowhere dense sets

Definition 1.1. *Let X be a metric space. A subset $M \subset X$ is called* <u>*nowhere dense in X*</u> *if the closure has empty interior, i.e. $int(\overline{M}) = \emptyset$.*

It follows straight from the definition that a subset of nowhere dense set is nowhere dense, and also that the closure of a nowhere dense set is nowhere dense. From this observation we immediately conclude

- A subset $M \subset X$ is nowhere dense in X if and only if it is contained in a closed set with empty interior.
- A subset $M \subset X$ is nowhere dense in X if and only if its complement contains an open dense set.

Proposition 1.1. *Let X be a metric space. The union of finitely many nowhere dense sets is still nowhere dense in X.*

Proof. It clearly suffices to prove the above statement for *two* nowhere dense sets $A_1 \subset X$ and $A_2 \subset X$. We will actually prove "A_1, A_2 *closed* and nowhere dense $\implies$ $A_1 \cup A_2$ is closed and nowhere dense". This implies the general statement since $A_1 \cup A_2 \subset \overline{A_1} \cup \overline{A_2}$ and hence $A_1 \cup A_2$ is contained in a closed nowhere dense set. To prove the statement in quotation marks, we will prove (the equivalent statement) that the intersection of two dense open sets is open and sense dense. Let B_1 and B_2 be open and dense. Clearly $B_1 \cap B_2$ is open. To show that it is dense we show that any non-empty open set U intersects $B_1 \cap B_2$. Indeed $B_1 \cap U$ is open and non-empty (since B_1 is dense) and also $B_2 \cap (B_1 \cap U)$ is non-empty (since B_2 is dense). Hence $(B_2 \cap B_1) \cap U \neq \emptyset$. □

Note that the above Proposition becomes wrong if *finitely many* is replaced by *countably many* as you can see from looking at $\mathbb{Q} \subset \mathbb{R}$.

Corollary 1.1. *Let X be a metric space. If X contains no isolated points, then every finite subset is nowhere dense.*

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Proof. No isolated points implies that for any $x \in X$ the one-point set $\{x\}$ is nowhere dense. $\square$

Note that if $x \in X$ was an isolated point of X, then the closed set $\{x\}$ is also open and hence has non-empty interior. Can you give an example?

Exercise 1.1. *Explain why the Cantor set is nowhere dense in* $\mathbb{R}$.

2 Meager and non-meager sets

Definition 2.1. *Let X be a metric space. We say that a subset $M \subset X$ is*

1. *meager (of first category) in X if M can be written as a countable union of nowhere dense sets*
2. *non-meager (of second category) in X if it is not meager*
3. *residual (generic) in X if its complement is meager*

For instance, $\mathbb{Q} \subset \mathbb{R}$ is of first category (what about $\mathbb{Q}$ as a metric space in itself?). We will soon see that the complement $\mathbb{R} \setminus \mathbb{Q}$ is non-meager! We will also discuss more examples below.

The basic idea of the above definition is to give a (purely topological) size to sets. The notion of category of sets and the notion of Lebesgue measure of sets are actually independent. In particular, as the next example shows, it is possible to write $\mathbb{R}$ as the union of a set of measure zero and a meager set.

Example 2.1. *Let*

$$X = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n} \right) \subset \mathbb{R}$$

with q_n an enumeration of the rational numbers. X is open and dense (why?) and has finite Lebesgue measure (why?). The complement has therefore infinite Lebesgue measure in $\mathbb{R}$ but is meager (why?). Conversely, the set

$$Y = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{m \cdot 2^n}, q_n + \frac{1}{m \cdot 2^n} \right) \subset \mathbb{R}$$

has zero measure (why?) but is a countable intersection of dense open sets. Therefore the complement Y^c is a countable union of nowhere dense closed sets.

3 The Baire category theorem

Theorem 3.1. *Let X be a <u>complete</u> metric space. Then*

1. *A meager set (still) has empty interior.*
2. *The complement of a meager set is dense. (Hence a residual set is dense.)*
3. *The intersection of countably many open dense sets is (still) dense.*

In applications we will actually only need the following corollaries

Corollary 3.1. *A non-empty complete metric space is non-meager in itself, i.e. it cannot be written as a countable union of nowhere dense sets.*

Proof. If it was meager the metric space $X = int(X)$ would be empty by 1. of the Theorem. $\square$

Corollary 3.2. *If a non-empty complete metric space is written as the countable union of closed sets, then at least one of the closed sets must contain an open ball.*

To prove the Baire category theorem we will make use of the following Lemma due to Georg Cantor.

Lemma 3.1. *Let X be a complete metric space and $... \subset F_3 \subset F_2 \subset F_1$ be a nested sequence of non-empty closed sets $F_n \subset X$ with $diam(F_n) \to 0$ as $n \to \infty$. Then there is a $x \in X$ with $\bigcap_{n=1}^{\infty} F_n = \{x\}$*

Proof. Left as an exercise. Strategy: Pick x_n in each set F_n and argue that the sequence $\{x_n\}$ is Cauchy. Use completeness of X to deduce a limit $x \in X$ and show that $x \in F_n$ for any n. To show x is the only point in the intersection assume the existence of another point y with $d(x, y) = \delta > 0$ and deduce a contradiction. $\square$

Proof of Baire category theorem. Let $M = \cup_n M_n$ be meager. Step 1: We first show that 1. and 2. are equivalent. If 1. holds, the complement $M^c = \cap_{n=1}^{\infty} (M_n)^c$ has to be dense. (If it wasn't, M would contain a ball that doesn't intersect M^c which contradicts 1.) Conversely, if 2. holds and hence M^c is dense, M cannot have interior points (otherwise there would be an open ball not intersecting M^c contradicting density).
Step 2: To prove 2., we consider $M_1, M_2, ...$ a sequence of nowhere dense sets. Clearly, $\overline{M_1}, \overline{M_2}, ...$ is then a sequence of nowhere dense *closed* sets and $\cup_{n=1}^{\infty} M_n \subset \cup_{n=1} \overline{M_n}$. Hence it suffices to show that the complement of the countable union of nowhere dense *closed* sets is dense. But this is equivalent to showing that the countable intersection of dense open sets is dense, which is 3.
Step 3: To prove 3. let $N_1, N_2, ...$ be a sequence of open dense sets. We pick an arbitrary non-empty open set $U \subset X$ and show it intersects $\cap_n N_n$. Let (a_n) be a sequence of real numbers with $a_n = 1/n$.

We start with the observation that $U \cap N_1$ is open and non-empty, hence contains a non-empty closed ball $\overline{B}(x_1, r_1)$ with $0 < r_1 < a_1$. Then $B(x_1, r_1) \cap N_2$ is open and non-empty, allowing us to pick a non-empty closed ball $\overline{B}(x_2, r_2) \subset B(x_1, r_1) \cap N_2$. Then $B(x_2, r_2) \cap N_3$ is open and non-empty, allowing us to pick a non-empty closed ball etc. Continuing inductively we obtain a sequence of nested non-empty closed balls with diameter going to zero and $\overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap N_n \subset U \cap N_n$. Applying Cantor's Lemma we conclude that $\bigcap_{n=1}^{\infty} U \cap N_n$ is non-empty. $\square$

4 Exercises

Exercise 4.1. *Show that a closed proper subspace in a normed linear space is nowhere dense (hence of first category).*

Exercise 4.2. *Prove that in a Banach space every non-empty open set is of the second category.*

Exercise 4.3. *Is every set of the second category the complement of a set of the first category?*

Exercise 4.4. *Is the closure of a set of the first category also of the first category?*

Exercise 4.5. *What is the category of the set of all polynomials in* $C[0,1]$*?*