

# Functional Analysis (Revision Sheet)

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## 1 General Things

In the exam, you shall be free to use all results proven in class as well as in the assessed coursework, unless you are being explicitly asked to (re)produce a proof. Obviously, the solution of a problem will never be just “This was proven in class”.

The PDE chapter (Chapter 10 in the current version of the notes) is for mastery students only.

Problems in the notes which have a (\*) are more advanced problems. You can go into the exam confidently without having attempted or solved these problems.

## 2 Useful things to revise

1. Statement and key idea in the proof of the big theorems:  
Hahn-Banach, Banach-Steinhaus, open mapping & closed graph, spectral theorem for compact self-adjoint operators on Hilbert spaces.
2. Section 6, Statements of Section 7, Section 8: be comfortable with computations in orthonormal bases, make sure you understand all the details in 8.3
3. Worksheet 2
4. the exams of the previous two years

## 3 Some more problems to practise

1. Write down a bounded and an unbounded linear operator. Write down a closed operator. Write down a compact operator.
2. Compute the norm of the functional  $f : (C[0, 1], \max) \rightarrow \mathbb{R}$  given by

$$f(x) = \left[ \int_0^1 dx(t) \right] - x\left(\frac{1}{2}\right).$$

3. Let  $T : X \rightarrow X$  be a bounded linear operator on a Banach space  $X$ . If  $\|T\| < 1$ , then the operator  $id - T$  is invertible and

$$(id - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

HINT: Set  $S_n = \sum_{k=0}^n T^k$  and prove  $S_n$  is Cauchy.

4. Suppose  $T$  is a linear operator on a Hilbert space  $X$  such that  $D(T) = X$  and

$$\langle x, Ty \rangle = \langle Tx, y \rangle \quad \text{for all } x, y \in X$$

Show that  $T$  is bounded. (HINT: Exercise 6 of Week 4 might be useful.)

5. Let  $x_n \rightarrow x$  in a Banach space  $X$ . Show that there is a linear combination of the  $x_n$  which strongly converges to  $x$ . (HINT: Hahn-Banach, more precisely Theorem 5.5 of the notes!)
6. Suppose  $T : H \rightarrow H$  with  $H$  a Hilbert space is bijective and compact. Can  $H$  be infinite dimensional?
7. What is the Fredholm alternative?