Lecture Notes Vienna

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Abstract

These notes accompany a course taught at the ESI-EMS-IAMP Summer school on Mathematical Relativity, Vienna, July 28th-August 1st.

1 Structure

- Lecture 1-1.5: Motivation. Wave equation on Minkowski space; energy momentum tensor, decay estimates from vectorfield multipliers and commutators
- Lecture 1.5-3: Boundedness of solutions to the wave equation on Schwarzschild; the red shift effect
- Lecture 4: Decay for solutions to the wave equation on Schwarzschild (statement, some ideas of the proof, the trapping phenomenon)
- Lecture 5: Outlook: The Kerr case (statement + difficulties), the wave equation on Schwarzschild (anti-) de Sitter space, progress on the stability problem

2 Motivation and Reminder

In the L5 course you noticed that the Einstein vacuum equations

$$R_{\mu\nu}\left[g\right] = 0 \tag{1}$$

are hyperbolic and can, in so-called *harmonic gauge*, be thought of as a system of quasi-linear wave equations. You also learned that this (constrained) system is locally well-posed.

A natural follow-up question is to ask about the *global* behaviour of solutions arising from the Cauchy problem. That is, for general initial data satisfying the constraint equations we may (for example) ask whether trapped surfaces, black holes and/ or singularities form in evolution.

This general question turns out to be very hard due to the non-linear nature of the Einstein vacuum equations (1). Instead it may be more reasonable to ask

1. Can we understand the evolution near "known" stationary solutions of the Einstein equations: Minkowski space, Schwarzschild or Kerr (Stability)
2. Can we construct (hopefully large classes of) special data for which we can guarantee the formation of trapped surfaces/ black holes?

We will focus here on the first question. As it turns out, even to show the stability of the trivial solution, Minkowski space, is a very hard problem and required a 500 page book (Christodoulou-Klainerman, 1989) to get resolved. The stability of the Kerr family of black holes, on the other hand, is an open problem.

A natural strategy to address a non-linear stability problem is

1. Linearize! Understand the linear problem very well, i.e. show that solutions *decay*
2. Understand the non-linear problem; this usually requires to identify some structure in the non-linearity

Naively linearising the Einstein equations near Minkowski space yields (decoupled!) wave equations, so this part is easy! For any other background, in particular black hole solutions, the metric is not flat and even the linearisation is non-trivial! Nevertheless, for most of the course we shall be concerned with the problem of understanding solutions ψ to the covariant wave equation

$$\Box_g \psi := g^{\mu\nu} \nabla_\mu \nabla_\nu \psi = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \psi \right) = 0$$

for g a black hole spacetime (which will be Schwarzschild for us).

Related problems concern Maxwell's equations on black hole backgrounds and the full system of gravitational perturbations.

3 The wave equation in Minkowski space

It is fruitful to develop and illustrate the techniques we will need in the black hole case first for the simpler case of Minkowski space, for which we have better geometric intuition. We have

$$\Box_\eta \psi := \eta^{\mu\nu} \partial_\mu \partial_\nu \psi = -\partial_t^2 \psi + \Delta \psi = 0 \,. \tag{2}$$

As you know, the wave equation is well-posed after specifying appropriate initial data, for instance $\psi(0, \mathbf{x}) = f(\mathbf{x})$ and $\partial_t \psi(0, \mathbf{x}) = g(\mathbf{x})$ (suitably smooth[1]) along the initial hypersurface $t = 0$. Moreover, because Minkowski space is so symmetric (Lorentz invariance – make sure you understand what is meant by the Lorentz invariance of the wave operator (2)), one can – and you probably have in a PDE course – write down an explicit solution formula (known as Kirchhoff's formula) for the solution in terms of the data f and g. We have

$$\psi(\mathbf{x}, t) = \frac{1}{4\pi t^2} \int_{|\mathbf{y}-\mathbf{x}|=t} \left(t g(\mathbf{y}) + f(\mathbf{y}) + \sum_i f_{y_i} (y_i - x_i) \right) dS_{\mathbf{y}} \,. \tag{3}$$

Proposition 3.1. *Any C^2 solution of the initial value problem for the wave equation is given by the above formula. Conversely given $f \in C^3$ and $g \in C^2$ the above defines a C^2 solution of the wave equation assuming the prescribed initial data at $t = 0$.*

Many features of the wave equation can be read off from this representation

- domain of dependence, domain of influence, Huygens principle, finite speed of propagation

[1] I avoid making precise statements in Sobolev spaces at this point.

- regularity loss at the C^k level
- decay of solutions (at least $1/t$ for solutions of compact support).

When we consider more complicated backgrounds, like Schwarzschild or Kerr (or perturbations thereof!), the nice and explicit representation formula above breaks down. We would like to develop methods which capture the above features in a more robust fashion than reading them off from a representation formula.

3.1 The energy estimate

Let us assume that we have a classical C^2 solution of $\Box\psi = 0$ on $[0,T]\times\mathbb{R}^d$ with "data" $\psi(0,x) = u_0(x)$ and $\partial_t\psi(0,x) = u_1(x)$. Multiplying the wave equation by $-\partial_t\psi$ yields

$$0 = -\Box\psi\cdot\partial_t\psi = \frac{1}{2}\partial_t(\partial_t\psi)^2 - \nabla_x(\partial_t\psi\nabla_x\psi) + \nabla_x\partial_t\psi\cdot\nabla_x\psi = 0$$

or

$$0 = \frac{1}{2}\partial_t\left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] - \nabla_x(\partial_t\psi\nabla_x\psi)\ . \tag{4}$$

If we integrate this over the spacetime slab $[0,T]\times\mathbb{R}^d$, then assuming that u decays sufficiently rapidly near infinity (more on this below!) we would obtain the energy conservation law

$$\int_{t=T} d^dx\left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] = \int_{t=0} d^dx\left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right]$$

As this works for any $\tau\le T$ we obtain

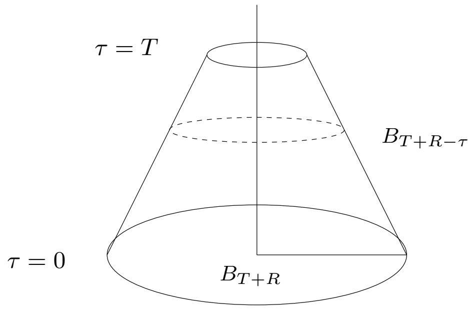
$$\|\partial_t\psi(\tau,\cdot)\|_{L^2_x} + \|\psi(\tau,\cdot)\|_{\mathring{H}^1_x} = \|u_1\|_{L^2} + \|u_0\|_{\mathring{H}^1}\ . \tag{5}$$

In order to derive this identity we have assumed that u is C^2 and that it vanishes sufficiently rapidly near spatial infinity in order to make the boundary term arising from $\nabla_x(\partial_t\psi\nabla_x\psi)$ vanish. We will now see that we can do much better if we suitably localize the estimate.

Fix $T>0$, $R>0$ and consider a region

$$K = \bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau} \tag{6}$$

where $B_{R+T-\tau}$ is the ball of radius $R+T-\tau$ centered at the origin.



You may think of this region as a cut-off (at $t=0$ and $t=T$) past light cone with tip at $\left(T+R,\vec{0}\right)$. We will denote the boundary of $B_{R+T-\tau}$ in $\mathbb{R}^3$ by $S_{R+T-\tau}$ and the unit outward normal to this boundary by N.

Integrating (4) over the region K then yields

$$\begin{aligned} & \frac{1}{2}\int_{\{t=T\}\times B_R} d^dx \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] \\ + \int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} & \left[\frac{1}{2}(\partial_t\psi)^2 + \frac{1}{2}|\nabla_x\psi|^2 - \partial_t\psi\cdot N\psi\right] d\sigma_{S_{R+T-\tau}} \\ & = \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^dx \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] \end{aligned} \tag{7}$$

It is not hard to see using Cauchy-Schwarz that the integrand in the second line is non-negative. We can actually obtain something more quantitative. Let us denote the induced gradient on the spheres $S_{R+T-\tau}$ by $\nabla\!\!\!/$ (i.e. the derivatives *tangent* to these $d-2$ dimensional spheres). We may decompose

$$\partial_t = N + V$$

where V is a derivative tangent to the wall of the cone[2] Then, from the easily verified identities

$$\begin{aligned} -\partial_t u N u &= -(Nu)^2 - Nu\cdot Vu \\ \frac{1}{2}\partial_t u\partial_t u &= \frac{1}{2}(Nu)^2 + Nu\cdot Vu + \frac{1}{2}(Vu)^2 \\ \frac{1}{2}|\nabla_x u|^2 &= \frac{1}{2}(Nu)^2 + \frac{1}{2}|\nabla\!\!\!/ u|^2 \end{aligned}$$

we see that (7) becomes

$$\begin{aligned} & \frac{1}{2}\int_{\{t=T\}\times B_R} d^dx \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] \\ + \int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} & \left[\frac{1}{2}(V\psi)^2 + \frac{1}{2}|\nabla\!\!\!/\psi|^2\right] d\sigma_{S_{R+T-\tau}} \\ & = \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^dx \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] \end{aligned} \tag{8}$$

This identity is truly remarkable and illustrates the domain of dependence property of the wave equation. Indeed, we certainly have

$$\int_{\{t=T\}\times B_R} d^dx \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] \le \int_{\{t=0\}\times B_{R+T}} d^dx \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2\right] \tag{9}$$

and hence

Corollary 3.1. *Suppose $\psi = 0$ in $\{t=0\}\times B_{R+T}$. Then $\psi = 0$ in $\bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau}$.*

Corollary 3.2. *Two C^2 solutions u and v in $K = \bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau}$ that satisfy $u = v$ and $\partial_t u = \partial_t v$ on $\{t=0\}\times B_{R+T}$ have to agree in all of K.*

Exercise 3.1. *Can you generalize this domain of dependence/ uniqueness properties to more general wave equations? Hint: Gronwall's inequality.*

[2] In polar coordinates $\partial_t = \partial_r + (\partial_t - \partial_r)$ since the wall of the cone is given by zero set of $H(t, x_1, \ldots, x_d) = t + \sqrt{x_1^2 + \ldots x_d^2} - R - T = t - T + r - R$, so that indeed $(\partial_t - \partial_r)H = 0$.

Let us understand a bit better the underlying geometry of this computation. The expression (4) is apparently a boundary term and it will induce different expressions dependent on the geometry of the boundary hypersurfaces. What is useful in the estimates is if the expressions induced are non-negative, as it was the case for the hypersurfaces of constant t and the characteristic hypersurfaces discussed above.

Exercise 3.2. *Obtain the energy estimate for two homologous spacelike hypersurfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, i.e. spacelike hypersurfaces with common boundary $\partial\mathcal{S}_1 = \partial\mathcal{S}_2$ bounding a region.*

3.2 The energy momentum tensor

We now turn to a more systematic and geometric interpretation of the above computation, which will also pave the way for further insights about the wave equation.

We define the energy momentum tensor[3]

$$T_{\mu\nu}\left[\psi\right] = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2}\eta_{\mu\nu}\left(\partial\psi\right)^2 .$$

If ψ solves the wave equation $\Box_\eta \psi = 0$, then

$$\partial_\mu T^{\mu\nu} = 0 \, ,$$

i.e. the energy momentum tensor is divergence-free. Consequently, if X is a spacetime vectorfield, we have

$$\partial^\mu \left(T_{\mu\nu} X^\nu\right) = \frac{1}{2} T_{\mu\nu}\left(\partial^\mu X^\nu + \partial^\nu X^\mu\right) = T_{\mu\nu}{}^{(X)}\pi^{\mu\nu} \, , \tag{10}$$

thereby defining the deformation tensor associated with the vectorfield X

$$2^{(X)}\pi^{\mu\nu} = \partial^\mu X^\nu + \partial^\nu X^\mu .$$

Note that this is precisely the Lie-derivative of the metric η! In particular,

If X is a Killing field, then ${}^{(X)}\pi^{\mu\nu} = 0$.

Exercise 3.3. *Can you understand the Poincare group of special relativity in this context?*

Using the definitions

$$J^{(X)}_\mu\left[\psi\right] = T_{\mu\nu}\left[\psi\right] X^\nu \qquad \text{and} \qquad K^X\left[\psi\right] = T_{\mu\nu}\left[\psi\right] {}^{(X)}\pi_{\mu\nu} \, ,$$

we can now revisit our above computation with $X = \partial_t$ and integrate over a spacetime region using Stokes theorem. Note in particular $T_{tt} = \frac{1}{2}|D\psi|^2 := \frac{1}{2}\left(\partial_t\psi\right)^2 + \frac{1}{2}|\nabla_x\psi|^2$. This is a manifestation of a much more general property of the energy momentum tensor:

Proposition 3.2. *(Positivity property)*

$$\boxed{T_{\mu\nu}X^\mu Y^\nu \geq 0 \ \ \textit{for } X, Y \textit{ both future directed causal} \quad \textit{(positivity property)}}$$

As the proof shows we actually have

$$T_{\mu\nu}X^\mu Y^\mu \geq b|D\psi|^2 \quad \text{for } X, Y \text{ future directed timelike.}$$

[3] One can motivate this from the Lagrangian formulation....

Exercise 3.4. *Proof the above property. Outline: Let X, Y future directed causal. If they are collinear the result is a really simple exercise, so let's assume they are not. Use the following two hints.*

Hint 1: If L and $\underline{L}$ are two future directed non-collinear null-vectors, normalised such that $g(L, \underline{L}) = -2$, then we have

$$T(L, L) = (L\psi)^2 \quad , \quad T(L, \underline{L}) = [e_1(\psi)]^2 + [e_2(\psi)]^2 \quad , \quad T(\underline{L}, \underline{L}) = (\underline{L}\psi)^2$$

where e_1, e_2 are two spacelike unit-vectors making $(L, \underline{L}, e_1, e_2)$ a null-frame.

Hint 2: We can write X and Y as linear combinations of two future directed causal null vectors L and $\underline{L}$ with positive coefficients.

Remark 3.1. *It is of course a natural question to ask what happens for other vectorfields (or, in terms of the "old" language, what happens for multiplying the wave equation with other objects and integrating by parts). We see that*

- *all Killing fields are nice candidates and lead to conservations laws but only the timelike ones have the positivity property when integrating (10) between spacelike slices.*
- *conformal Killing fields $K = u^2\partial_u + v^2\partial_v$ is timelike, satisfies ${}^{(K)}\pi \sim g$ and leads to something useful (more later)*

3.3 The wave equation on a general $(\mathcal{M}, g)$

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. We define

$$\Box_g \psi := g^{\mu\nu}\nabla_\mu\nabla_\nu\psi = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\psi\right) = 0\,,$$

which can be derived from the action

$$S[\psi] = \int_{\mathcal{M}} g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi d\mu_g\,.$$

The energy momentum tensor is defined as

$$T_{\mu\nu} := \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi\right)$$

and satisfies the energy identity

$$\nabla^\mu\left(T_{\mu\nu}X^\mu\right) = {}^{(X)}\pi_{\mu\nu}T^{\mu\nu} \tag{11}$$

for the deformation tensor

$$2^{(X)}\pi_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu\,.$$

Again we will use the notation

$$J^{(X)}_\mu[\psi] = T_{\mu\nu}[\psi]X^\nu \quad \text{and} \quad K^X[\psi] = T_{\mu\nu}[\psi]\,{}^{(X)}\pi^{\mu\nu}$$

and the main identity

$$\nabla^\mu J^{(X)}_\mu[\psi] = K^X[\psi] \tag{12}$$

Exercise 3.5. *Repeat the proof of the all important positivity property, Proposition 3.2.*

Integrating the identity over a spacetime region $\mathcal{R}$ we obtain

$$\int_{\mathcal{R}} J_\mu^{(X)}[\psi]\, n^\mu_{\partial\mathcal{R}} = \int_{\mathcal{R}} K^{(X)}[\psi] \tag{13}$$

where the normal vector to the boundary is defined by the following figure

3.4 An integrated decay estimate

Another fundamental idea is to exploit the identity (10) in a different way. Note that after integration both sides just depend on ONE derivative of ψ. Hence we can also try to estimate the right hand side from the left.

Our goal is to construct a multiplier which yields

Proposition 3.3. *Any C^2 solution of the wave equation $\Box_\eta \psi = 0$ satisfies the estimate*

$$\int_{t=0}^{\infty} dt \int_{\Sigma_t} \frac{1}{r}|\nabla\!\!\!\!/\,\psi|^2 \leq CE_0[\psi] := C\left[\int_{\Sigma_0} |\partial_t\psi|^2 + |\partial_x\psi|^2 + |\partial_y\psi|^2 + |\partial_z\psi|^2\right].$$

for a uniform constant C.

Note that the above estimate already captures in some sense that the angular derivatives of ψ have to decay in time. Similar estimates can then be derived for the other derivatives of ψ and the decay statements can then be improved further.

Proof. [Morawetz] Note first that we have on Σ_t by a Hardy inequality and energy conservation (ψ of compact support)

$$\int_0^\infty \int_{S^2} \psi^2 dr d\omega \leq C \int_0^\infty \int_{S^2} r^2 (\partial_r\psi)^2 dr d\omega \leq CE_0[\psi]\,. \tag{14}$$

We choose

$$X = \partial_r$$

and apply the energy identity (10) in a region

$$\mathcal{M} = \mathcal{M}(0,T) \setminus \{(0,t)\times B_\epsilon\}\,.$$

Draw picture, identify normals... We compute

$$K^X[\psi] = \left(\frac{1}{r}\right)(\partial_t\psi)^2 + \left(-\frac{1}{r}\right)(\partial_r\psi)^2 \tag{15}$$

and

$$J_\mu^{(X)}(\partial_t)^\mu = T[\psi](\partial_r,\partial_t) = \partial_t\psi\partial_r\psi \leq (\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!\!/\,\psi|^2$$

$$J_\mu^{(X)}(\partial_r)^\mu = T[\psi](\partial_r,\partial_r) = \frac{1}{2}(\partial_t\psi)^2 - \frac{1}{2}|\nabla\!\!\!\!/\,\psi|^2 + \frac{1}{2}|\partial_r\psi|^2$$

For a smooth function h depending on r we have the identity ($dvol = dtdr\sin\theta d\theta d\phi$)

$$
\begin{aligned}
&\int_{\mathcal{M}} dvol\partial^\mu \left(h\psi\partial_\mu\psi - \frac{1}{2}\left(\psi^2\partial_\mu h\right)\right) \\
&= \int_{\mathcal{M}} dvol\left[h\left((-\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right) - \frac{1}{2}\psi^2\cdot\Box_\eta h\right] \qquad (16)
\end{aligned}
$$

Choosing $h = \frac{1}{r}$ and adding the two identities yields

$$
\int_{\mathcal{M}} dvol\frac{1}{r}|\nabla\!\!\!/\psi|^2 + \int_0^T dt\int_{S^2} d\omega\psi^2\,(r=0) \leq CE_0\,[\psi]\ . \qquad (17)
$$

after realising that

- The boundary terms on constant t can be estimated by $E_0\,[\psi]$, in particular the term $\int_{\Sigma_t}\frac{1}{r}\psi\partial_t\psi$ coming from (16) can be handled using Cauchy Schwarz and (14).
- The boundary term at infinity vanish. The one on the cylinder has unit normal $-\partial_r$ leading to a positive term and terms that vanish in the limit $\epsilon \to 0$.

□

4 The wave equation on Schwarzschild

4.1 Schwarzschild: basic geometry

Recall the Penrose diagram of Schwarzschild that you have seen in the introductory lecture. The crucial new feature here is the presence of the horizon $\mathcal{H}^+$, a null-hypersurface. We write the Schwarzschild metric in $(t^\star, r, \theta, \phi)$ coordinates

$$
g_S = -\left(1-\frac{2M}{r}\right)(dt^\star)^2 + \frac{4M}{r}dt^\star dr + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\sigma^2
$$

covering the exterior region.

Exercise 4.1. *Restricting to the region $r > 2M$, show that the Schwarzschild metric takes its "familiar" form after doing the coordinate transformation*

$$
t^\star = t + 2M\log(r-2M)\ .
$$

The components of the inverse metric of g_S are

$$
\begin{aligned}
g^{t^\star t^\star} &= -\left(1+\tfrac{2M}{r}\right), & g^{t^\star r} &= \tfrac{2M}{r}, \\
g^{rr} &= 1-\tfrac{2M}{r}, & g^{AB} &= g\!\!\!/^{AB}.
\end{aligned} \qquad (18)
$$

The volume element is

$$
d\eta = r^2 dt^\star dr d\omega,
$$

where $d\omega$ is the volume element on the unit 2-sphere.

Let $\Sigma_{t^\star}$ denote the hypersurface of constant t. It has unit normal given by

$$
\begin{aligned}
n &= \sqrt{-g^{t^\star t^\star}}\frac{\partial}{\partial t^\star} - \frac{g^{t^\star r}}{\sqrt{-g^{t^\star t^\star}}}\frac{\partial}{\partial r}, \\
n^\flat &= -\frac{1}{\sqrt{-g^{t^\star t^\star}}}dt^\star,
\end{aligned} \qquad (19)
$$

and an induced volume element

$$dS_{\Sigma_t} = \sqrt{-g^{t^\star t^\star}} r^2 dr d\omega \sim r^2 dr d\omega. \tag{20}$$

Note that $\Sigma_{t^\star}$ is a regular spacelike hypersurface, even as it approaches the horizon.

Exercise 4.2. *Show that the hypersurface $r = 2M$ is null. What is its normal?*

4.2 The energy estimate

The vectorfield $T = \partial_{t^\star}$ is Killing. Therefore

$$K^{(T)}[\psi] = 0$$

and

$$\begin{aligned} J^{(T)}_\mu [\psi] n^\mu &= T[\psi] \left(\partial_{t^\star}, \sqrt{-g^{t^\star t^\star}} \partial_{t^\star} - \frac{g^{t^\star r}}{\sqrt{-g t^\star t^\star}} \partial_r \right) \\ &= \frac{1}{2} \left(-g^{t^\star t^\star} (\partial_{t^\star}\psi)^2 + g^{rr} (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) r^2 dr d\omega \end{aligned} \tag{21}$$

$$J^{(T)}_\mu [\psi] T^\mu \Big|_{\mathcal{H}^+} = (\partial_{t^\star}\psi)^2 \,. \tag{22}$$

Our main identity (12) hence provides the energy identity

$$\begin{aligned} &\int_{\Sigma_{t^\star_2}} \left(\left(1 + \frac{2M}{r}\right) (\partial_{t^\star}\psi)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) r^2 dr d\omega \\ &\qquad\qquad + 2 \int_{\mathcal{H}^+ \left(t^\star_1, t^\star_2\right)} (\partial_{t^\star}\psi)^2 \, dt^\star d\omega \\ &= \int_{\Sigma_{t^\star_1}} \left(\left(1 + \frac{2M}{r}\right) (\partial_{t^\star}\psi)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) r^2 dr d\omega \,. \end{aligned}$$

The following can be immediately observed

- Away from the horizon we control all derivatives, however the control degenerates near the horizon (for the transversal derivative)
- On the horizon itself, we only control the $\partial_{t^\star}$ derivative.
- energy on spacelike slices non-increasing (energy leaves through the horizon)

Note that the above identity is consistent with ψ blowing up along the event horizon in evolution!

Can we improve this estimate? Our goal will be to prove the following

Proposition 4.1. *For any sufficiently smooth solution of the wave equation on Schwarzschild, we have the estimate*

$$\begin{aligned} &\int_{\Sigma_{t^\star}} r^2 dr d\omega \left[(\partial_{t^\star}\psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] \\ &\leq C \int_{\Sigma_{t^\star = 0}} r^2 dr d\omega \left[(\partial_{t^\star}\psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] \end{aligned} \tag{23}$$

for any $t^\star \geq 0$.

4.3 The redshift

4.3.1 The heuristics

Consider two stationary observers A and B following sitting at fixed $r_A > 2M$ and $r_B > r_A > 2M$ respectively (i.e. following the orbits of the timelike Killing field $\partial_{t^\star}$). If A sends two signals at coordinate time $t_1^\star$ and $t_2^\star$ then since $\gamma_A(t^\star) = (t^\star, r_A, \theta, \phi)$ the *proper time* elapsed between the two signals is $s_A = \int_{t_1^\star}^{t_2^\star} dt^\star \sqrt{-g(\dot{\gamma}, \dot{\gamma})} = \sqrt{1 - \frac{2M}{r_A}} (t_2^\star - t_1^\star)$. Similarly, for B, the *proper time* elapsed between receiving the two signals is $s_B = \sqrt{1 - \frac{2M}{r_B}} (t_2^\star - t_1^\star)$. Interpreting this result in terms of waves and the two signals as measuring the time between two wave crests, we see that light is redshifted (i.e. the frequency energy *decreases*) as it travels away from the horizon. If we are willing to invoke that quantum mechanically $E = \hbar\omega$ we conclude that the energy decreases (which is also clear from the photon needing energy to escape from the gravitational field).

4.3.2 The redshift multiplier

Consider the vectorfield

$$N = g(r) T - f(r) \partial_r$$

for functions g and f to be determined. The idea will be to choose g and f very carefully near the horizon so that the following Proposition is true, which is taken due to Dafermos-Rodnianski. The formulation of the Proposition and the proof is taken from the Zurich Nachdiplom lecture notes of Dafermos:

Proposition 4.2. *There exists a vectorfield N such that*

1. $(\phi_\tau)_\star N = N$ *(the vectorfield is $t^\star$-time-independent)*
2. *N is future directed timelike*
3. *$N = T$ in $\Sigma_0 \setminus B$ where B is compact*
4. *There exists an $r_0 > 2M$ such that on $\Sigma_{t^\star} \cap \{r \leq r_0\}$ we have*

$$K^N[\psi] \geq b J_\mu^{(N)} N^\mu \tag{24}$$

Proof. It suffices to construct a timelike vectorfield N_0 along Σ_0 which satisfies property 4 on $\Sigma_0 \cap \{r \leq r_1\}$ for some $r_1 > 2M$. This is because given such N_0 we define

$$r_0 = 2M + \frac{r_1 - 2M}{2} < r_1$$

and a cut-off function χ which is 1 for $2M < r < r_0$ and zero for $r > r_1$. The vectorfield

$$\tilde{N}_0 = \chi N_0 + (1 - \chi) T$$

is then future directed timelike (why?) and satisfies property 4 for $r \leq r_0$, as well as property 3. Pushing $\tilde{N}_0$ forward along the integral curves of $\partial_{t^\star}$ hence yields the desired N.

Actually, by continuity it suffices to ensure that (24) holds on the horizon sphere $S_0 = \Sigma_{t^\star=0} \cap \{r = 2M\}$ (why?).

Note that the vectorfield $-2\partial_r + 2\partial_{t^\star}$ is null on $\mathcal{H}^+$ (as $g_{rr} - 2g_{rt^\star} + g_{t^\star t^\star} = 2 - 2 + 0 = 0$ on $\mathcal{H}^+$) and satisfies $g(-2\partial_r + 2\partial_{t^\star}, T) = -2$.

We would like to extend the vectorfield $-2\partial_r + 2\partial_{t^\star}$ off the horizon to a vectorfield Y satisfying that

$$\nabla_Y Y = -\sigma\left(Y + T\right) \quad \text{holds } \underline{\text{on } S_0}$$

for a large positive constant σ. To do this, take

$$Y = \left[-2 + k_1\left(1 - \frac{2M}{r}\right)\right]\partial_r + \left[2 + k_2\left(1 - \frac{2M}{r}\right)\right]\partial_{t^\star}$$

and choose the constants k_1 and k_2 appropriately. We claim that $N_0 = Y + T$ satisfies (24) near S_0 on Σ_0.

To show this last part, we perform computations in a null-frame (T, Y, E_1, E_2) at a point on the sphere S_0. We compute at p

$$\begin{aligned}
\nabla_T Y &= -\kappa Y + a^1 E_1 + a^2 E_2 \\
\nabla_Y Y &= -\sigma T - \sigma Y \\
\nabla_{E_1} Y &= h_1^1 E_1 + h_1^2 E_2 - \frac{1}{2} a^1 Y \\
\nabla_{E_2} Y &= h_2^1 E_1 + h_2^2 E_1 - \frac{1}{2} a^2 Y
\end{aligned}$$

These formulae are easily verified: The second holds by definition, for the third and fourth we only observe $g\left(\nabla_{E_1} Y, Y\right) = 0$, for the first $g\left(\nabla_T Y, Y\right) = 0$. (Why does the same a^1, a^2 appear in the first and third and fourth?). The κ appearing in the first equation is the surface gravity of the Schwarzschild horizon, $\kappa = \frac{1}{4M}$, which was defined in previous lectures ($\nabla_T T = \kappa T$). We now compute

$$2^{(Y+T)}\pi\left(V, W\right) = 2^{(Y)}\pi\left(V, W\right) = g\left(\nabla_W Y, V\right) + g\left(\nabla_V Y, W\right)$$

$$\begin{aligned}
K^N = K^Y &= \frac{1}{2}\left[\mathbb{T}\left(Y, Y\right)\kappa + \mathbb{T}\left(T, Y\right)\sigma + \mathbb{T}\left(T, T\right)\sigma\right] \\
&\quad - \frac{1}{2}\left(\mathbb{T}\left(E_1, Y\right) a^1 + \mathbb{T}\left(E_2, Y\right) a^2\right) \\
&\quad + \mathbb{T}\left(E_1, E_2\right) h_1^1 + \mathbb{T}\left(E_2, E_2\right) h_2^1 + \mathbb{T}\left(E_1, E_2\right)\left(h_1^2 + h_2^1\right) \qquad (25)
\end{aligned}$$

The terms in the first line are all good and control all derivatives, in fact we have σ as a largeness parameter at our disposal. The only derivative not controlled with a σ largeness is $|Y\psi|^2$. The terms in lines two and three will be absorbed by the terms in the first line. For this to be possible, note that no $|Y\psi|^2$-terms appear at all in the second and third line! This is because ($\mathbb{T}\left(V, W\right) = V\psi W\psi - \frac{1}{2} g\left(V, W\right)\left[-Y\psi T\psi + |\nabla\!\!\!/\psi|^2\right]$). We hence see that

$$K^N \geq b\left(|Y\psi|^2 + |T\psi|^2 + |\nabla\!\!\!/\psi|^2\right) \geq b J_\mu^{(N)} N^\mu\,.$$

$\square$

4.4 Proof of boundedness

Given the multiplier N we can complete the proof of boundedness. Write the N-identity as

$$\begin{aligned}
&\int_{\Sigma_{t_2^\star}} J_\mu^{(N)} n_\Sigma^\mu + \int_{\mathcal{H}^+\left(t_1^\star, t_2^\star\right)} J_\mu^{(N)} n_{\mathcal{H}}^\mu + \int_{\mathcal{M}\left(t_1^\star, t_2^\star\right)\cap\{r \leq r_0\}} K^N\left[\psi\right] \\
&\qquad = -\int_{\mathcal{M}\left(t_1^\star, t_2^\star\right)\cap\{r_0 \leq r \leq r_1\}} K^N\left[\psi\right] + \int_{\Sigma_{t_1^\star}} J_\mu^{(N)} n_\Sigma^\mu\,.
\end{aligned}$$

Using the above Proposition (and dropping the second (good) term), we find

$$\int_{\Sigma_{t_2^\star}} J_\mu^{(N)} n_\Sigma^\mu + b \int_{\mathcal{M}\left(t_1^\star, t_2^\star\right) \cap \{r \leq r_0\}} J_\mu^{(N)} n_\Sigma^\mu$$

$$\leq B \int_{\mathcal{M}\left(t_1^\star, t_2^\star\right) \cap \{r_0 \leq r \leq r_1\}} J_\mu^{(N)} n_\Sigma^\mu + \int_{\Sigma_{t_1^\star}} J_\mu^{(N)} n_\Sigma^\mu . \tag{26}$$

To estimate the "bad" term on the right hand side we recall the T identity

$$\int_{\Sigma_{t^\star}} J_\mu^{(T)} n_\Sigma^\mu + \int_{\mathcal{H}^+\left(t_0^\star, t^\star\right)} J_\mu^{(T)} n_\Sigma^\mu = \int_{\Sigma_{t_0^\star}} J_\mu^{(T)} n_\Sigma^\mu ,$$

from which we derive

$$b \int_{\Sigma_{t^\star} \cap \{r \geq r_0\}} J_\mu^{(N)} n_\Sigma^\mu \leq \int_{\Sigma_{t^\star}} J_\mu^{(T)} n_\Sigma^\mu \leq \int_{\Sigma_{t_0^\star}} J_\mu^{(T)} n_\Sigma^\mu , \tag{27}$$

for b a constant depending on r_0. Integrating the previous estimate in $t^\star$

$$b \int_{\mathcal{M}\left(t_1^\star, t_2^\star\right) \cap \{r \geq r_0\}} J_\mu^{(N)} n_\Sigma^\mu \leq \left(t_2^\star - t_1^\star\right) \cdot \int_{\Sigma_{t_0^\star}} J_\mu^{(T)} n_\Sigma^\mu . \tag{28}$$

Combining (28) with (26), it follows that for any $t_2^\star \geq t_1^\star \geq t_0^\star$ we have

$$\int_{\Sigma_{t_2^\star}} J_\mu^{(N)} n_\Sigma^\mu + b \int_{\mathcal{M}\left(t_1^\star, t_2^\star\right)} J_\mu^{(N)} n_\Sigma^\mu \leq B \left(t_2^\star - t_1^\star\right) \int_{\Sigma_{t_0^\star}} J_\mu^{(T)} n_\Sigma^\mu + \int_{\Sigma_{t_1^\star}} J_\mu^{(N)} n_\Sigma^\mu .$$

Writing

$$f\left(t^\star\right) := \int_{\Sigma_{t^\star}} J_\mu^{(N)} n_\Sigma^\mu \quad \text{and the data quantity} \quad D := \int_{\Sigma_{t_0^\star}} J_\mu^{(T)} n_\Sigma^\mu ,$$

we have

$$f\left(t_2^\star\right) + b \int_{t_1^\star}^{t_2^\star} dt^\star f\left(t^\star\right) \leq f\left(t_1^\star\right) + \hat{B} \cdot D \left(t_2^\star - t_1^\star\right) . \tag{29}$$

This implies (Exercise[4]):

$$f\left(t^\star\right) \leq B f\left(t_0^\star\right) . \tag{30}$$

4.5 Pointwise boundedness

In the previous section we showed that the L^2-based *energy* remains bounded on spacelike slices. Suppose we wanted to prove that ψ itself is *pointwise* uniformly bounded by initial data. The way to do this is via Sobolev embedding. We just quote two standard results here

Proposition 4.3. *Consider the Riemannian manifold $\left(S^2_{t^\star, r}, \not{g}_{AB}\right)$. For $u\left(t^\star, r, \theta, \phi\right)$ smooth we have*

$$\sup_{S^2} |u|^2 = \int_{S^2} \sin\theta d\theta d\phi \left[|r^2 \not\nabla \not\nabla u|^2 + |r \not\nabla u|^2 + |u|^2 \right] .$$

Consider a slice $\Sigma_{t^\star}$. For $u\left(t^\star, r, \theta, \phi\right)$ smooth and of compact support in r we have

$$\sup_{\Sigma_\tau} |u| \leq C \left[\|u\|_{\mathring{H}^2(\Sigma_\tau)} + \|u\|_{\mathring{H}^1(\Sigma_\tau)} \right]$$

with C independent of the size of the support.

[4] Hint: Use the fundamental theorem of calculus to derive $f'(t) + b f(t) \leq \hat{B} \cdot D$.

You can find these results in any textbook on PDEs which has a chapter on Sobolev spaces, e.g. Evans. The results can be interpreted as follows. Pointwise uniform boundedness follows if we can control higher derivatives in L^2.

To achieve this goal we first note that the wave equation commutes with Killing fields, i.e. if K is Killing and ψ satisfies $\Box_g \psi = 0$, then

$$\Box_g K\psi = 0\,.$$

In particular, we can repeat our boundedness proof for $(\partial_{t^\star})^k (\Omega_i)^l \psi$ for any i (with Ω_i denoting a basis of the $SO(3)$ Lie-algebra on the sphere) and any k and l. It is an easy computation to check that for $\not{g} = r^2\gamma_{AB}$ the round metric on a sphere of radius r

$$r^2 \not{g}^{AB} \partial_A \psi \partial_B \psi = r^2 |\nabla\!\!\!/ \psi|^2 \leq C \sum_{i=1}^{3} |\Omega_i \psi|^2\,,$$

that is all angular derivatives of ψ are controlled once we control $\Omega_i\psi$ for all i. In particular, in this way we control

$$\int_{\Sigma_{t^\star}} \left(r^4 |\partial_r \nabla\!\!\!/ \nabla\!\!\!/ \psi|^2 + r^2 |\partial_r \nabla\!\!\!/ \psi|^2 + |\partial_r \psi|^2 + |\nabla\!\!\!/ \psi|^2 + r^2 |\nabla\!\!\!/ \nabla\!\!\!/ \psi|^2\right) r^2 dr d\omega$$
$$\leq C \left(E_0[\psi] + \sum_i E_0[\Omega_i \psi] + \sum_{i,j} E_0[\Omega_i \Omega_j \psi] \right)$$

with E_0 denoting the non-degenerate N-energy.

Now observe that on $\Sigma_{t^\star}$ we have

$$\int_R^\infty \int_{S^2} \psi^2 dr d\omega + \int_{S^2_{t^\star,R}} \psi^2 r d\omega \leq C \int_R^\infty \int_{S^2} (\partial_r \psi)^2 r^2 dr d\omega$$

for every $r \geq 2M$ and ψ of compact support on $\Sigma_{t^\star}$ by a simple Hardy inequality.

Combining the above with the Sobolev inequality on the sphere yields a pointwise bound on ψ.

What if the background is not spherically symmetric? How do you obtain bounds for higher order (transversal) derivatives? The key is to use elliptic estimates away from the horizon and a commutation with the redshift vectorfield near the horizon.

4.5.1 Elliptic Estimates

Away from the horizon one can do elliptic estimates. Let us write the wave equation as

$$\frac{1}{r^2}\partial_r \left(r^2 (1-\mu) \partial_r \psi\right) + \Delta\!\!\!/ \psi = \left(1 + \frac{2M}{r}\right) \partial_{t^\star}^2 \psi - \frac{2M}{r} \partial_{t^\star} \partial_r \psi - \frac{2M}{r^2} \partial_{t^\star} \psi$$

Strictly away from the horizon, the left hand side is a uniformly elliptic operator on Σ_τ and the right hand side in in L^2 by previous estimates. This allows to estimate *all* spatial derivatives away from the horizon.

$$\|\psi\|^2_{\mathring{H}^2(\Sigma_{t^\star} \cap \{r \geq r_0\})} + \|\psi\|^2_{\mathring{H}^1(\Sigma_{t^\star} \cap \{r \geq r_0\})} \leq C \left(E_0[\psi] + E_0[T\psi]\right)$$

This works whenever ∂_t is timelike.

4.5.2 The redshift commutation

To obtain all derivatives near the horizon (or even higher transversal derivatives in the Schwarzschild case, which can never be obtained via commutation by $\partial_{t^\star}$ or Ω_i alone!), one needs to commute with the redshift vectorfield. (The underlying general idea is: We can commute by vectorfields which are not Killing just as we can apply the basic vectorfield identity (12) for non-Killing fields. We can hope that for geometrically chosen vectorfields the error made by the commutation is "good" in the sense that the term will contribute the correct sign in the energy identity.)

Let us define $\hat{Y} = -\partial_r$ and recall that this is the missing transversal derivative on the horizon. We have

Proposition 4.4. *Let ψ satisfy $\Box_g \psi = 0$. Then*

$$\Box_g \left(\hat{Y}\psi\right) = \left(\frac{2}{r} - \frac{2M}{r^2}\right)\hat{Y}\left(\hat{Y}\psi\right) - \frac{4}{r}\left(\hat{Y}\left(T\psi\right)\right) + P_1\psi\,,$$

where $P_1\psi = \frac{2}{r^2}\left(T\psi - \hat{Y}\psi\right)$ is a first order operator.

Note that the first term has a *positive* coefficient on (and hence near) the horizon! If you were doing this computation for a general horizon, you would see the surface gravity κ appearing here!

Proof. Direct computation. $\square$

We now show that one can control second transversal derivatives near the horizon. By the boundedness statement we have

$$\int_{\mathcal{M}(0,\tau)\cap\{r\le r_0\}} K^N\left[\psi\right] \le BD\tau \tag{31}$$

and commuting the wave equation with T (and repeating the boundedness proof) we have also

$$\int_{\mathcal{M}(0,\tau)\cap\{r\le r_0\}} \left(\hat{Y}T\psi\right)^2 \le \int_{\mathcal{M}(0,\tau)\cap\{r\le r_0\}} K^N\left[T\psi\right] \le BD\tau\,, \tag{32}$$

where D is an initial data quantity (more precisely, $E_0\left[\psi\right]$ and $E_0\left[T\psi\right]$).

We now commute the wave equation with $\hat{Y}$ and apply the N multiplier.[5] This produces

$$\begin{aligned}\int_{\Sigma_\tau} J^N_\mu\left[\hat{Y}\psi\right]n^\mu_\Sigma + \int_{\mathcal{H}^+(0,\tau)} J^N_\mu\left[\hat{Y}\psi\right]n^\mu_{\mathcal{H}} + \int_{\mathcal{M}(0,\tau)\cap\{r\le r_0\}} K^N\left[\hat{Y}\psi\right] \\ = \int_{\Sigma_0} J^N_\mu\left[\hat{Y}\psi\right]n^\mu_\Sigma + \int_{\mathcal{M}(0,\tau)\cap\{r_0\le r\le r_1\}}\left(-K^N\right)\left[\hat{Y}\psi\right] \\ + \int_{\mathcal{M}(0,\tau)\cap\{r\le r_0\}} \mathcal{E}^N\left[\hat{Y}\psi\right] + \int_{\mathcal{M}(0,\tau)\cap\{r_0\le r\le r_1\}} \mathcal{E}^N\left[\hat{Y}\psi\right]\,,\end{aligned}$$

with

$$\mathcal{E}^N\left[\hat{Y}\psi\right] = -N\hat{Y}\left[\left(\frac{2}{r} - \frac{2M}{r^2}\right)\hat{Y}\left(\hat{Y}\psi\right) - \frac{4}{r}\left(\hat{Y}\left(T\psi\right)\right) + P_1\psi\right]. \tag{33}$$

Now recall

$$N = Y + T = \left(2 + k_1\left(1-\mu\right)\right)\hat{Y} + \left(1 + k_2\left(1-\mu\right)\right)T$$

[5]Note that if $\Box_g\psi = F$ for some right hand side F, the identity (12) generalises to $\nabla^\mu\left(T_{\mu\nu}X^\nu\right) = FX\left(\psi\right) + T_{\mu\nu}{}^{(X)}\pi^{\mu\nu}$.

(for k_1 and k_2 fixed). So the $\left(\hat{Y}\hat{Y}\psi\right)^2$ has a good sign near the horizon in the energy identity!

We have for $r \leq r_0$

$$\mathcal{E}^N\left[\hat{Y}\psi\right] \geq \frac{1}{4M}|\hat{Y}\hat{Y}\psi|^2 - C|\hat{Y}T\psi||\hat{Y}\hat{Y}\psi| - C|\hat{Y}T\psi|^2 - .... \tag{34}$$

(the point being that all terms following the first involve at least one T derivative (or are first order) and are hence a-priori controlled by (31) and (32)!) We finally obtain

$$\int_{\Sigma_\tau} J^N_\mu\left[\hat{Y}\psi\right] n^\mu_\Sigma + \frac{1}{2}\int_{\mathcal{M}(0,\tau)} J^N_\mu\left[\hat{Y}\psi\right] n^\mu_\Sigma = \int_{\Sigma_0} J^N_\mu\left[\hat{Y}\psi\right] n^\mu_\Sigma + BD\tau \tag{35}$$

where we have dropped the horizon term, used the fact that away from the horizon we already control all derivatives from the initial $E_0[\psi]$ and $E_0[T\psi]$ energies through elliptic estimates.

Repeating the argument in the boundedness proof yields an L^2 bound on all second derivatives which have at least one $\hat{Y}$ or one T in them. Now one can do elliptic estimates on the spheres (write the wave equation so that only angular derivatives remain on the left and use the L^2 control on the right hand side).

4.6 Decay

For the linear wave equation we saw that waves actually decay in time. I advertised that the robust way of capturing this (which will also work for metrics close to Minkowski) is via an integrated decay estimate. Can we do something similar, i.e find a multiplier which produces a positive $K^X[\psi]$ and whose boundary terms are controlled by the energy? It turns out there is an important obstruction which goes back to the existence of trapped null-geodesics on black hole backgrounds.

In order to capture decay, we have to slightly alter our foliation by spacelike slices

draw picture with old $\Sigma_{t^\star}$-foliation and new $\tilde{\Sigma}_{t^\star}$ foliation (hyperboloidal slices ending on null-infinity). We let the $\Sigma_{t^\star}$ agree with the $\tilde{\Sigma}_{t^\star}$ for $r = R > 10M$ for convenience.

Exercise 4.3. *Find an explicit parametrisation of the $\tilde{\Sigma}_{t^\star}$-slices in the picture. Does the boundedness proof work for the $\tilde{\Sigma}_{t^\star}$-foliation? How does the energy on $\tilde{\Sigma}_{t^\star}$ and on null-infinity look like?*

4.6.1 The decay statement

Proposition 4.5. *Any sufficiently smooth solution of the wave equation on Schwarzschild satisfies the degenerate estimate*

$$\int_{\tilde{\mathcal{M}}(t_1^\star,t_2^\star)} \frac{1}{r^2}\left(1-\frac{3M}{r}\right)^2\left[(\partial_{t^\star}\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right] \leq C\cdot\tilde{E}[\psi](t_1^\star)$$

as well as the non-degenerate estimate

$$\int_{\tilde{\mathcal{M}}(t_1^\star,t_2^\star)} \frac{1}{r^2}\left[(\partial_{t^\star}\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right] \leq \tilde{E}[\psi](t_1^\star) + \tilde{E}[T\psi](t_1^\star)$$

Of course, our main task will be to explain the degeneration at $r = 3M$!

4.6.2 A few words about the proof

At the technical level the proof proceeds quite similarly to the proof we have seen for Minkowski space. One may start with a multiplier $X = f(r)\,\partial_r$, compute $K^X[\psi]$ and add an appropriate h-identity. One will be forced to let f degenerate at $r = 3M$ in order to make $K^X[\psi]$ manifestly non-negative. Finding f and h from scratch is extremely subtle. (For large angular momentum ℓ for ψ (using the spherical symmetry of the background) there is a simple construction described in the Lecture Notes of Dafermos and Rodnianski).

4.6.3 From integrated decay to pointwise decay

There is a certain black box approach, developed recently by Dafermos and Rodnianski, which allows us to derive from the integrated decay estimate of Proposition 4.5 a statement of decay of the energy (and eventually, if so desired, pointwise bounds).

Theorem 4.1. *We have*

$$\int_{\tilde{\Sigma}_{t^\star}} J^N_\mu[\psi]\, n^\mu \lesssim (t^\star)^{-2} \int_{\tilde{\Sigma}_0} r^2 \left(J^N_\mu[TT\psi] + J^N_\mu[T\psi] + J^N_\mu[T\psi]\right) n^\mu$$

We won't go into this here but I recommend the very readable 14page paper of Dafermos and Rodnianski, arXiv: 0910.4957.

4.6.4 Null-geodesics on Schwarzschild

Let $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a null-gedoesic. We can do the computation in these coordinates because we are most interested in the behaviour away from the horizon. We have

$$-(1-\mu)\,\dot{t}^2 + (1-\mu)^{-1}\,\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 0$$

Since γ is a geodesic and ∂_t is Killing, the quantity

$$g_{\mu\nu}\dot{\gamma}^\mu (\partial_t)^\nu = E$$

is constant along γ (why?). Similarly

$$g_{\mu\nu}\dot{\gamma}^\mu (\partial_\phi)^\nu = L$$

is constant. Finally, we can wlog assume $\theta = \pi/2$ (why?). This yields

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2}\right) = \frac{1}{2}E^2$$

or

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2 \quad \text{with} \quad V = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2}\right)$$

a one-dimensional scattering problem. Note

$$V'(r) = \frac{L^2}{r^4}(3M - r)$$

so the potential has a unique global maximum at $r = 3M$. We already see the existence of null-geodesics remaining stationary at $r = 3M$ (choose

$\frac{L^2}{E^2} = 27M^2$). (Note the geodesic is of course non-radial!) This is an obstruction to decay in the geometric optics (=high frequency) approximation where wave propagation can be approximated by null-geodesics. However, using elementary ODE analysis, one can also see that the phenomenon is unstable.

4.6.5 A quantitative statement

Theorem 4.2 (Sbierski)**.** *There exists a sequence of initial data for the wave equation with corresponding solution ψ_n such that*

- $E_0[\psi_n] = 1$ *for all n and*
- *there exists a positive constant $b > 0$ (not depending on n) such that given any time $T^\star > 0$, there exists an n such that*

$$\int_{\Sigma_{t^\star} \cap \{|r-3M| < \frac{1}{10}\}} \left[(\partial_{t^\star}\psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] \geq b E_0[\psi_n]$$

In other words, the energy can remain concentrated near $r = 3M$ for arbitrary long times without increasing the H^1-norm of the initial data. Of course localising the solution very close to $r = 3M$ will make the H^2-norm large!

Sbierski's theorem tells you why there has to be a degeneration in the first estimate of Proposition 4.5. [Sketch: If there wasn't, then using a dyadic decomposition $t_i^\star = 2^i t_{i-1}^\star$, we could find a sequence of good slices $\tilde{\Sigma}_{\bar{t}_i^\star}$ (with $t_i^\star \leq \bar{t}_i^\star \leq t_{i+1}^\star$) on which in particular the estimate $\int_{\Sigma_{\bar{t}_i^\star} \cap \{|r-3M| < \frac{1}{10}\}} \left[(\partial_{t^\star}\psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] \leq C\left(\bar{t}_i^\star\right)^{-1} \tilde{E}_0[\psi]$ holds. As this estimate holds for any ψ_n (the $\bar{t}_i^\star$ will change but always $t_i^\star \leq \bar{t}_i^\star \leq t_{i+1}^\star$), we can for any solution find a (sufficiently late) slice where the energy near $3M$ is ϵ of the initial energy. This violates Sbierski's theorem.]

5 Instability: Waves on extremal RN backgrounds

Let us recapitulate what we have learned so far. We considered the wave equation on a spherically-symmetric, static background. The time-like Killing field gave us a conservation law and already a boundedness statement away from the horizon. We then used the redshift vectorfield both as a multiplier and as a commutator to obtain L^2 bounds for all derivatives. The latter computation relied on the positivity of the surface gravity of the event horizon. The next example, studied in detail by Stefanos Aretakis shows that our result breaks down in the case of a degenerate horizon.

5.0.6 The Reissner-Nordstroem metric

We consider the Einstein-Maxwell equations

$$R_{\mu\nu} - \frac{1}{2} R_{\mu\nu} = 8\pi T_{\mu\nu}$$

for

$$T_{\mu\nu} = F_{\mu\sigma} F_{\nu}{}^{\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}$$

and F the Maxwell two-form. The following spherically-symmetric metric is a solution of the Einstein-Maxwell system

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\omega^2 \qquad (36)$$

Here Q is the charge of the black hole and $M \geq |Q|$ is the mass. Let us look at the Penrose diagram (again we are interested in the exterior only).

In (v, r, θ, ϕ)-coordinates (horizon penetrating), the metric reads ($v = t + r^\star$)

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dv^2 + 2dvdr + r^2 d\omega^2 \, .$$

Write

$$D = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{1}{r^2}\left(r - r_+\right)\left(r - r_-\right) \quad r_+ \geq r_- \text{ (equality if } |Q| = M)$$

The vectorfield $V = \partial_v$ is Killing and null on the horizon $\mathcal{H}^+$. The surface gravity κ on the event horizon is

$$\kappa = \left(\nabla_V V\right)^v \Big|_{\mathcal{H}^+} = \Gamma^v{}_{vv}\Big|_{\mathcal{H}^+} = \frac{1}{2} g^{vr}\left(-g_{vv}\right)_r \Big|_{r=r_+} = \frac{1}{2} D_{,r}\Big|_{r=r_+} = \frac{1}{2}\frac{r_+ - r_-}{r_+^2}$$

Hence it **vanishes** in the extremal case and is positive in the non-extremal case.

Recall that $\kappa > 0$ was crucial for our boundedness statement! Here we again have boundedness of the *degenerate* energy (why?) but the second (transversal) derivative of ψ actually blows up along the horizon! This can be seen quite easily. Let now $M = |Q|$. Restricting the wave equation

$$\Box_g \psi = D\partial_r\partial_r\psi + 2\partial_v\partial_r\psi + \frac{2}{r}\partial_v\psi + \left(D_{,r} + \frac{2D}{r}\right)\partial_r\psi + \not{\Delta}\psi$$

to the horizon we find

$$\partial_v\left(\partial_r\psi + \frac{1}{M}\psi\right) = -\not{\Delta}\psi \, .$$

Hence $\int_{S^2} \partial_r\psi + \frac{1}{2M}\psi$ is constant along $\mathcal{H}^+$ and does not decay!

Commuting the wave equation by ∂_r and restricting again to the horizon yields (extremal case)

$$|\partial_r\partial_r\psi| \to \infty$$

along $\mathcal{H}^+$.

6 The Wave Equation on Kerr

only heuristics; explain superradiance, small a-case and separation of superradiant and trapped modes (see slides!)