Measure and Integration: Example Sheet 3

Fall 2016 [G. Holzegel]

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1 Properties of $\limsup$ and $\liminf$

Let (a_n) be a sequence of numbers in the extended real numbers $\overline{\mathbb{R}}$. In lectures we defined

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{k\geq n} a_k \quad \text{and} \quad \liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k$$

and observed that while $\lim_{n\to\infty} a_n$ does not necessarily exists, the $\limsup$ and the $\liminf$ always do (why?).

a) Give an example of a sequence for which $\sup a_n$, $\inf a_n$, $\limsup a_n$ and $\liminf a_n$ are all different.

b) Show that if $\limsup_{n\to\infty} a_n = A$ then (a_n) has a subsequence converging to A and A is the largest number with this property.

c) Let (a_n) and (b_n) be sequences in $\overline{\mathbb{R}}$. Show that

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty}(a_n)\,.$$

Show also that (provided none of the sums below is of the form $\infty - \infty$) one has

$$\limsup_{n\to\infty}\,(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n\,.$$

2 G_δ and F_σ sets

a) Show that a closed set is a G_δ and an open set an F_σ. Hint: If F is closed, consider $U_n = \{x \mid d\,(x, F) < \frac{1}{n}\}$.

b) *Give an example of an F_σ that is not a G_δ.
Hint: Consider the rationals in $\mathbb{R}$. In the proof you might need the Baire Category Theorem ("A complete metric space (here $\mathbb{R}$) cannot be written as a countable union of nowhere dense sets.")]

c) *Give an example of a Borel set which is neither a G_δ nor an F_σ.

3 Measurable functions

a) Prove that a monotone function $f : \mathbb{R} \to \mathbb{R}$ is measurable (in fact Borel measurable).

b) Let $f : \mathbb{R}^d \to \mathbb{R}$. Prove that if $\{x \mid f\,(x) \geq r\}$ is measurable for every $r \in \mathbb{Q}$ then f is measurable.

c) If $(f_n) : \mathbb{R}^d \to \mathbb{R}$ is a sequence of measurable functions on $\mathbb{R}^d$, then the set $\{x \mid \lim f_n\,(x) \text{ exists}\}$ is measurable.

4 Approximating measurable functions by continuous ones

Prove that every measurable function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is the limit a.e. of a sequence of continuous functions.
Hint: Recall the approximation theorems (and their proof) given in lectures.

5 The Cantor Function revisited

Recall the function F from Example Sheet 1 (Problem 1), which we constructed as a continuous surjective function $F : \mathfrak{C} \to [0,1]$ from the Cantor Set onto $[0,1]$.

a) Show that F can be extended as a continuous function $f : [0,1] \to [0,1]$.
Hint: If $x \in \mathfrak{C}^c = \bigcup_n U_n$, then x lies in one of the disjoint open intervals of U_N for some N, say (a,b). Show that $F(a) = F(b)$ (using the properties of the Cantor function) and define f to be equal to that number on the interval (a,b). This defines f on all of $[0,1]$ and it is not hard to see that f is continuous.

b) By considering the inverse image of $\mathcal{N} \subset [0,1]$ (the non-measurable set from lectures) with respect to f conclude that a continuous function can map a measurable set to a non-measurable set.

c) By considering the function $g : [0,1] \to [0,1]$ defined by

$$g(y) = \inf\{x \in [0,1] \mid f(x) = y\}$$

conclude that a monotone function can map a non-measurable set to a measurable set.
Hint: First show that g is monotone, injective and maps $[0,1]$ to $\mathfrak{C}$. Now take $\mathcal{N}$ to be the non-measurable set constructed in lectures.

Remark: Part c) can be seen as the reason why we only require the Borel sets to be pulled back to measurable sets in the definition of a measurable function. Indeed, if we included the Lebesgue measurable sets we would (by the above) have to exclude monotone functions, which is clearly undesirable. (Working a bit harder one can also give examples of continuous functions mapping a non-measurable to a measurable set.)