Measure and Integration: Example Sheet 1

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On this example sheet we look at the Cantor Set and its cousins. The exercises are taken from the first chapter of Stein-Shakarchi [SS], mildly edited.

1 The Cantor Set

To construct the Cantor Set $\mathfrak{C}$, let $C_0 = [0,1]$ be the unit interval. The set C_1 is obtained from C_0 by tri-secting C_0 and removing from it the middle-third open interval, i.e. $C_1 = [0,1/3] \cup [2/3,1]$. The C_i for $i \geq 2$ are defined recursively by always deleting the open middle-third of the (disjoint) intervals from C_{i-1}. The Cantor set $\mathfrak{C}$ is finally obtained as the intersection

$$\mathfrak{C} := \bigcap_{n=1}^{\infty} C_n \, .$$

a) Prove that $\mathfrak{C}$ is compact and non-empty.

b) Prove that $\mathfrak{C}$ is totally disconnected, i.e. given x and y in $\mathfrak{C}$ with $x \neq y$ there is a $x < z < y$ with $z \notin \mathfrak{C}$.

c) Prove that $\mathfrak{C}$ does not have isolated points.

d) Prove that $m_\star(\mathfrak{C}) = 0$.

e) Show that we can write C_n as

$$C_n = \bigcup_{a_1,\ldots,a_n \in \{0,2\}} \left[\sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{n} a_k 3^{-k} + \frac{1}{3^n} \right] .$$

Hints: How many intervals of what length does C_n contain? Use induction to prove that the left endpoints of the intervals in C_n are given by $\sum_{k=1}^{n} a_k 3^{-k}$ for the $a_k \in \{0,2\}$.

f) Show that $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \in \{0,2\} \Leftrightarrow x \in \mathfrak{C}$.

g) Define the Cantor-Lebesgue function $F : \mathfrak{C} \to [0,1]$ as

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{for } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \text{ where } b_k = \frac{a_k}{2}.$$

Show that F is well-defined and in fact continuous on $\mathfrak{C}$. Show also that F is surjective. Conclude that $\mathfrak{C}$ is uncountable.

2 Fat Cantor Sets

Construct a closed set $\hat{\mathfrak{C}}$ analogous to the Cantor set as follows: This time we remove at the k^{th} stage 2^{k-1} centrally situated open intervals each of length $\ell_k = \frac{1}{4^k}$. The set $\hat{\mathfrak{C}}$ is again defined as the intersection of the closed sets $\hat{\mathfrak{C}}_k$ appearing at stage k.

a) Show that $m_\star\left(\hat{\mathfrak{C}}\right) = \frac{1}{2}$ and conclude that $\hat{\mathfrak{C}}$ is uncountable.

b) Show that $\hat{\mathfrak{C}}$ is again compact, totally disconnected and has no isolated points.

3 *Characterisation of Riemann integrable functions

Prove that a bounded function on an interval $[a, b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.

This is a hard problem. You should look up the outline of the proof in Stein-Shakarchi (p.47) or elsewhere in the literature. You can use the result at the end of Exercise 4.

4 Limits of continuous functions $f : [0,1] \to \mathbb{R}$ (Exercise 10 of [SS])

In this exercise we construct the sequence of continuous functions promised in the first lecture.

Let $\hat{\mathfrak{C}}$ denote the Fat Cantor Set constructed in Exercise 2. We define the function F_1 to be a piecewise linear continuous function which is equal to 1 on the complement of the first interval $I_1 = [3/8, 5/8]$ removed from $[0,1]$ in the construction of $\hat{\mathfrak{C}}$ and zero at the point at the centre of I_1. Similarly we construct F_2 to be a piecewise linear continuous function which is equal to 1 on the complement of the open intervals removed at the second stage and zero at the centre of these intervals. Continuing in this way we can define F_n for all n and finally

$$f_n = F_1 \cdot F_2 \cdot \ldots \cdot F_n$$

a) Show that for all $n \geq 1$ and all $x \in [0,1]$ one has $0 \leq f_n(x) \leq 1$ and $f_n(x) \geq f_{n+1}(x)$. We conclude that $f_n(x)$ converges to a limit which we denote by $f(x)$.

b) Show that the function f is discontinuous at every point of $\hat{\mathfrak{C}}$. Conclude that f is not Riemann integrable (despite the sequence $s_n = \int f_n$ converging).
Hint: Note that $f(x) = 1$ if $x \in \hat{\mathfrak{C}}$ and find a sequence of points $\{x_n\}$ so that $x_n \to x$ and $f(x_n) = 0$.

5 The outer Jordan content of a set (Exercise 14 of [SS])

The outer Jordan content $J_\star(E)$ of a bounded set E in $\mathbb{R}$ is defined by

$$J_\star(E) = \inf \sum_{j=1}^{N} |I_j|$$

where the infimum is taken over all *finite* coverings $E \subset \bigcup_{j=1}^{N} I_j$ by intervals I_j.

a) Prove that $J_\star(E) = J_\star\left(\overline{E}\right)$ for every set E. Here $\overline{E}$ denotes the closure of E.

b) Exhibit a countable subset $E \subset [0,1]$ such that $J_\star(E) = 1$ while $m_\star(E) = 0$.