On this example sheet we look at the Cantor Set and its cousins. The exercises are taken from the first chapter of Stein-Shakarchi [SS], mildly edited.

1 The Cantor Set

To construct the Cantor Set $C$, let $C_0 = [0, 1]$ be the unit interval. The set $C_1$ is obtained from $C_0$ by tri-secting $C_0$ and removing from it the middle-third open interval, i.e. $C_1 = [0, 1/3] \cup [2/3, 1]$. The $C_i$ for $i \geq 2$ are defined recursively by always deleting the open middle-third of the (disjoint) intervals from $C_{i-1}$. The Cantor set $\mathcal{C}$ is finally obtained as the intersection

$$\mathcal{C} := \bigcap_{n=1}^{\infty} C_n.$$ 

a) Prove that $\mathcal{C}$ is compact and non-empty.

b) Prove that $\mathcal{C}$ is totally disconnected, i.e. given $x$ and $y$ in $\mathcal{C}$ with $x \neq y$ there is a $x < z < y$ with $z \notin \mathcal{C}$.

c) Prove that $\mathcal{C}$ does not have isolated points.

d) Prove that $m_*(\mathcal{C}) = 0$.

e) Show that we can write $C_n$ as

$$C_n = \bigcup_{a_1, \ldots, a_n \in \{0, 2\}} \left[ \sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{n} a_k 3^{-k} + \frac{1}{3^n} \right].$$

Hints: How many intervals of what length does $C_n$ contain? Use induction to prove that the left endpoints of the intervals in $C_n$ are given by $\sum_{k=1}^{n} a_k 3^{-k}$ for the $a_k \in \{0, 2\}$.

f) Show that $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \in \{0, 2\} \Leftrightarrow x \in \mathcal{C}$.

g) Define the Cantor-Lebesgue function $F : \mathcal{C} \to [0, 1]$ as

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \text{ for } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \text{ where } b_k = \frac{a_k}{2}.$$ 

Show that $F$ is well-defined and in fact continuous on $\mathcal{C}$. Show also that $F$ is surjective. Conclude that $\mathcal{C}$ is uncountable.
2 Fat Cantor Sets

Construct a closed set \( \hat{C} \) analogous to the Cantor set as follows: This time we remove at the \( k^{th} \) stage \( 2^{k-1} \) centrally situated open intervals each of length \( \ell_k = \frac{1}{4^k} \). The set \( \hat{C} \) is again defined as the intersection of the closed sets \( \hat{C}_k \) appearing at stage \( k \).

a) Show that \( m_* (\hat{C}) = \frac{1}{2} \) and conclude that \( \hat{C} \) is uncountable.

b) Show that \( \hat{C} \) is again compact, totally disconnected and has no isolated points.

3 *Characterisation of Riemann integrable functions

Prove that a bounded function on an interval \([a, b]\) is Riemann integrable if and only if its set of discontinuities has measure zero.

This is a hard problem. You should look up the outline of the proof in Stein-Shakarchi (p.47) or elsewhere in the literature. You can use the result at the end of Exercise 4.

4 Limits of continuous functions \( f : [0, 1] \to \mathbb{R} \) (Exercise 10 of [SS])

In this exercise we construct the sequence of continuous functions promised in the first lecture.

Let \( \hat{C} \) denote the Fat Cantor Set constructed in Exercise 2. We define the function \( F_1 \) to be a piecewise linear continuous function which is equal to 1 on the complement of the first interval \( I_1 = [3/8, 5/8] \) removed from \([0, 1]\) in the construction of \( \hat{C} \) and zero at the point at the centre of \( I_1 \). Similarly we construct \( F_2 \) to be a piecewise linear continuous function which is equal to 1 on the complement of the open intervals removed at the second stage and zero at the centre of these intervals. Continuing in this way we can define \( F_n \) for all \( n \) and finally

\[
F_n = F_1 \cdot F_2 \cdot \ldots \cdot F_n
\]

a) Show that for all \( n \geq 1 \) and all \( x \in [0, 1] \) one has \( 0 \leq f_n (x) \leq 1 \) and \( f_n (x) \geq f_{n+1} (x) \). We conclude that \( f_n (x) \) converges to a limit which we denote by \( f (x) \).

b) Show that the function \( f \) is discontinuous at every point of \( \hat{C} \). Conclude that \( f \) is not Riemann integrable (despite the sequence \( s_n = \int f_n \) converging).

Hint: Note that \( f (x) = 1 \) if \( x \in \hat{C} \) and find a sequence of points \( \{ x_n \} \) so that \( x_n \to x \) and \( f (x_n) = 0 \).

5 The outer Jordan content of a set (Exercise 14 of [SS])

The outer Jordan content \( J_* (E) \) of a bounded set \( E \) in \( \mathbb{R} \) is defined by

\[
J_* (E) = \inf \sum_{j=1}^{N} |I_j|
\]

where the infimum is taken over all finite coverings \( E \subset \bigcup_{j=1}^{N} I_j \) by intervals \( I_j \).

a) Prove that \( J_* (E) = J_* (\overline{E}) \) for every set \( E \). Here \( \overline{E} \) denotes the closure of \( E \).

b) Exhibit a countable subset \( E \subset [0, 1] \) such that \( J_* (E) = 1 \) while \( m_* (E) = 0 \).