Measure and Integration: Example Sheet 2

Fall 2016 [G. Holzegel]

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Exercises 1 and 2 supply small results that have been used in Lectures. Exercises 4-6 are taken from the first chapter of Stein-Shakarchi [SS], mildly edited.

1 Tonelli's Theorem for sequences

Let $x_{m,n}$ be a doubly infinite sequence of **non-negative** real numbers. In lectures we often used the following equality

$$\sum_{(m,n)\in\mathbb{N}^2} x_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{m,n} \tag{1}$$

with the understanding that if the right hand side is infinite then so is left hand side and conversely. Prove this inequality. Can you spot the occasions where we used it in lectures?

Hints: Note that the left hand side is defined as

$$\sum_{(m,n)\in\mathbb{N}^2} x_{m,n} := \sup_F \sum_F x_{m,n}$$

with the sum (and the sup) taken over all *finite* subsets $F \subset \mathbb{N}^2$. Prove the $\leq$ and $\geq$-directions separately. For the former, note $\sum_F x_{m,n} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{m,n}$ holds for all F.

2 Distances between sets

Let E and F be closed disjoint sets in $\mathbb{R}^d$.

a) Give an example of $d(E,F) = 0$.

b) Show that if one of the sets is *compact*, then necessarily $d(E,F) > 0$.

3 Approximation of measurable sets

Let E be a Lebesgue measurable set E. In lectures we saw that for any $\epsilon > 0$

- there exists an open set $\mathcal{U}$ containing E such that $m(\mathcal{U}\setminus E) \leq \epsilon$ (by definition)
- there exists a closed set F contained in E such that $m(E \setminus F) \leq \epsilon$ (by Lemma 2.3 of the notes)

a) Show that in general, given a measurable set E it is not possible to find for any $\epsilon > 0$ an open set $\mathcal{U}$ contained in E such that $m(E\setminus\mathcal{U}) \leq \epsilon$. (Hint: Try the Fat Cantor Set from last week.)

b) Let E be measurable and $m_\star(E) < \infty$. Show that for any $\epsilon > 0$ there exists a compact set K with $K \subset E$ and $m(E\setminus K) \leq \epsilon$. [Hint: Use the fact that this holds for closed F and exhaust by cubes using the regularity properties of the measure.]

c) Let E be measurable and $m_\star(E) < \infty$. Show that for any $\epsilon > 0$ there exists a *finite* union $F = \cup_{j=1}^{N} Q_j$ of closed cubes with $m(E\Delta F) \leq \epsilon$. Here $E\Delta F := (E\setminus F)\cup(F\setminus E)$ is the symmetric difference.

4 The Borel-Cantelli Lemma [Exercise 16 from [SS]]

Suppose $(E_k)_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ such that

$$\sum_{k=1}^{\infty} m\left(E_k\right) < \infty\,.$$

Let

$$E = \{x \in \mathbb{R}^d \mid x \in E_k \text{ for infinitely many } k\}.$$

Show that E is measurable and that $m\left(E\right) = 0$. Hint: Use that $E = \cap_{n=1}^{\infty} \cup_{k \geq n} E_k$.

5 Failure of finite additivity on all sets for the exterior measure

We proved in lectures that the exterior measure is countably subadditive but claimed that one cannot in general conclude that $m_\star\left(E_1\right) + m_\star\left(E_2\right) = m_\star\left(E_1 \cup E_2\right)$ holds for an arbitrary disjoint union $E = E_1 \cup E_2$ of sets. [Note, however, we proved this in the case where $d\left(E_1, E_2\right) > 0$.] Using the non-measurable set $\mathcal{N} \subset I := [0,1]$ constructed in lectures, we may now verify this claim along the following lines:

a) If E is a measurable subset of $\mathcal{N}$ then $m\left(E\right) = 0$. Hint: Consider the sets $E_k = E + r_k \subset \mathcal{N}_k$.

b) Show that $\mathcal{N}^c := I \setminus \mathcal{N}$ satisfies $m_\star\left(\mathcal{N}^c\right) = 1$. Hint: Argue by contradiction and use part a).

c) Conclude that

$$m_\star\left(\mathcal{N}\right) + m_\star\left(\mathcal{N}^c\right) \neq m_\star\left(\mathcal{N} \cup \mathcal{N}^c\right)\,.$$

6 Fun Stuff [Exercise 11 from [SS]]

Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 in their decimal expansion. Find $m\left(A\right)$.