Partial Differential Equations (Week 1)

Gustav Holzegel

January 23, 2019

Abstract

Notes under construction. All comments and corrections welcome.

1 Introduction and Classification of PDEs

A partial differential equation (PDE) for a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a relation of the form

$$F\left(x_1, x_2, ..., u, u_{x_1}, u_{x_2}, ..., u_{x_1 x_1}, u_{x_1 x_2}, ...\right) = 0\,, \tag{1}$$

where F is a given function of the variables $x_1, ..., x_n$, the unknown u and a *finite* number of partial derivatives of u. We will sometimes write this shorthand as

$$F\left(\mathbf{x}, u, Du, D^2u, ..., D^k u\right) = 0\,. \tag{2}$$

The *order* of the PDE is the number of highest derivatives appearing in (1). A *classical solution* of the k^{th}-order PDE is a k-times continuously differentiable function $u\left(x_1, ..., x_n\right)$ satisfying (2).

Which F's give rise to "interesting" PDE? Can one develop a theory for all reasonable F's? It turns out that the latter point of view is too general to lead to powerful results.[1] Instead, physics provides some of the most interesting[2] PDEs and this was also how historically, PDE developed as a subject. The theories of electromagnetism, quantum mechanics, fluid dynamics etc all rest on (systems of) PDEs. As is often the case, the equations are relatively easy to state but statements about general behavior solutions can be extremely hard to obtain. As you can imagine, if the F above is complicated, one will not be able to write down "explicit" solutions but merely hope to prove existence of a suitable class of solutions and, perhaps, properties of such solutions in the large. Fortunately for us, most PDEs in physics are first or second order and for this course we'll certainly focus on those.

Before we turn to concrete examples, let us classify the PDEs further.

[1] Although this approach goes *some* way, as we shall see during the first few weeks.

[2] both from the point of view of applications (obviously) and from the point of view of leading to interesting mathematics

We will say that the PDE is *linear* if it is of the form

$$\sum_{|\alpha|\leq k} a_{\alpha}\left(x_{1},...,x_{n}\right) D^{\alpha} u=f\left(x\right) \tag{3}$$

where the sum runs over all multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ of size 0 to k. If also $f = 0$, the equation in called homogeneous. In the latter case we have the important *superposition principle* for solutions (in particular, adding two solutions produces another). If the α_a are constant, we call the equation a *constant-coefficient* linear PDE.

The next level of difficulty is given by *semilinear* equations. These are equations of the form

$$\sum_{|\alpha|=k} a_{\alpha}\left(\mathbf{x}\right) D^{\alpha} u+\tilde{F}\left(\mathbf{x}, u, D u, ..., D^{k-1} u\right)=0\,, \tag{4}$$

i.e. the relation is purely linear in the top-order derivatives.

Moving on, we get to *quasilinear* equations, which are of the form

$$\sum_{|\alpha|=k} a_{\alpha}\left(\mathbf{x}, u, D u, ..., D^{k-1} u\right) D^{\alpha} u+\tilde{F}\left(\mathbf{x}, u, D u, ..., D^{k-1} u\right)=0\,, \tag{5}$$

i.e. the top-derivatives still enter only linearly in the sense that their coefficients are allowed to depend only on all strictly lower derivatives.

Finally, there are *fully non-linear* PDEs, for which no special structure is assumed. Such equations will not make a big appearance in the seminar although there are certainly "interesting" such PDEs (see the examples below)!

Here are some examples:

1. if $n = 1$ in (1) we arrive at an ODE. I will review basic ODE theory in the second lecture.
2. if $n \geq 2$ and $k = 1$ we get first order equations. The simplest is perhaps the transport equation: $\partial_t u + a\partial_x u = 0$ for a constant $a > 0$. This equation is a linear constant coefficient equation and is easily understood (see below). The general case will be addressed in Week 2.
3. Laplace/ Poisson equation, $\Delta u = \sum_i \partial_i^2 u = f$
4. The wave equation $-\partial_t^2 u + \Delta u = 0$
5. The heat equation $\partial_t u - \Delta u = 0$
6. A semi-linear wave equation: $-\partial_t^2 u + \Delta u = u^3$
7. A quasi-linear wave equation: $-\partial_t^2 u + (1 + \phi(u_t))\,\Delta u = 0$ for some function $\phi : \mathbb{R} \to \mathbb{R}$ (which is small in L^∞, say)
8. The eikonal equation $u_x^2 + u_y^2 = 1$ appearing in geometric optics is first-order fully non-linear.

9. Let $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$: $\det\left(D^2 u\right) = f(x, u, Du)$ with $f : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ given, the Monge-Ampere equation (appears in conformal geometry and when trying to find a hypersurface with prescribed Gaussian curvature). This is an example of a fully non-linear equation.

As we will see, a good understanding of the linear equations is essential to make progress with non-linear equations (simply because one can understand a lot by linearizing!).

The above concerns PDEs for a *scalar* function. We can clearly generalize the above to systems of PDEs. The unknown u then becomes vector valued $(u_1, ..., u_m)$ and n (not necessarily equal to m) PDE's are specified to form a system of PDEs. The classical (linear) example is of course Maxwell's equations (in vacuum):

$$div E = 0 \quad , \quad div B = 0 \quad , \quad \partial_t B + \vec{\nabla} \times E = 0 \quad , \quad \partial_t E - \vec{\nabla} \times B = 0 \tag{6}$$

for vectorfields $E : \mathbb{R}^4 \to \mathbb{R}^3$ and $B : \mathbb{R}^4 \to \mathbb{R}^3$. If you write this out in components you find a first-order linear system of 8 equations for 6 unknowns. (Hence it seems overdetermined, we'll understand why this is not a problem when we talk about constraints later).

Exercise 1.1. *Given a C^2 solution of Maxwell's equations, show that the components of E and B satisfy wave equations.*

An example of a non-linear system are the Euler equations for an incompressible fluid (of constant density) with velocity vector u^μ and pressure p

$$(u_t)_\nu + u^\mu \partial_\mu u_\nu = -\partial_\nu p \tag{7}$$
$$div\ u = 0\,. \tag{8}$$

As I already mentioned, every PDE is somehow different (which is not surprising given the manifold of different physical phenomena they describe!) and different techniques are available for different PDE. However, there are a couple of common themes that are worth pointing out:

1. well-posedness. It is easily seen that an equation like Laplace's equation has infinitely many solutions. Is there a useful way to parametrize the set of solutions to a PDE? Can we supplement the above PDE with initial/ boundary conditions so that there *exists* a *unique* solution which *depends continuously on the data* prescribed? What function spaces do the solutions live in? Does it make sense to consider (only) analytic solutions?

2. Connect the mathematical analysis with the physical phenomena (e.g. finite speed of propagation for the wave equation, dispersion, dissipation, formation of shocks), "derivation" of PDEs from physics

3. asymptotic behavior of solutions: as they approach the boundary or "large times", local vs global existence of solutions. Blow-up phenomena.

4. conservation laws: are there (perhaps "almost") conserved quantities in the evolution? Do they help to control the solution? For instance, given the Schroedinger equation $u_t = i\Delta u$ for $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, suppose we have a solution which decays sufficiently fast near infinity one can formally show that the L^2-norm is conserved in time. One of our tasks will be to make such formal computations rigorous.

5. a-priori estimates

2 ODEs refresher

2.1 Basic functional analysis: Function spaces

Recall the notion of a *metric space* (E, d), i.e. a set E for whose elements (called points) one has a distance function $d : E \times E \to \mathbb{R}_0^+$ which satisfies

- $d(x, y) \geq 0 \quad$ with equality iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

A metric space (E, d) is called *complete* if every Cauchy sequence (with respect to d) converges to an element of E.

A *normed vectorspace* $(E, \|\cdot\|)$ is a vectorspace equipped with a function (the norm) $\|\cdot\| : E \to R_0^+$ such that for any $x, y \in E$ we have

- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
- $\|tx\| = |t|\|x\| \quad$ for any $t \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

The norm induces a canonical metric $d(x, y) := \|x - y\|$ on E, as one readily checks.

A *Banach space* is a normed vectorspace which is complete with respect to its canonical metric.

A Hilbert space has even more structure. To define it, we first define the notion of an *inner product space* $(E, \langle \cdot, \cdot \rangle)$, sometimes called a pre-Hilbert space. This is a vector space E (say over $\mathbb{R}$) equipped with an inner product, i.e. a function $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$ which satisfies

- $\langle x, x \rangle > 0$ if $x \neq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

Note that the inner-product defines a norm via $\|x\| := \sqrt{\langle x, x \rangle}$, as one readily checks. Consequently, every inner product space is a normed vectorspace.

A *Hilbert space* is an inner-product space which is complete.

Examples

1. Consider the space $C^0[a,b]$, the space of real valued functions which are continuous on the interval $[a,b]$. This is clearly a vectorspace. Consider the following two norms on $C^0[a,b]$:

$$\|x(t)\|_{L^2} = \left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}}$$
$$\|x(t)\|_\infty = \sup_t |x(t)| \tag{9}$$

   and their associated canonical metrics d_2 and d_∞. Show that $C^0[a,b]$ is incomplete with respect to d_2 but complete with respect to d_∞ (hint: uniform convergence). Can you generalize to $C^k[a,b]$?

2. The Lebesgue L^p spaces ($1 \leq p < \infty$). We define $L^p[a,b]$ to be the space of measurable functions $f : [a,b] \to \mathbb{R}$ for which

$$\|f\|_{L^p} = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} < \infty\,. \tag{10}$$

   Identifying functions which agree almost everywhere, the left hand side becomes a norm on this (quotient) space. For $p = \infty$ one defines

$$\|f\|_{L^\infty} = \sup\ ess_{t\in(a,b)}|f(t)|. \tag{11}$$

   One can show that $L^p[a,b]$ is a Banach space for $1 \leq p \leq \infty$. The case $p = 2$ is special. The space L^2 has a Hilbert space structure inherited from the inner-product

$$\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt\,. \tag{12}$$

   In the course, we will mostly work with L^2-based spaces. On the one hand because the additional Hilbert space structure (and the notion of orthogonality that comes along with it) is extremely useful, on the other because for the wave equation L^2-estimates are actually the only type of estimates which don't lose regularity.

2.2 Banach's fixed point theorem

Theorem 2.1. *Let $X \subset E$ be a non-empty closed subset of a complete metric space (E,d). Let $A : X \to X$ be a contraction map, i.e.*

$$d(Ax, Ay) \leq q \cdot d(x,y) \qquad \text{for some } q < 1 \text{ fixed} \tag{13}$$

Then A has a unique fixed point, i.e. a point $\tilde{x}$ with $A\tilde{x} = \tilde{x}$.

Proof. Note that A is continuous. Choose $x_0 \in X$ arbitrary and define recursively a sequence $x_{n+1} = Ax_n$. Claim: This sequence is Cauchy. Indeed,

$$d\left(x_{n+k}, x_n\right) = d\left(A^{n+k}x_0, A^n x_0\right) \leq q^n d\left(A^k x_0, x_0\right) \tag{14}$$

and we have to show that the right hand side is smaller than some given ϵ for $n \geq N$ sufficiently large and all $k \geq 0$. To see this, note that by the triangle inequality

$$d\left(A^k x_0, x_0\right) \leq \sum_{i=1}^{k} d\left(A^i x_0, A^{i-1} x_0\right) \leq \sum_{i=1}^{k} q^{i-1} d\left(Ax_0, x_0\right) \leq \frac{1}{1-q} d\left(Ax_0, x_0\right)$$

In particular, we have

$$d\left(x_{n+k}, x_n\right) \leq \left[q^n \sum_{i=1}^{k} q^{i-1}\right] d\left(Ax_0, x_0\right) \leq \frac{q^n}{1-q} d\left(Ax_0, x_0\right) . \tag{15}$$

Obviously, we can choose n so large that the right hand side becomes smaller than any prescribed $\epsilon > 0$. This shows that x_n is indeed Cauchy.

Since a closed subset of a complete metric space is also complete, x_n converges in X to some $\tilde{x}$. As A is continuous we can take the limit $n \to \infty$ of the expression $x_{n+1} = Ax_n$ obtaining $\tilde{x} = A\tilde{x}$ as desired.

To conclude the uniqueness note that if we had x_1, x_2 with $Ax_1 = x_1$ and $Ax_2 = x_2$ we would have

$$d\left(x_1, x_2\right) = d\left(Ax_1, Ax_2\right) \leq q d\left(x_1, x_2\right) \tag{16}$$

which is a contradiction unless $x_1 = x_2$. □

Remark 2.2. *Inspecting the proof we see that it actually suffices to have*

$$d\left(A^n x, A^n y\right) \leq \alpha_n d\left(x, y\right) \qquad \text{with } \alpha_n \geq 0 \text{ and } \sum_n \alpha_n < \infty$$

to conclude the existence of a unique fixed point. This generalization is convenient, as we will see in the next subsection.

2.3 The Theorem of Picard and Lindelöf

Let $t_0 \in \mathbb{R}$ and $\phi_0 \in \mathbb{R}^n$ be constant (to be thought of as an initial time and an initial condition/ state vector). Fix positive constants a, b and let $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$ be a function which is continuous on the cylinder

$$R = \left\{(t, \phi) \in \mathbb{R}^{1+n} \quad | \quad |t - t_0| \leq a \quad , \quad \|\phi - \phi_0\| \leq b\right\} ,$$

and in addition Lipschitz in the second component x, i.e.

$$\|f\left(t, \phi\right) - f\left(t, \psi\right)\| \leq L\|\phi - \psi\|$$

holds for some positive $L < \infty$ and all $\left(t, \phi\right), \left(t, \psi\right) \in R$.

Theorem 2.3. *With ϕ_0, f and R as above, set $M = \sup_{(t,\phi)\in R} \|f(t,\phi)\|$ and consider the ODE*

$$\begin{aligned} \frac{d}{dt}\phi(t) &= f(t,\phi(t)) \\ \phi(t_0) &= \phi_0 \end{aligned} \tag{17}$$

for an unknown function $\phi : \mathbb{R} \to \mathbb{R}^n$. Claim: There exists a unique classical solution $\phi(t)$ to the above ODE on the time interval

$$J = \left\{ t \in \mathbb{R} \;\middle|\; |t - t_0| \leq \alpha \quad \text{where } \alpha = \min\left(a, \frac{b}{M}\right) \right\}. \tag{18}$$

Proof. We first note that the statement claimed is equivalent to the statement that

$$\phi(t) = \phi_0 + \int_{t_0}^{t} f(\tau,\phi(\tau))\, d\tau \tag{19}$$

has a unique solution $\phi(t)$ which is continuous on J. Indeed, one direction follows directly by integrating the ODE and the fundamental theorem of calculus. The reverse follows from noting that for continuous $\phi(t)$ the right hand side of (19) is actually differentiable in t and applying the fundamental theorem of calculus.

Hence let us prove that (19) has a unique solution. We will formulate this as the statement that a certain map has a fixed point. We define the closed subset

$$C_b(J) = \{\phi \in C(J) \;\Big|\; \sup_{t\in J} \|\phi(t) - \phi_0\| \leq b\} \subset C(J) \tag{20}$$

of the space of functions continuous on J (equipped with the sup-norm). Intuitively, these are functions whose graph remains in the cylinder R. We now consider the map

$$\begin{aligned} A : C_b(J) &\to C_b(J) \\ \phi(t) &\mapsto \phi_0 + \int_{t_0}^{t} f(\tau,\phi(\tau))\, d\tau\,. \end{aligned} \tag{21}$$

We need to check that this is indeed a map into $C_b(J)$. This means to show that

$$\| \int_{t_0}^{t} f(\tau,\phi(\tau)) \| \leq b$$

holds for all $t \in J$, which clearly follows from

$$\| \int_{t_0}^{t} f(\tau,\phi(\tau)) \| \leq |t - t_0| M \leq \alpha M \leq b\,.$$

We claim that A is a contraction. Indeed, by induction we show

$$\|A^n\phi(t) - A^n\psi(t)\| \leq \frac{L^n |t-t_0|^n}{n!} \sup_{t\in J} \|\phi(t) - \psi(t)\|\,,$$

the latter claim being a consequence of the induction step

$$\begin{aligned}\|A^n\phi(t) - A^n\psi(t)\| &\le \int_{t_0}^{t} \|f\left(t, A^{n-1}\phi(\tau)\right) - f\left(t, A^{n-1}\psi(\tau)\right)\| d\tau \\ &\le L \int_{t_0}^{t} \|A^{n-1}\phi(\tau) - A^{n-1}\psi(\tau)\| d\tau \\ &\le \frac{L^n}{(n-1)!} \sup_{t\in J} \|\phi(t) - \psi(t)\| \int_{t_0}^{t} |\tau - t_0|^{n-1} d\tau \\ &\le \frac{L^n |t-t_0|^n}{n!} \sup_{t\in J} \|\phi(t) - \psi(t)\| .\end{aligned}$$

By Remark 2.2 following the fixed point theorem we conclude that A has a unique fixed point.[3] But this fixed point is a solution of (19) and hence of our ODE. □

A couple of remarks are in order. First note that the Lipschitz condition is essential (at least for the uniqueness). The ODE

$$\frac{d}{dt} u(t) = \sqrt{u(t)} \qquad u(0) = 0$$

has the obvious solution $u = u_1 = 0$ identically. However the function u_2 defined by being identically zero for $t < 0$ and equal to $u(t) = t^2/4$ for $t \geq 0$ is also a classical solution. Of course, $\sqrt{u}$ is not Lipschitz in u near zero.

The second remark concerns the *local* character of the above theorem. The ODE

$$\frac{d}{dt} u(t) = [u(t)]^2 \qquad u(0) = 1$$

blows up in finite time, the solution being $u(t) = \frac{1}{1-t}$. We will see later that, in general, the only way for the solution to cease to exist for times $t > t_{fin}$ for some $t_{fin} > t_0$ is that u blows up as t_{fin} is approached.

Finally, let us remark how to convert an n^{th} order scalar autonomous ODE of the form

$$\partial_t^n \phi(t) = \mathcal{F}\left(\phi(t), \partial_t \phi(t), ..., \partial_t^{n-1}\phi(t)\right) \tag{22}$$

to a first order ODE for a system. To do this, define for $i = 0, ..., n-1$ the quantity $u_i = \partial_t^i \phi$. We then have the system

$$\begin{aligned}(u_i)_t &= u_{i+1} \qquad \text{for } i = 0, ..., n-2 \\ (u_{n-1})_t &= \mathcal{F}(u_0, u_1, ..., u_{n-1})\end{aligned} \tag{23}$$

Clearly, for (22) one has to specify the first $n-1$-derivatives of ϕ at the initial time t_0 to obtain a well-posed problem, which is of course equivalent to specifying u_0, ..., u_{n-1} in the familiar (23).

Using a similar trick, one can convert a non-autonomous ODE into an autonomous one (Exercise).

[3] Of course, one could apply Banach's fixed point theorem in its original form. However, this is at the cost of (possibly) shrinking the time interval which now depends also on the Lipschitz constant.

2.4 Gronwall's inequality

Theorem 2.4. *Let $\phi : [t_0, t_1] \to \mathbb{R}^+$ be continuous and non-negative. Suppose that ϕ obeys*

$$\phi(t) \leq A + \int_{t_0}^{t} B(s)\,\phi(s)\,ds \qquad \text{for all } t \in [t_0, t_1] \tag{24}$$

where $A \geq 0$ and $B : [t_0, t_1] \to \mathbb{R}^+$ is continuous and non-negative. Then

$$\phi(t) \leq A \cdot \exp\left(\int_{t_0}^{t} B(s)\,ds\right). \tag{25}$$

Proof. We will assume $A > 0$ and obtain the case $A = 0$ as a limit. Look at

$$\frac{d}{dt}\left(A + \int_{t_0}^{t} B(s)\,\phi(s)\,ds\right) = B(t)\,\phi(t) \leq B(t)\left(A + \int_{t_0}^{t} B(s)\,\phi(s)\,ds\right).$$

Since $A > 0$, we can write

$$\frac{d}{dt}\log\left[A + \int_{t_0}^{t} B(s)\,\phi(s)\,ds\right] \leq B(t)$$

Integrating this in t yields

$$\phi(t) \leq A + \int_{t_0}^{t} B(s)\,\phi(s)\,ds \leq A\exp\left(\int_{t_0}^{t} B(s)\,ds\right)$$

which is the result. $\square$

Remark 2.5. *Note that (25) is sharp in that $\psi(t) = A\exp\left(\int_{t_0}^{t} B(s)\,ds\right)$ satisfies (24) with equality.*

There is also a differential form of Gronwall's inequality. For $\phi : [t_0, t_1] \to \mathbb{R}^+$ a C^1 non-negative function satisfying

$$\partial_t \phi(t) \leq B(t)\,\phi(t) \qquad \text{for all } t \in [t_0, t_1]$$

for $B : [t_0, t_1] \to \mathbb{R}$ continuous (but not necessarily non-negative), we have the estimate

$$\phi(t) \leq \phi(t_0)\ \exp\left(\int_{t_0}^{t} B(s)\,ds\right) \qquad \text{for all } t \in [t_0, t_1]$$

Proof. Define the non-negative C^1 function $\psi(t) = \phi(t)\exp\left(-\int_{t_0}^{t} B(s)\,ds\right)$ and observe that $\psi'(t) \leq 0$. $\square$

2.5 Applications

2.5.1 Uniqueness and continuous dependence on the data

To see how Gronwall's inequality helps us in understanding our ODE problem, we consider the ODE for the difference of two solutions (not necessarily arising from the same data). Suppose we are given two C^1 solutions ϕ_1 and ϕ_2 of the ODE

$$\frac{d}{dt}\phi_i\left(t\right) = f\left(\phi_i\left(t\right)\right) \qquad \phi_i\left(0\right) = \phi_{0i}$$

for $i = 1, 2$ and both defined for $[t_0, T]$. Then their difference satisfies

$$\frac{d}{dt}\left(\phi_1\left(t\right) - \phi_2\left(t\right)\right) = f\left(\phi_1\left(t\right)\right) - f\left(\phi_2\left(t\right)\right) ,$$

which we can integrate between t_0 and $t_0 \leq t \leq T$ to obtain

$$\phi_1\left(t\right) - \phi_2\left(t\right) = \phi_1\left(t_0\right) - \phi_2\left(t_0\right) + \int_{t_0}^{T} \left[f\left(\phi_1\left(\tau\right)\right) - f\left(\phi_2\left(\tau\right)\right)\right] d\tau$$

leading to the inequality

$$\|\phi_1\left(t\right) - \phi_2\left(t\right)\| \leq \|\phi_1\left(t_0\right) - \phi_2\left(t_0\right)\| + \int_{t_0}^{t} \|f\left(\phi_1\left(\tau\right)\right) - f\left(\phi_2\left(\tau\right)\right)\| d\tau .$$

Inserting the Lipschitz condition on f we find

$$\|\phi_1\left(t\right) - \phi_2\left(t\right)\| \leq \|\phi_1\left(t_0\right) - \phi_2\left(t_0\right)\| + L\int_{t_0}^{t} \|\phi_1\left(\tau\right) - \phi_2\left(\tau\right)\| d\tau$$

to which Gronwall's inequality can be applied

$$\|\phi_1\left(t\right) - \phi_2\left(t\right)\| \leq \|\phi_1\left(t_0\right) - \phi_2\left(t_0\right)\| \; e^{L(t-t_0)} \tag{26}$$

This estimate implies a uniqueness statement: If two classical solutions agree at t_0, they have to agree for all times. Moreover, the estimate implies what we call *continuous dependence on the data*. Given a solution $\phi\left(t\right)$ for $[0, T)$ arising from initial data $\phi\left(0\right) = \phi_0$ and some prescribed $\epsilon > 0$, there exists a time $0 < T^\star < T$ and a $\delta > 0$ such that for any data ψ_0 satisfying $\|\phi_0 - \psi_0\| < \delta$ we have $\|\phi\left(t\right) - \psi\left(t\right)\| < \epsilon$ in $[0, T^\star]$.

Note that the continuous dependence on the data can be strengthened if we assume that the right hand side $f\left(\phi\right)$ has more regularity. Suppose, for instance, that f is C^2. Then

$$\begin{aligned}\partial_t\left(\partial_t\phi_1\left(t\right) - \phi_2\left(t\right)\right) &= f'\left(\phi_1\right)\partial_t\phi_1\left(t\right) - f'\left(\phi_2\right)\partial_t\phi_2\left(t\right) \\ &= f'\left(\phi_1\right)\left[\partial_t\phi_1\left(t\right) - \partial_t\phi_2\left(t\right)\right) + \partial_t\phi_2\left(t\right)\left(f'\left(\phi_1\right) - f'\left(\phi_2\right)\right) .\end{aligned}$$

Now integration will produce the estimate

$$\begin{aligned}\|\partial_t\phi_1\left(t\right) - \partial_t\phi_2\left(t\right)\| &\leq \|\partial_t\phi_1\left(t_0\right) - \partial_t\phi_2\left(t_0\right)\| \\ &\quad + \int_{t_0}^{t} C_1\|\partial_t\phi_1 - \partial_t\phi_2\| + \int_{t_0}^{t} C_2\|\phi_1 - \phi_2\| .\end{aligned} \tag{27}$$

Adding the old estimate for $\|\phi_1(t) - \phi_2(t)\|$ itself will produce a Gronwall inequality for the C^1-norm and hence C^1-dependence of the solution on the data.

2.5.2 Blow up

Let us finally return to the problem of blow-up of solutions and the claim that if the maximal time of existence is t_{fin}, then $\lim_{t \to t_{fin}} \|\phi(t)\| = \infty$ has to hold. Consider for simplicity

$$\frac{d}{dt}\phi = f(\phi) \quad , \quad \phi(t_0) = \phi_0 \tag{28}$$

with f *locally Lipschitz* (so in particular Lipschitz on compact domains).

Proposition 2.6. *If the solution to (28) only exists for $t_0 < t_{fin} < \infty$ then necessarily $\lim_{t \to t_{fin}} \|\phi(t)\| = \infty$. A similar statement holds for the past direction.*

Proof. Suppose for contradiction that there was a sequence of times $t_n \to t_{fin}$ along which $\|\phi(t_n)\| \leq C$ for all t_n. To obtain the contradiction, we observe that we can infer from Theorem 2.3 the following statement: Given a bound on f in a ball of radius $2C$, i.e. $\sup_{\phi \in \overline{B}_{2C}(0)} \|f(\phi)\| \leq M$, there is for any initial data in a ball of radius C, $\|\phi_0\| \leq C$ a *uniform* time of existence δ for the solution, with $\delta > 0$ depending only on M (but not on the particular choice of data in the ball). Since f is assumed to be locally Lipschitz we have in particular a uniform bound on f in a ball of radius $2C$ and hence obtain $\delta > 0$ as described in the previous sentence. We now pick a t_N which is at least $\frac{\delta}{2}$-close to t_{fin}, i.e. $|t_N - t_{fin}| < \frac{\delta}{2}$. Next we solve the ODE (28) with the data at t_N, which by assumption lies in a ball of radius C. Thereby we obtain a solution defined on $[t_N, t_N + \delta)$ and glue it to the old solution defined on $[t_0, t_N]$. This new glued solution agrees with the old solution for $t_0 \leq t < t_{fin}$ by the Gronwall-uniqueness estimate of the previous section and extends past t_{fin}. Contradiction. $\square$

2.6 Bootstrap arguments

We will encounter bootstrap arguments as a powerful technique to study the global behavior of solutions to PDE in the last part of the course. We introduce the main idea at this point and illustrate it with an example below.

The setting is the following. Suppose you have a classical solution $\phi(t)$ of our ODE arising from data ϕ_0 at $t = t_0$ and a (continuous) norm $\|\cdot\|$ measuring the size of the solution. We know the solution exists for t close to t_0 but we do not know, in general, about the global behavior. Suppose we have two statements:

$$\|\phi(t)\| \leq 2C \quad \text{Statement } A(t)$$

$$\|\phi(t)\| \leq C \quad \text{Statement } B(t)$$

Obviously, if $B(t)$ holds at t, then $A(t)$ also holds. Consider now the following set

$$\Omega = \{t \in I = [t_0, \infty) \quad | \quad \|\phi(t)\| \leq 2C\} \tag{29}$$

which is the set of all t such that statement $A(t)$ holds. Clearly this set is closed by the continuity of the norm, and, with an appropriate (depending on the data) choice of C also non-empty. Suppose, we could establish in addition the following statement:

$$\text{If } A(t) \text{ holds for the solution, then } B(t) \text{ holds also.} \tag{30}$$

Then we claim that $\Omega = [t_0, \infty)$ and the solution satisfies $\|\phi(t)\| < C$ for all times. Clearly, the first claim would follow if we could show that Ω is also open. Namely, together with the above this would imply that $\Omega \subset [t_0, \infty)$ is an open, closed, non-empty subset of a connected set I and therefore $\Omega = I$. (Otherwise $I = \Omega \cup (I \setminus \Omega)$ would express the connected I as a disjoint union of non-empty open sets, which is a contradiction.) So is Ω open? Yes, because suppose for $t \in \Omega$ the above statement implies $\|\phi(t)\| \leq C$ and by continuity, there is an open neighborhood of t for which $\|\phi(t)\| < 2C$ holds.

In applications of this, the hard part of the argument will typically be to prove (30), i.e. to show that a certain bound on the solution can be bootstrapped. Let us illustrate this with an example (taken from Tao's book "Non-linear Dispersive Equations").

Proposition 2.7. *Let $V \in C^\infty(\mathbb{R}^n \to \mathbb{R})$ be such that $V(0) = 0$, $\nabla V(0) = 0$ and $\nabla^2 V(0)$ strictly positive definite. Then for all $\phi_0, \phi_1 \in \mathbb{R}^n$ sufficiently close to zero there is a unique global classical solution $\phi \in C^\infty(\mathbb{R} \to \mathbb{R}^n)$ to the Cauchy problem*

$$\partial_t^2 \phi(t) = -\nabla V(\phi(t)) \quad , \quad \phi(0) = \phi_0 \quad , \quad \phi_t(0) = \phi_1 \,. \tag{31}$$

Moreover, the solution remains bounded uniformly in t.

Proof. Transform the above system to a first order system to establish that a solution exists locally near $t = 0$ and that the solution can only blow up if the norm

$$\|\phi(t)\|_N^2 = \|\partial_t \phi(t)\|^2 + \|\phi(t)\|^2 \tag{32}$$

approaches infinity. To understand the global behavior, note that the energy

$$E(t) = \frac{1}{2}\|\partial_t \phi\|^2 + V(\phi(t))$$

is conserved:

$$\partial_t E(t) = \langle \partial_t \phi, \partial_t \partial_t \phi \rangle + \langle \partial_t \phi, \nabla V(\phi(t)) \rangle = 0 \,.$$

Hence

$$E(t) = \frac{1}{2}\|\phi_1\|^2 + V(u_0)$$

for all times and moreover we can make this quantity as small as we desire by choosing the data small. However, we can not conclude that $\phi(t)$ is small for all times because away from 0, the potential could be very negative, allowing $\partial_t \phi$ to be large while $E(t)$ remains small. However, we can bootstrap the smallness of ϕ as follows. Let

$$\text{Statement} \quad A(t) \ : \quad \|\phi(t)\|_N < 2\epsilon$$

$$\text{Statement} \quad B(t) \ : \quad \|\phi(t)\|_N < \epsilon$$

and

$$\Omega = \{t \in I = [t_0, \infty) \quad | \quad \|\phi(t)\|_N \leq 2\epsilon\}\,.$$

As above, Ω is clearly closed and non-empty for any ϵ, the latter provided we choose the data sufficiently small. The difficulty is to show that $A(t)$ implies $B(t)$ for ϵ sufficiently small. Note that by Taylor's theorem, we have for $\phi(t)$ near 0,

$$\begin{aligned} V(\phi(t)) &= V(0) + \langle \nabla V(0), \phi(t)\rangle + \langle \phi(t), \nabla^2 V(0)\phi(t)\rangle + \mathcal{O}\left(\|\phi\|^3\right) \\ &\geq c\|\phi(t)\|^2 - \mathcal{O}\left(\epsilon^3\right) \end{aligned} \tag{33}$$

for some constant c provided $A(t)$ holds for sufficiently small ϵ (this implies that $\|\phi(t)\| = \mathcal{O}(\epsilon)$). But this means that

$$E(0) = E(t) \geq \frac{1}{2}\|\partial_t\phi(t)\|^2 + c\|\phi(t)\|^2 - \mathcal{O}\left(\epsilon^3\right)\,,$$

and hence

$$\begin{aligned} \|\partial_t\phi(t)\|^2 + \|\phi(t)\|^2 &\leq \max\left(2, c^{-1}\right)\left[\frac{1}{2}\|\partial_t\phi(t)\|^2 + c\|\phi(t)\|^2\right] \\ &\leq \max\left(2, c^{-1}\right)\left[E_0 + \mathcal{O}\left(\epsilon^3\right)\right] \leq \epsilon^2 \end{aligned} \tag{34}$$

for sufficiently small ϵ, which is the statement $B(t)$. We have shown that $A(t)$ implies $B(t)$ for sufficiently small ϵ. Therefore Ω is also open and hence $\Omega = [t_0, \infty)$ and the solution is globally uniformly bounded, as claimed. □

3 Exercises

1. Do the exercises in the text.

2. Classify the following PDEs (linear?, semi-linear? quasi-linear?):

   (a) $div\left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right) = 0$ for $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$, the equation for a minimal surface. Can you "derive" this equation?

   (b) $u_t - \Delta u = u^2$ for $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, a so-called reaction-diffusion equation

(c) $u_t + H(Du) = 0$ for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where $H : \mathbb{R}^n \to \mathbb{R}$ is smooth. This is (a simple form of) the Hamilton-Jacobi equation from classical mechanics.

3. Show that the conclusion of Theorem 2.4 fails if B is allowed to be negative.

4. Suppose $f : [0, \infty) \to \mathbb{R}^+$ is C^1 and satisfies

$$f(t_2) + c \int_{t_1}^{t_2} d\bar{t} f(\bar{t}) \leq C \cdot f(t_1)$$

for all $0 \leq t_1 < t_2 < \infty$ and positive constants c, C. Show that f has to decay exponentially in time.

5. Can the solution to a linear ODE blow up in finite time?

6. Consider the non-linear ODE

$$\frac{d}{dt} u(t) = -\left(u^T u\right) u$$

for $u : \mathbb{R} \to \mathbb{R}^n$. Prove that for all initial data $u(t_0) = u_0$, the solution exists globally for $t > t_0$. Does the solution remain bounded? Does it decay in time?

7. Give an alternative proof of Gronwall's inequality using a bootstrap argument. HINT: Bootstrap the estimate $\phi(t) \leq (1+\epsilon) A \exp\left(\int_{t_0}^t (1+\epsilon) B(s) ds\right)$ for $\epsilon > 0$.

8. (Osgood's uniqueness theorem.) Let I be an interval of $\mathbb{R}$ and $F : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. Let Ω be an open subset of $\mathbb{R}^d$ (as usual endowed with the Euclidean norm), $t_0 \in I$, $u_0 \in \Omega$. Let $\omega : C(\mathbb{R}^+, \mathbb{R}^+)$ be an increasing continuous function which satisfies

$$\omega(0) = 0 \ , \ \omega(\sigma) > 0 \ \text{ for all } \sigma > 0 \ , \ \lim_{\epsilon \to 0} \int_\epsilon^a \frac{1}{\omega(\sigma)} d\sigma = \infty \ \text{ for all } a > 0.$$

Suppose that

$$\|F(t, y_1) - F(t, y_2)\| \leq \omega(\|y_1 - y_2\|) \quad \text{for all } (t, y_1, y_2) \in I \times \Omega \times \Omega.$$

Let $u_1, u_2 : I \to \Omega$ be two differentiable functions which are solutions to the Cauchy problem

$$\frac{d}{dt} u(t) = F(t, u(t)) \quad u(t_0) = u_0$$

(a) Show that $u_1 = u_2$ on I.

(b) Why is the uniqueness statement stronger than the one discussed in lectures?