Partial Differential Equations (Week 6+7)
Elliptic Regularity and Weak Solutions

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1 Elliptic Regularity

We will now entertain the following questions. Say u is C^2 and satisfies

$$\Delta u = f \tag{1}$$

in Ω. How regular is u depending on f? What is the minimum regularity on f so that we can actually find a C^2 solution of (1)? What about weak solutions? Our formula (**??**) is already suggestive as far as the relation of the regularity of f and the regularity of u is concerned. Let's take for simplicity the case of the ball for which we found a nice Green's function for the Dirichlet problem:

$$u(\xi) = \int_\Omega G(x,\xi) f(x)\, dx \tag{2}$$

Now since the Green's function is symmetric in x and ξ what is suggested by (2) is that we can gain (almost) two derivatives (explain!).[1]

We will now try to quantify this regularity gain but not at the level of the Hoelder spaces $C^{k,\alpha}$ (which would be an option leading to *Schauder regularity theory*) but at the level of the Sobolev spaces H^k.

Recall that $H^k(\Omega)$ is defined as the space of $L^1_{loc}(\Omega)$ functions, which have k weak derivatives and for which

$$\|u\|^2_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^2 dx < \infty\,. \tag{3}$$

We also define the homogeneous Sobolev "semi"-norm

$$\|u\|^2_{\dot{H}^k(\Omega)} = \sum_{|\alpha|= k} \int_\Omega |D^\alpha u|^2 dx\,, \tag{4}$$

[1]There are counter examples of solutions of $\Delta u = f$ with with f continuous but u not C^2. The right spaces to quantify the vague statement made above are the Hoelder spaces $C^{k,\alpha}$ (which are Banach spaces), for which one can derive the estimates of the type $\|u\|_{C^{2,\alpha}} \leq C\|f\|_{C^{0,\alpha}}$. from (2). Such an estimate is a regularity estimate. Search for *Schauder estimates* in the literature.

which unlike (3) behaves well under scaling of the coordinates (in case that $\Omega = \mathbb{R}^n$). The spaces $H^k(\Omega)$ are Hilbert spaces with the inner-product

$$< u, v >_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v \; dx \, .$$

Remark 1.1. *There are more general Sobolev spaces $W^{k,p}(\Omega)$ (which become the $H^k(\Omega)$ for $p = 2$). Those are still Banach spaces. We won't need them in this course.*

The associated notion of orthogonality for the H^k and the fact that these spaces are complete make them useful to do analysis. Moreover, these spaces are actually suggested by the equation themselves, more precisely, the estimates that one can derive from them.

Indeed, suppose for a moment that both f and u in (1) were smooth and of compact support in Ω. Then the elementary computation

$$\int_\Omega f^2 = \int_\Omega |\Delta u|^2 dx = \sum_{i,j} \int_\Omega \partial_i^2 u \partial_j^2 u = \sum_{i,j} \int_\Omega (\partial_i \partial_j u)(\partial_i \partial_j u) \, dx = \|u\|^2_{\dot{H}^2(\Omega)}$$

with the boundary terms vanishing in the integration by parts by the assumption of compact support, we would have

$$\|u\|^2_{\dot{H}^2(\Omega)} = \|f\|^2_{H^0(\Omega)} := \|f\|^2_{L^2(\Omega)} \, , \tag{5}$$

expressing the "gain" of two-derivatives at the level of Sobolev spaces. Now we can commute the Laplace equation by ∂_{x_i} for $i = 1, ..., d$ and by the same computation obtain

$$\|u\|^2_{\dot{H}^{2+k}(\Omega)} = \|f\|^2_{H^k(\Omega)} \tag{6}$$

for all integers $k > 0$. I claim that these identities will be extremely useful in helping us to prove

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in C^2(\Omega)$ and $\Delta u = f$ in Ω with $f \in C^\infty(\Omega)$. Then u is also C^∞ in Ω.*

Unfortunately we *used* the smoothness (and assumed compact support) of u to derive the identity (6). The idea of the following proof is to turn things around and to prove the smoothness of u using the identity (6).

Preliminaries

Let $\chi : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$ be a smooth radial cut-off function which satisfies

- *supp* $\chi \subset \overline{B(0,1)}$
- $\int_{\mathbb{R}^d} \chi dx = 1$

- $\chi = 1$ for $|x| \leq \frac{1}{2}$.

Then we know that $\chi_\epsilon(x) := \epsilon^{-d}\chi\left(\frac{x}{\epsilon}\right)$ it supported in the ball $B(0,\epsilon)$ and satisfies $\int_{\mathbb{R}^d} \chi_\epsilon dx = 1$

For a given $x_0 \in \Omega \subset \mathbb{R}^d$ we pick a ball around x_0 which is compactly contained in Ω. Let this ball have radius 2ρ. We define

$$\tilde{u} = \chi\left(\frac{x - x_0}{\rho}\right) u$$

which localizes the u to the ball of radius ρ around x_0, and the associated

$$\tilde{f} := \Delta\tilde{u} = \chi f + 2\nabla\chi\nabla u + \Delta\chi\ u\,. \tag{7}$$

We note that $\tilde{f}$ is supported in $B(x_0,\rho)$ and agrees with f in $B\left(x_0, \frac{\rho}{2}\right)$.

In addition, we define the sequence of mollifications

$$\tilde{u}_m = \tilde{u} \star \chi_{\frac{\rho}{m}} \quad \text{for any } m > 1 \tag{8}$$

Note that $\tilde{u}_m$ is smooth and supported in $B(x_0, 2\rho)$. Also, the mollifications

$$\tilde{f}_m = \tilde{f} \star \chi_{\frac{\rho}{m}} \quad \text{for any } m > 1 \tag{9}$$

are smooth and supported in $B(x_0, 2\rho)$, hence in $H^k = H^k(\mathbb{R}^n)$ for all k.

The proof

We first note that we have by definition for any $\epsilon > 0$

$$\Delta_x \tilde{u}_m(x) = \Delta \int_{\mathbb{R}^d} \epsilon^{-d}\chi\left(\frac{x-y}{\epsilon}\right)\tilde{u}(y)\,dy = \Delta\int_{\mathbb{R}^d} \epsilon^{-d}\chi\left(\frac{y}{\epsilon}\right)\tilde{u}(x-y)\,dy\,.$$

Since $\tilde{u}$ is compactly supported, the Laplacian goes through the integral, which – in view of its translation invariance – produces

$$\Delta\tilde{u}_m = \tilde{f}_m \tag{10}$$

1. Step 1. Since $f \in C^\infty$ and u is in C^2, the identity (7) gives that $\tilde{f}$ is C^1. Since $\tilde{f}$ is also of compact support in Ω, $\tilde{f} \in H^1$.

2. Step 2. We now recall an exercise about mollifiers from the previous week, which implies that the mollifications $\tilde{f}_m$ defined in (9) converge to $\tilde{f}_m \to \tilde{f}$ in H^1 as $m \to \infty$. In particular, $\tilde{f}_m$ is Cauchy in H^1. By the same argument, we know that $\tilde{u}_k \to \tilde{u}$ in H^2 and hence $\tilde{u}_k$ Cauchy in H^2.

3. Step 3. Since $\Delta(\tilde{u}_k - \tilde{u}_l) = \tilde{f}_k - \tilde{f}_l$ and both differences are smooth and of compact support, the identity (6) is now valid and takes the form

$$\|\tilde{u}_k - \tilde{u}_l\|^2_{\dot{H}^{2+k}} = \|\tilde{f}_k - \tilde{f}_l\|^2_{\dot{H}^k}\,. \tag{11}$$

Since we already know that $\tilde{u}_k$ is Cauchy in H^2 we can apply the above with $k=1$ to conclude that $\tilde{u}_k$ is actually Cauchy in H^3. Since we already know the limit is $\tilde{u} \in H^2$ we infer $\tilde{u} \in H^3$. Noting that $\tilde{u}$ agrees with u on $B\left(x_0, \frac{\varrho}{2}\right)$ and that we can perform the above steps around any point x_0, we conclude $u \in H^3$ (no tilde!) on any compact subset of Ω.

4. Step 4. We now repeat the procedure. We pick x_0 and the balls as above, and revisiting (7), we see that now $\tilde{f}$ is actually H^2. Now go to Step 2 replacing H^s by H^{s+1} everywhere.

By induction, this shows that $u \in H^k_{loc}(\Omega)$ for all positive integers k. By Sobolev embedding (see Exercises) we conclude that $u \in C^{k-\frac{n}{2}-\delta}(\Omega)$.

Reflecting on this proof, we see that it required the translation invariance of the Laplacian (true for all constant coefficient linear operators) and the availability of the fundamental estimate for the H^k-norm arising from the equation. We already note at this stage that only the second ingredient is fundamental, the first is technical.

Exercise 1.3. *Below we will show existence of weak solutions of $\Delta u = f$ with $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$. It's worth revisiting the above proof after that.*

2 Existence of weak solutions for $\Delta u = f$

2.1 The Energy Estimate

From Green's identities we know that for $u \in C^2(\Omega) \cap C^1\left(\bar{\Omega}\right)$ we have the identity

$$\int_\Omega \nabla u \cdot \nabla u = -\int_\Omega uf$$

for solutions $\Delta u = f$ of Poisson's equation which vanish on the boundary of Ω (which we assume to be sufficiently regular): $u = 0$ on $\partial\Omega$. Applying Cauchy-Schwarz yields

$$\int_\Omega \nabla u \cdot \nabla u \leq \epsilon \int_\Omega u^2 + \frac{1}{\epsilon}\int_\Omega f^2 \tag{12}$$

We have the following fundamental *Poincare inequality*

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ smooth. Let $u \in C^1\left(\bar{\Omega}\right)$ vanish on $\partial\Omega$. Then*

$$\int_\Omega u^2 dx \leq C_\Omega \int_\Omega |\nabla u|^2 dx \tag{13}$$

holds for a constant C_Ω depending only on Ω.

Proof. We first enclose Ω in a big cube $\Gamma := \{x \in \mathbb{R}^n \mid |x_i| \leq a \ \text{ for } i = 1, ..., n\}$. Continue u to be zero in Γ outside Ω. Then, for any $(x_1, ..., x_n) \in \Gamma$ we have

$$u^2(x) = \left(\int_{-a}^{x} \partial_{x_1} u(\xi, x_2, ...x_n) \, d\xi_1\right)^2 \leq 2a \int_{-a}^{a} d\xi_1 \left[\partial_{x_1} u(\xi_1, x_2, ...x_n)\right]^2 \quad (14)$$

Hence

$$\int_{-a}^{a} dx_1 u^2(x_1, ..., x_n) \leq 4a^2 \int_{-a}^{a} d\xi_1 \left[\partial_{x_1} u(\xi_1, x_2, ...x_n)\right]^2 \quad (15)$$

Integrating over $x_2, ..., x_n$ yields the desired inequality with $C_\Omega = 4a^2$. $\square$

Remark 2.2. *The proof shows that $\Omega \subset \mathbb{R}^n$ just needs to be bounded in one direction and $u \in H_0^1(\Omega)$ (see below).*

As a consequence of Poincare's inequality, we obtain from (12) that

$$\int_\Omega |\nabla u|^2 dx \leq C_\Omega \int_\Omega f^2 dx \quad (16)$$

holds for all $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ which vanish on the boundary and satisfy Poisson's equation $\Delta u = f$ in Ω. In view of Poincare's inequality,

$$\|u\|^2_{H^1(\Omega)} \leq C_\Omega \int_\Omega f^2 dx \quad (17)$$

2.2 Construction of Weak solutions

We now address the problem of constructing weak solutions to Dirichlet's problem. We shall work with the Sobolev space $H_0^1(\Omega)$, which can be characterized as the closure of the space of smooth functions of compact support in Ω with respect to the $H^1(\Omega)$ norm. By density, (13) holds for $u \in H_0^1(\Omega)$ and implies that the $H^1(\Omega)$ norm restricted to $u \in H_0^1(\Omega)$ is equivalent to the $\dot{H}^1(\Omega)$ norm.

It will be favorable to continue working with the $\dot{H}^1(\Omega)$-inner product on $H_0^1(\Omega)$ functions

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \ dx \quad (18)$$

We will call a $u \in H_0^1(\Omega)$ such that

$$\langle u, v\rangle = - < f, v >_{L^2} \quad (19)$$

holds for all $v \in H_0^1(\Omega)$ a weak solution. Equivalently, we could formulate this as $u \in H_0^1(\Omega)$ satisfying

$$< u, \Delta v >_{L^2} = < f, v >_{L^2} \quad (20)$$

for all test functions $v \in C_0^\infty(\Omega)$. (Recall that the latter are dense in $H_0^1(\Omega)$. This is the distributional formulation we've seen earlier.

Remarkably, the existence of a weak solution of the Dirichlet problem follows directly from the Riesz Representation theorem:

Theorem 2.3. *Let $(H, \langle \cdot \rangle)$ be a Hilbert space. Every bounded linear functional ϕ on H can be represented uniquely in the form $\phi[u] = \langle u, v \rangle$ for some $v \in H$.*

Proof. The kernel of ϕ is a closed linear subspace of H. We have the decomposition $H = ker\phi \oplus (ker\phi)^{\perp}$.[2] If $\phi(u) = 0$ for all u, then $v = 0$ will work. If not, there will be a $w \in (ker\phi)^{\perp}$ with $\|w\| = 1$ satisfying $\langle w, u \rangle = 0$ for all u with $\phi(u) = 0$. It's not hard to check that for any u the vector $\phi(u) w - \phi(w) u$ is in the kernel of ϕ. Hence $\langle w, \phi(u) w - \phi(w) u \rangle = 0$ which yields

$$\phi(u) = \langle u, \frac{\overline{\phi(w)w}}{< w, w >} \rangle = \langle u, \overline{\phi(w)} w \rangle$$

The uniqueness is a simple exercise. □

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. Let $f \in L^2(\Omega)$. Then there exists a unique $u \in H_0^1(\Omega)$ such that u is a weak solution of $\Delta u = f$.*

Proof. Note that $\phi[v] := -\langle f, v \rangle_{L^2}$ defines a bounded linear functional on $H_0^1(\Omega)$ equipped with inner product (18). (Why?) Then apply Riesz-Theorem to conclude that (19) holds. □

This proof almost seems too easy and it's good to recall what goes into the Riesz Representation theorem. The variational character can also be understood from the following proposition which is known as *Dirichlet's principle*

Proposition 2.5. *Suppose that $u \in C^1(\bar{\Omega})$, $u = 0$ on $\partial\Omega$ and $f \in L^2(\Omega)$. Then the following are equivalent*

1. *$\Delta u = f$ in $\mathcal{D}'(\Omega)$.*
2. *$J[u] \leq J[w]$ for all $w \in C^1(\overline{\Omega})$ with $w = 0$ on $\partial\Omega$ for J being the functional*
$$J[w] = \int_{\Omega} \left[|\nabla w(x)|^2 + 2w(x) f(x) \right] dx$$
3. *u is a critical point of J in the sense that*
$$\frac{d}{dt} J[u + t\phi] \Big|_{t=0} = 0 \qquad \textit{holds for all } \phi \in C^1(\bar{\Omega}) \textit{ with } \phi = 0 \textit{ on } \partial\Omega$$

Proof. Exercise. □

The proposition suggests to find a distributional solution of $\Delta u = f$ via proving the existence of a minimizer of a variational problem. This is a very important technique, as many PDEs can be shown to admit a variational formulation.

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[2] This uses the variational principle: Given a closed subspace $B \subset H$ and a $v \in H$, there is an element $\tilde{v}$ in B which is closest to v in the sense that $\|x - \tilde{v}\| = \inf_{x \in B} \|x - v\|$ for $x \in B$. Moreover, this $v - \tilde{v}$ is orthogonal to B in that $\langle v - \tilde{v}, x \rangle$ holds for any $x \in B$. Once this is established (Exercise. Hint: Parallelogram identity!), write any $x \in H$ as $x = x_0 + (x - x_0)$ with $x_0 \in B$ and, by the previous result, $x - x_0$ in $B^{\perp}$.

2.3 More Regularity

With Theorem 2.4 you should now revisit the proof of Proposition 1.2 (cf. Exercise 1.3) and try to prove that the weak solution is more regular in Ω (in fact, classical), given more regularity for f.

3 General Second Order Elliptic Equations

How much does all this depend on the fact that we were dealing with Laplace's equation? As we will mentioned, the crucial ingredients were the a-priori estimates that we derived on the solution. For this section I will follow closely Evans, Chapter 6.2.

We consider the following general linear operator

$$Lu = -\sum_{i,j}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)\, u_{x_i} + c(x)\, u \tag{21}$$

written in divergence form, with suitable regularity assumptions on the coefficients (made precise below) and $a^{ij} = a^{ji}$ symmetric. We consider the Dirichlet problem

$$\begin{aligned} Lu &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned} \tag{22}$$

Definition 3.1. *L is called <u>uniformly elliptic</u> if there exists a constant $\theta > 0$ such that*

$$\sum_{i,k=1}^{n} a^{ij}(x)\, \xi_i \xi_j \geq \theta |\xi|^2 \tag{23}$$

for almost every $x \in U$ and all $\xi \in \mathbb{R}^n$.

In other words, the symmetric matrix a^{ij} is uniformly positive definite at (almost) every point of U with the smallest eigenvalue being larger or equal to θ.

The next two definitions are obvious generalizations of our treatment of the Laplace equation:

Definition 3.2. *The bilinear form $B[\cdot]$ associated with L is defined to be*

$$B[u,v] := \int_U \left[\sum_{i,j}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cu \cdot v \right] dx \tag{24}$$

for $u, v \in H_0^1(U)$.

Definition 3.3. *We say that $u \in H_0^1(U)$ is a weak solution of the boundary value problem (22) if*

$$B[u,v] = (f,v)_{L^2(U)} \tag{25}$$

holds for all $v \in H_0^1(U)$.

3.1 Existence

Let $(H, < \cdot >)$ be a Hilbert space. We need a generalization of the Riesz representation theorem for the general bilinear form (24).

Theorem 3.4. *Assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear map satisfying the following two conditions:

$$|B[u,v]| \leq \alpha \|u\| \|v\| \quad \text{for some } \alpha > 0 \text{ and all } u, v \in H \ \text{(boundedness)}$$

$$\beta \|u\|^2 \leq B[u,u] \quad \text{for some } \beta > 0 \text{ and all } u \in H \ \text{(coercivity)}$$

Assume further that $f : H \to \mathbb{R}$ is a bounded linear function on H. Then there exists a unique element $u \in H$ such that

$$B[u,v] = f[v] \quad \text{holds for all } v \in H$$

This is the famous Lax-Milgram theorem. Note that if B was in addition symmetric, the Riesz representation theorem would provide again a two line proof. The general case is more difficult and the content of

Exercise 3.5. *Prove this theorem using the following outline (or otherwise). For fixed u, apply Riesz to $v \to B[u,v]$, which is a bounded linear functional. Get $B[u,v] =< Au, v >$ where A maps u to the element promised by Riesz. Show that A is a bounded linear operator, which is one-to-one and has range all of H. Then apply Riesz again.*

With this at hand, all we need to do to conclude the existence of a weak solution of (22), is to check whether the linear elliptic operator L (or rather its associated bilinear form (24)) satisfies the two conditions of Theorem 3.4. Let us assume that the coefficients a, b, c of L are in $L^\infty(U)$.

The first (boundedness) condition is very easily checked (Exercise). For the second, we start with the ellipticity condition

$$\theta \int_U |Du|^2 dx \leq \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} = B[u,u] - \int_U \left[\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right] dx \,. \quad (26)$$

Using now that b is in L^∞ and applying Cauchy's inequality with ϵ to the mixed term we easily obtain

$$\frac{\theta}{2} \int_U |Du|^2 dx \leq B[u,u] + C \int_U u^2 dx \quad (27)$$

for a constant C. By the Poincare inequality for $u \in H_0^1$, the left hand side also controls the L^2 norm of u and we are done, obtaining

$$\beta \|u\|^2_{H_0^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)} \quad (28)$$

for a (possibly small) constant β and a (possibly large) constant γ, which can be computed from the L^∞ bounds on the coefficients (in particular, the ellipticity constant θ) and the geometry of U only. The estimate (28) is not quite the desired coercivity condition unless $\gamma = 0$.

Exercise 3.6. *State conditions on the coefficients a, b, c which would allow one to derive the estimate (28) with $\gamma = 0$.*

The following example shows that $\gamma > 0$ can be a true obstruction to finding a unique weak solution:

Example 3.7. *Let $L = -\partial_x^2 - \partial_y^2 - 2\pi^2$ and U the unit square. Then $u = 0$ and $u(x, y) = \sin(\pi x)\sin(\pi y)$ both solve $Lu = 0$.*

We will understand this better in due course. In any case, if we add a large enough zeroth order term to L we always have:

Theorem 3.8. *There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each $f \in L^2(U)$ the following statement holds. There exists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem*

$$\begin{aligned} Lu + \mu u &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned} \tag{29}$$

Proof. Observe that $B_\mu[u, v] = B[u, v] + \mu(u, v)_{L^2(U)}$ satisfies the assumptions of the Lax-Milgram theorem. $\square$

3.2 Interior Regularity: From H^1 to H^2_{loc}

We begin by showing that a weak solution in U in actually in H^2_{loc}. As we will derive regularity statements only in the interior, it will suffice to only assume $u \in H^1$ (and not H_0^1). Note that for the Laplacian you could repeat the proof of Proposition 1.2 to arrive at the same conclusion.

Theorem 3.9. *Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$. Suppose $u \in H^1(U)$ is a weak solution of the elliptic PDE $Lu = f$ in U. Then $u \in H^2_{loc}(U)$ and for each open $V \subset\subset U$ we have the estimate*

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

for a constant depending only on V, U and the coefficients a, b, c of L.

For the proof we will need the notion of a finite difference quotient. For $u \in L^1_{loc}(U)$ and $V \subset\subset U$ we define

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h} \tag{30}$$

for $x \in V \subset\subset U$ and h a real number satisfying $0 < |h| < dist(V, \partial U)$ and e_k the k^{th} unit vector in $\mathbb{R}^n$. The $D_k^h u$ is called the k^{th} difference quotient of size h. We also define

$$D^h u := \left(D_1^h u, ..., D_n^h u\right) \tag{31}$$

Exercise 3.10. *Verify the following properties of difference quotients. Let $u, v, w \in H_0^1(U)$ be compactly supported in $V \subset\subset U$. Then*

1. *$D_k^h u$ is in $H_0^1(U)$ for sufficiently small h (regularity)*
2. *$\int_U D_k^h v \cdot w = -\int_U v D_k^{-h} w$ for h sufficiently small (integration by parts formula)*
3. *$D_k^h(vw) = v^h D_k^h w + \left(D_k^h v\right) w$ with $v^h = v(x + he_k)$ (product rule)*

More difficult are the following crucial properties

Exercise 3.11. *If $u \in H^1(U)$, then*

$$\|D^h u\|_{L^2(V)} \leq C\|Du\|_{L^2(U)} \quad \text{holds for all } 0 < |h| < \frac{1}{2} dist(V, \partial U) \tag{32}$$

(which you can prove for smooth u and then argue by density). Very important is the converse property: If

$$\|D^h u\|_{L^2(V)} \leq C \quad \text{for all } 0 < |h| < \frac{1}{2} dist(V, \partial U) \tag{33}$$

then $u \in H^1(V)$ with $\|Du\|_{L^2(V)} \leq C$. Hence if we can bound the difference quotients uniformly in h, then we can conclude the existence of a weak derivative!

Proof of Theorem 3.9. Fix a $V \subset\subset U$ open and choose a W open with $V \subset\subset W \subset\subset U$. Select a smooth cut-off function ζ which is equal to 1 in V and equal to zero in $\mathbb{R}^n \setminus W$. Since u is a weak solution, we have

$$\boxed{A} := \sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} dx = \int_{\mathcal{U}} \left(f - \sum_i b^i u_{x_i} - cu \right) v \, dx =: \boxed{B} \tag{34}$$

If we knew u was smooth we would set, $v = -\zeta^2 u_{kk}$ and integrate by parts. However, at this point we only know $u \in H^1(U)$! Therefore we set

$$v = -D_k^{-h}\left(\zeta^2 D_k^h u\right) \tag{35}$$

and use the integration by parts formula for difference quotients. The goal will be to bound the difference quotient $\|D_k^h Du\|$ in L^2 on V and then to apply the second property of Exercise 3.11, which will guarantee that $u \in H^2(V)$.

Noting that difference quotients commute with weak derivatives, we have

$$\begin{aligned}
\boxed{A} &= -\int_{\mathcal{U}} a^{ij} u_{x_i} \left[D_k^{-h}\left(\zeta^2 D_k^h u\right)\right]_{x_j} dx \\
&= \int_{\mathcal{U}} D_k^h\left(a^{ij} u_{x_i}\right) \left[2\zeta\zeta_{x_j} D_k^h u + \zeta^2 D_k^h u_{x_j}\right] dx \\
&= \int_{\mathcal{U}} \left(a^{ij,h} D_k^h u_{x_i} + \left(D_k^h a^{ij}\right) u_{x_i}\right) \left[2\zeta\zeta_{x_j} D_k^h u + \zeta^2 D_k^h u_{x_j}\right] dx \\
&= \int_{\mathcal{U}} \left(a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j}\right) \zeta^2 dx + \text{error-terms}
\end{aligned} \tag{36}$$

Therefore,

$$\boxed{A} \geq \theta \int_{\mathcal{U}} \zeta^2 |D_k^h Du|^2 dx - |\text{error-terms}| \tag{37}$$

Observe now that (note this uses $a^{ij} \in C^1(U)$ – where?)

$$\begin{aligned} |\text{error-terms}| &\leq C \int_{\mathcal{U}} \left[|D_k^h Du||D_k^h u|\zeta|D\zeta| + |Du||D_k^h Du|\zeta^2 + |Du||D_h^k u|\zeta|D\zeta| \right] dx \\ &\leq \frac{\theta}{2} \int_{\mathcal{U}} \zeta^2 |D_k^h Du|^2 dx + C \int_{\mathcal{U}} |Du|^2 dx \qquad (38) \end{aligned}$$

Combining (37) and (38) and using that $\boxed{A} = \boxed{B}$ we have

$$\frac{\theta}{2} \int_{\mathcal{U}} \zeta^2 |D_k^h Du|^2 dx \leq |\boxed{B}| + C \int_{\mathcal{U}} |Du|^2 dx \tag{39}$$

Finally,

$$|\boxed{B}| \leq C \int_U \left(|f| + |Du| + |u|\right) |v| dx \leq \frac{C^2}{\epsilon} \int_U \left(|f| + |Du| + |u|\right)^2 dx + \epsilon \int_U |v|^2 dx$$

where v is as defined in (35). We easily compute

$$\begin{aligned} \int_U |v|^2 \leq C \int_U |D\left(\zeta^2 D_k^h u\right)|^2 dx &\leq C \int_W \left(\zeta^2 |D_k^h Du|^2 + |D_k^h u|^2\right) dx \\ &\leq C \int_U \left[\zeta^2 |D_k^h Du|^2 + |Du|^2\right] dx \qquad (40) \end{aligned}$$

and obtain, choosing the ϵ above appropriately,

$$|\boxed{B}| \leq C \int_U \left(|f| + |Du| + |u|\right)^2 dx + \frac{\theta}{4} \int_U |v|^2 dx \tag{41}$$

Combining (39) with (41) we finally obtain

$$\int_V |D_k^h Du|^2 dx \leq \int_U \zeta^2 |D_k^h Du|^2 dx \leq C \int_U \left(|f|^2 + |Du|^2 + |u|^2\right) dx \tag{42}$$

for h sufficiently small and $k = 1, 2, ..., n$.

By the properties of the difference quotients and the fact that $u \in H^1(U)$ and $f \in L^2(U)$ we already conclude $u \in H^2_{loc}(U)$. We can refine the above estimate further. To do this, we observe that the same argument as above shows that for any $V \subset\subset W \subset\subset U$ we have

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right) . \tag{43}$$

Now choose a new cut-off function χ which is equal to 1 in W and equal to zero in a neighborhood of the boundary of U. Setting $v = \zeta^2 u$ in the formula for the generalized solution produces (Exercise)

$$\int_U \zeta^2 |Du|^2 dx \leq C \left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)} \right) . \tag{44}$$

Combining this with (43) we obtain the estimate claimed in the Theorem. $\square$

It is clear that this argument can be iterated in the sense that if $f \in H^k(U)$ and appropriate assumptions hold on the coefficients, then every weak solution $u \in H_0^1(U)$ is actually in $u \in H_{loc}^{k+2}(U)$. You may want to formulate this statement precisely and do a proof by induction or at least look it up in the literature, e.g. in the book of Evans.

3.3 Boundary Regularity: From H^1 to H^2

In the previous section we proved (in particular) that if $u \in H_0^1(U)$ is a weak solution of $Lu = f$ in U, then u is in $H_{loc}^2(U)$ under very mild conditions on the coefficients. Note that this already implies that the equation $Lu = f$ is satisfied almost everywhere (Exercise) or, if f is sufficiently regular, that u is a classical solution in the interior.

We now turn to the issue of whether a weak solution $u \in H_0^1(U)$ remains smooth up to the boundary. For the following theorem we hence assume that the boundary of U is smooth.

Theorem 3.12. *Assume $a^{ij} \in C^1\left(\bar{U}\right)$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$. Suppose $u \in H_0^1(U)$ is a weak solution of the elliptic PDE $Lu = f$ in U, $u = 0$ on ∂U. Then $u \in H^2(U)$ with the estimate*

$$\|u\|_{H^2(U)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

for a constant depending only on U and the coefficients a, b, c of L.

The proof has two steps, of which we are only going to sketch the first. Namely, we first prove the statement for U being a half-ball in $\mathbb{R}^n$, i.e. $U = B(0,1) \cap \{x_n > 0\}$ with straight boundary $\{x_n = 0\} \cap B(0,1)$. The second step is to construct – for arbitrary U – a diffeomorphism which maps the neighborhood of a point on ∂U to the half-ball and to carry the estimates over. (This step requires certain regularity of the boundary.)

To carry out the first step, as mentioned we let $U = B(0,1) \cap \mathbb{R}_+^n$. We set $V := B(0,1/2) \cap \mathbb{R}_+^n$ and choose a cut-off function which is equal to 1 on $B(0,1/2)$ and vanishes identically on $\mathbb{R}^n \setminus B(0,1)$. In particular, it vanishes on the curved part of ∂U.

The idea is to repeat the "interior" computation with

$$v = -D_k^{-h}\left(\zeta^2 D_k^h u\right)$$

where $k = 1, ...n-1$ involves only difference quotients in the *tangential* directions. This v is well-defined and actually in $H_0^1(U)$ (why?). Repeating the "interior" computation we obtain the analogue of (43)

$$\sum_{\substack{k,l=1 \\ k+l<2n}}^{n} \|u_{x_k x_l}\|_{L^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right) \tag{45}$$

We are cheating here slightly. The reason is that what we proved for finite difference quotients (the second property of Exercise 3.11) is slightly weaker than what is need here.

Exercise 3.13. *State and prove the property of finite differences needed to obtain the estimate (45).*

In view of (45), the only thing missing is an H^2 estimate for the second derivative $u_{x_n x_n}$. For this we recall that we can write the equation $Lu = f$ (satisfied almost everywhere in U!) as

$$-\sum_{i,j=1} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \left(b^i - \sum_{j=1}^{n} a^{ij}_{x_j} \right) u_{x_i} + cu = f \tag{46}$$

and moreover, as

$$a^{nn} u_{x_n x_n} = \sum_{\substack{k,l=1 \\ k+l<2n}}^{n} a^{kl} u_{x_k x_l} + \sum_{i=1}^{n} \left(b^i - \sum_{j=1}^{n} a^{ij}_{x_j} \right) u_{x_i} + cu - f \tag{47}$$

The uniform ellipticity condition yields $a^{nn} \geq \theta > 0$ and therefore

$$|u_{x_n x_n}| \leq C \left[\sum_{k,l=1,k+l<2n}^{n} |u_{x_k x_l}| + |Du| + |u| + |f| \right] \tag{48}$$

The right hand side is in $L^2(V)$ and hence finally

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right) \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right) \tag{49}$$

with the last inequality following from the basic energy estimate

$$\|u\|_{H^1_0(U)} \leq C \left[\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)} \right]$$

derived earlier for $u \in H^1_0(U)$.

Once the estimate is established for the half ball one constructs an appropriate diffeomorphism, i.e. introduces coordinates near a point on the boundary in which the boundary locally corresponds to $x_n = 0$. It is easy to see that the crucial ellipticity condition is preserved.

Again, this method can be iterated and used to establish that u is actually smooth up to the boundary provided f and all the coefficients are smooth in $\bar{U}$. We refer again to the book of Evans.

3.4 The Fredholm Alternative

Let us understand better the problem of non-uniqueness (and non-existence) of weak solutions for elliptic equation which was touched upon in Example 3.7. For this we need a brief digression reminding you of the properties of compact operators.

3.4.1 Compact Operators

Definition 3.14. *A bounded linear operator*

$$K : X \to Y$$

between two real Banach spaces X and Y is called compact provided for each bounded sequence $\{u_k\}_{k=1}^{\infty} \subset X$, the sequence $\{Ku_k\}_{k=1}^{\infty}$ is precompact in Y, i.e. there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $\{Ku_{k_j}\}_{j=1}^{\infty}$ converges in Y.

Exercise 3.15. *Find examples and non-examples. Prove that if K is compact, then it is bounded (hence continuous). Prove that K is compact if and only if the adjoint $K^{\star}$ is compact.*

We will mostly deal with compact operators between Hilbert spaces. That is a good occasion to do

Exercise 3.16. *Remind yourself of the spectral theorem for symmetric (self-adjoint) compact operators $K : H \to H$ in a Hilbert space.*

The property of compact operators most relevant to us goes by the name of the Fredholm alternative and concerns compact perturbations of the identity.

Theorem 3.17. *Let $K : H \to H$ be a compact linear operator. Then*

1. *$N(I - K)$ is finite dimensional*
2. *$R(I - K)$ is closed*
3. *$R(I - K) = N(I - K^{\star})^{\perp}$*
4. *$N(I - K) = \{0\}$ iff $R(I - K) = H$*
5. *$\dim N(I - K) = \dim N(I - K^{\star})$*

Proof. See appendix of Evans. □

Corollary 3.18. *Precisely one of the following statements holds:*

1. *for each $f \in H$ the equation $u - Ku = f$ has a unique solution*
2. *the homogeneous equation $u - Ku = 0$ has non-trivial solutions.*

Moreover, in the case of 2. then, the space of solutions of the homogenous equation is finite dimensional and $u - Ku = f$ has a solution if and only if $f \in N(I - K^{\star})^{\perp}$.

3.4.2 Application to $Lu = f$

Let us connect the theory of compact operators with our problem $Lu = f$ and assume for convenience that the coefficients a, b, c of L are actually in $C^\infty\left(\bar{U}\right)$.

Definition 3.19. *The formal adjoint (or transpose) associated with L is the operator $L^\star$ defined by*

$$L^\star v := -\sum_{i,j}\left(a^{ij}v_{x_j}\right)_{x_i} - \sum_i b^i v_{x_i} + \left(c - \sum_i b^i_{x_i}\right) v\,.$$

The adjoint bilinear form associated with L is defined to be

$$B^\star : H^1_0(U) \times H^1_0(U) \to \mathbb{R} \qquad B^\star[v,u] = B[u,v]\,.$$

Finally, we call $v \in H^1_0(U)$ a solution of the adjoint problem

$$L^\star u = f \quad \textit{in } U \qquad , \qquad v = 0 \quad \textit{on } \partial U$$

provided that $B^\star[v,u] = (f,u)_{L^2(U)}$ holds for all $u \in H^1_0(U)$.

Theorem 3.20. *Precisely one of the following statements holds:*

1. *For each $f \in L^2(U)$ there exists a unique weak solution u of the inhomogeneous problem*
$$(IP)\begin{cases} Lu = f & \textit{in } U \\ u = 0 & \textit{on } \partial U \end{cases} \tag{50}$$

2. *There exists a weak solution $u \neq 0$ of the homogeneous problem*
$$(HP)\begin{cases} Lu = 0 & \textit{in } U \\ u = 0 & \textit{on } \partial U \end{cases} \tag{51}$$

If the second alternative holds, then the dimension of the subspace $N \subset H^1_0(U)$ of weak solutions to (HP) is finite and equals the dimension of the subspace $N^\star \subset H^1_0(U)$ of weak solutions of the homogeneous adjoint problem

$$(AP)\begin{cases} L^\star v = 0 & \textit{in } U \\ v = 0 & \textit{on } \partial U \end{cases} \tag{52}$$

Finally, the problem (IP) has a weak solution if and only if $(f,v) = 0$ holds for all $v \in N^\star$.

Proof. Step 1: Relating the problem to Fredholm theory. We let

$$B_\gamma[u,v] = B[u,v] + \gamma(u,v)_{L^2}$$

with γ defined such that the associated operator $L_\gamma u = Lu + \gamma u$ has the property that $L_\gamma u = g$ has a unique weak solution $u \in H^1_0(U)$ for each $g \in L^2(U)$ (cf. Theorem 3.8), i.e.[3]

$$B_\gamma[u,v] = (g,v) \quad \text{for all } v \in H^1_0(U) \tag{53}$$

[3] We can assume $\gamma > 0$ wlog since for $\gamma = 0$ we already proved that the first statement and not the second applies.

We write $u = L_\gamma^{-1} g$ to denote the operator that maps a given $g \in L^2(U)$ to the unique $u \in H_0^1(U)$ satisfying (53).

By definition, $u \in H_0^1(U)$ is a weak solution of (IP) if and only if

$$B_\gamma[u,v] = (\gamma u + f, v) \quad \text{for all } v \in H_0^1(U) \tag{54}$$

and hence if and only if $u = L_\gamma^{-1}(\gamma u + f)$ holds.

We define $Kw = \gamma L_\gamma^{-1} w$ for $w \in L^2(U)$. The point of this is that we can write

$$(I - K)u = h := L_\gamma^{-1} f$$

which is of the form in the Fredholm alternative, if we can show that K is a linear, bounded, compact operator. Indeed, we have $K : L^2(U) \to H_0^1(U) \subset\subset L^2(U)$. The first part is bounded since in view of

$$\beta \|u\|^2_{H_0^1(U)} \le B_\gamma[u,u] = (g,u) \le \|g\|_{L^2(U)} \|u\|_{L^2(U)} \le \|g\|_{L^2(U)} \|u\|_{H_0^1(U)}$$

we have

$$\|u\|_{H_0^1(U)} = \|Kg\|_{H_0^1(U)} \le \|g\|_{L^2(U)} .$$

The second part (the inclusion into $L^2(U)$) is compact by Rellich's theorem.[4]

Step 2: Applying the Fredholm alternative. By the latter precisely one of the following statements hold:

1. for each $h \in L^2(U)$ the equation $u - Ku = h$ has a unique solution $u \in L^2(U)$.
2. the equation $u - Ku = 0$ has non-trivial solutions in $L^2(U)$.

If the first statement applies then – by the equivalence (54) – there is a unique solution to (IP). If the second statement applies then necessarily $\gamma \neq 0$ (why?) and the space of solutions to $u - Ku$ is finite and equals the dimension of the space $N^\star$ of solutions to $v - K^\star v = 0$. It is easy to see the following equivalences

$$u - Ku = 0 \;\text{ if and only if } u \in H_0^1(U) \text{ is a weak solution of } (HP).$$

and

$$v - K^\star v = 0 \;\text{ if and only if } v \in H_0^1(U) \text{ is a weak solution of } (AP).$$

To prove the last claim of the theorem, observe that by the Fredholm alternative $u - Ku = h$ has a solution if and only if $(h,v) = 0$ holds for all $v \in N(I - K^\star)$, i.e. for all v satisfying $v - K^\star v = 0$. Finally,

$$(h,v) = \left(L_\gamma^{-1} f, v\right) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^\star v) = \frac{1}{\gamma}(f,v)$$

$\square$

[4]Recall that $X \subset\subset Y$ means that $\|x\| \le C\|x\|_Y$ (continuity) holds and that each bounded sequence in X is precompact in Y (compactness). See the exercises.

We conclude with the following

Theorem 3.21. *There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary value problem*

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{55}$$

has a unique solution for each $f \in L^2(U)$ if and only if $\lambda \neq \Sigma$. If Σ is infinite, then $\Sigma = (\lambda_k)_{k=1}^{\infty}$ form a non-decreasing sequence with $\lambda_k \to \infty$.

The set Σ is called the spectrum of the operator L.

Corollary 3.22.

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{56}$$

has a non-trivial solution $w \neq 0$ if and only if $\lambda \in \Sigma$. In this case λ is an eigenvalue of L with associated eigenfunction w.

Proof of Theorem 3.21. Recall the constant $\gamma \geq 0$ which is always such that $L_\gamma = L + \gamma$ is invertible. Wlog we can assume $\gamma > 0$. We can also assume that $\lambda > -\gamma$ since if $\lambda + \gamma \leq 0$, the homogenous problem $Lu + \gamma u = (\lambda + \gamma) u$ the homogeneous problem is easily seen not to admit non-trivial solutions.

We then have the following equivalent statements:

- Problem (55) has a unique weak solution for a given λ
- The associated homogeneous problem has only the trivial solution.
- $Lu + \gamma u = (\gamma + \lambda) u$ has only the trivial solution.
- $u = L_\gamma^{-1} (\gamma + \lambda) u = \frac{\gamma+\lambda}{\gamma} K u$ has only the trivial solution
- $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K

As the collection of eigenvalues of a compact operator K is either finite or forms a sequence converging to zero (see exercises below) we obtain that (55) has a unique solution unless λ is the member of a sequence $\lambda_k \to \infty$. ☐

4 Exercises

1. Do the exercises in the text if you haven't already done so. Exercises 3.5 and 3.11 are particularly recommended.
2. Let u be a smooth solution of $Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = 0$ in U, where L is uniformly elliptic and the coefficients a^{ij} are in $C^1\left(\overline{U}\right)$.

   Set $v := |Du|^2 + \lambda u^2$. Show that

   $$Lv \leq 0 \text{ holds in } U \text{ for } \lambda \text{ sufficiently large}$$

Deduce

$$\|Du\|_{L^\infty(U)} \leq C\left(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}\right) .$$

Remark: The "deduce" step requires the maximum principle for general elliptic operators which we haven't shown in general, except in the 2-dimensional case (Exercise 5 of Week 5). You can assume it here or consider the proof another exercise. [Hint: For the maximum principle all consideration are local. Suppose you have an interior maximum at $\tilde{x} \in U$. You want to change coordinates at $\tilde{x}$ by a rotation such that diagonalises a^{ij}. For this you first need to understand how the PDE changes under linear coordinate transformations: $y_i = (\tilde{x})_i + \sum_j R_{ij}(x-\tilde{x})_j$ for a matrix R_{ij} which diagonalises a_{ij} at $\tilde{x}$.]

3. (Best constant in Poincare's inequality; F. John Chapter 5) Show that if there exists a function $u \in C^2\left(\overline{\Omega}\right)$ vanishing for $\partial\Omega$ for which the quotient

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

reaches its smallest value λ, then u is an eigenfunction to the eigenvalue λ, i.e. $\Delta u + \lambda u = 0$ in Ω. In fact λ must be the smallest eigenvalue belonging to an eigenfunction in $C^2\left(\overline{\Omega}\right)$.

4. Let $n = 3$ and Ω be the ball $|x| < \pi$. Show that a solution u of $\Delta u + u = w(x)$ with vanishing boundary values can only exist if

$$\int_\Omega w(x)\frac{\sin|x|}{|x|}dx = 0$$

5. *Dirichlet boundary condition for weak solutions.* Suppose we are given a weak solution $u \in H_0^1(\Omega)$ of the second order elliptic boundary value problem $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$ where Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary. It is natural to ask in what sense is "$u = 0$ on $\partial\Omega$"? Remember that functions in $H_0^1(\Omega)$ are only defined a.e. so it does not make sense to talk about the value of the function on the boundary unless the solution is more regular. The correct way to do this is via traces of functions in Sobolev spaces. Without introducing this technical machinery here, the following exercise describes a satisfactory notion of the solution vanishing on the boundary.

Let Ω be as above and $\Omega_\epsilon = \{x \in \Omega \mid dist(x, \partial\Omega) < \epsilon\}$. Then there is a constant C and an $\epsilon_0 > 0 > 0$ such that for all $u \in H_0^1(\Omega)$ and $\epsilon < \epsilon_0$ we have the estimate

$$\int_{\Omega_\epsilon} |u|^2 dx \leq C\epsilon^2 \int_{\Omega_\epsilon} |\nabla u|^2$$

Hint: Prove this estimate for functions in $C_0^\infty(\Omega)$ and argue by density. To do the former, construct suitable coordinates near the boundary and mimic the proof of the Poincare inequality.

Deduce that
$$\frac{1}{vol(\Omega_\epsilon)}\int_{\Omega_\epsilon}|u|^2dx = o(\epsilon)$$
holds as $\epsilon \to 0+$ and interpret your result.

6. Prove the following *basic version of the Banach Alaoglu theorem* (which we used in connection with the difference quotients): Let (u_k) be a bounded sequence in a separable Hilbert space H, i.e. $\|u_k\|_H \leq C$. Then there exists a subsequence which converges weakly in H. Hint: Use the following outline

   (a) Pick an ONB (e_k) and use a diagonal argument to show that for a subsequence of the (u_k), denoted $(u_n^{(n)})$ (arising from a Cantor diagonal argument) we have that
   $$\langle u_n^{(n)}, e_k\rangle \to v_k \in \mathbb{R} \quad \text{holds for all } e_k.$$

   (b) Show that $\sum_{k=1}^{\infty}|v_k|^2 < \infty$ and hence $v = \sum_k v_k e_k \in H$.

   (c) Show that $u_n^{(n)} \rightharpoonup v$.

7. (Rellich's theorem, simple version). Let (u_k) be a bounded sequence in $H_0^1(\Omega)$ with Ω as in lectures. Then there exists a subsequence converging strongly in $L^2(\Omega)$.
   Outline: Extend each u_k by zero outside Ω to get a sequence (u_k) in $H^1(\mathbb{R}^n)$. By Banach-Alaoglu, there exists a subsequence with $u_{n_k} \rightharpoonup u$. For fixed ξ the map $\mathcal{F}_\xi : H_0^1(\Omega) \to \mathbb{R}$ mapping $v \in H_0^1(\Omega)$ to $\hat{v}(\xi)$ (Fourier transform evaluated at ξ) is a bounded linear functional on $H_0^1(\Omega)$. Now use the Fourier definition of $L^2(\Omega)$ and $H^1(\Omega)$ to conclude.

8. Sobolev inequality on bounded domains [F. John, Chapter 5]

   Definition: A *conical sector* $\Gamma \subset \mathbb{R}^n$ is the intersection of a ball with a cone from its centre:
   $$\Gamma = \{y \mid y = x + t\xi \ ; \ 0 \leq t \leq h \ ; \ \xi \in \sigma\}$$
   where σ is a relatively open subset of the unit sphere in $\mathbb{R}^n$. We call x the *vertex* and h the *radius* of Γ. The *solid angle* ω of Γ is the $(n-1)$-dimensional measure of σ. An open set $\Omega \subset \mathbb{R}^n$ has the *cone property* if there exist positive numbers h, ω such that each $x \in \Omega$ is vertex of a conical sector $\Gamma \subset \Omega$ of radius h and solid angle ω.

   Show that for any $\Omega \subset \mathbb{R}^n$ with the cone property there exists a C (depending on Ω) such that for any $u \in C^s(\Omega)$ with $s = \frac{n}{2}+1$ if n is even and $s = \frac{n}{2}+\frac{1}{2}$ if n is odd, and any $x \in \Omega$ we have
   $$|u(x)| \leq C\|u\|_{H^s(\Omega)} \,. \tag{57}$$

Outline: Show that for $\phi \in C^s(\mathbb{R})$ with $\phi(t) = 0$ for $t > h$ we have

$$\phi(0) = \frac{(-1)^s}{(s-1)!} \int_0^h t^{s-1} \phi^{(s)}(t) dt \,.$$

Apply Cauchy-Schwarz to get

$$\phi^2(0) = C \int_0^h t^{n-1} (\phi^{(s)}(t))^2 dt \,. \tag{58}$$

Finally, let $\zeta(t) \in C^\infty(\mathbb{R})$ with $\zeta(0) = 0$ and $\zeta(t) = 0$ for $t > h$ and consider

$$\phi(t) = \zeta(t) u(x + t\xi) \,.$$

Integrate (58) over the set σ.

9. *(Spectral properties of compact operators on Hilbert spaces)* Let $K : H \to H$ be a compact operator with H a separable Hilbert space. We say that λ is an eigenvalue of K if $Ku = \lambda u$ holds for some $u \neq 0$. Show that K can have at most countably many eigenvalues and that they can only accumulate at 0.

   Outline: Show that for any $k > 0$ the number of distinct eigenvalues with $|\lambda| \geq k$ is finite. Assuming this is false there is a sequence (λ_n) of distinct eigenvalues and a corresponding sequence of eigenvectors (v_n). Let $Y_n := span(v_1, ..., v_n)$. Choose a sequence (y_n) with $\|y_n\| = 1$, $y_n \in Y_n$ and $y_n \in Y_{n-1}^\perp$. Show that $\|Ky_m - Ky_n\| \geq k$ for all $m > n$ and conclude.

5 Non-examinable exercises and further reading

1. Prove Theorem 3.17. (See the book of Evans or my notes on Functional Analysis.)

2. (See the book of Evans.) Formulate and prove a statement generalizing the maximum principle we discussed for the Laplacian to general elliptic operators

$$Lu = -a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu$$

   For the strong maximum principle, you will have to prove the following Lemma, due to E. Hopf:

   Lemma 5.1. *Let $u \in C^2(U) \cap C^1(\bar{U})$, $c \equiv 0$ in U and $Lu \leq 0$ for L as above. Suppose there exists a point $x^0 \in \partial U$ such that $u(x_0) > u(x)$ for all $x \in U$. Assume that U satisfies the interior ball condition at x^0 (there is an open ball $B \subset U$ with $x^0 \in \partial B$; this follows for instance if ∂U is C^2). Then the strict inequality*

$$\frac{\partial u}{\partial \nu}(x^0) > 0$$

   holds, where ν is the outer unit normal to B at x^0.