Partial Differential Equations (Week 5)
Laplace's Equation

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1 Laplace's equation

We start now with the analysis of one of the most important PDEs of mathematics and physics, the Laplace equation

$$\Delta u := \sum_{i=1}^{d} \partial_i^2 u = 0\,, \tag{1}$$

or, more generally, the Poisson equation, $\Delta u = f$, which has a prescribed inhomogeneity f on the right hand side.[1] Equation (1) appears naturally in electrostatics, complex analysis and many other areas. We have already seen that this PDE is *elliptic*. Our approach will be to understand the behavior of solutions to (1) in great detail first (which is doable in view of the highly symmetric form of the operator) before turning to more general elliptic operators and equations. In summary, our goals are

- "Solve" $\Delta u = f$. What formulation is well-posed and which one's are ill-posed? What are the conditions on f?
- What is the regularity of u? How does it depend on f?
- What happens for more general elliptic operators $Lu = f$?

I will follow very closely Fritz John's book, Chapter 4, for the first part.

1.1 Uniqueness for Dirichlet and Neumann problem

Let $\Omega \subset \mathbb{R}^d$ be an open bounded connected region of $\mathbb{R}^d$ whose boundary is sufficiently regular for Stokes theorem to apply. Recall the Green's identities

[1] In applications, f could be a charge-distribution whose electrostatic potential u is to be determined.

(which are special cases of our formula for the transpose of an operator), valid for $u, v \in C^2\left(\bar{\Omega}\right)$:

$$\int_\Omega v\Delta u = -\int_\Omega \sum_i v_{x_i} u_{x_i} dx + \int_{\partial\Omega} v\frac{du}{dn} dS\,, \tag{2}$$

$$\int_\Omega v\Delta u = \int_\Omega u\Delta v + \int_{\partial\Omega} \left[v\frac{du}{dn} - u\frac{dv}{dn}\right] dS\,, \tag{3}$$

and recall that $\frac{d}{dn}f = \sum_i \xi^i \partial_i f$ means differentiating in the direction of the *outward* unit normal ξ to the boundary $\partial\Omega$. Applying (3) with $v = 1$ we find

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{du}{dn} dS\,, \tag{4}$$

while applying (2) with $v = u$ produces

$$\int_\Omega \sum_i \left(\partial_i u\right)^2 + \int_\Omega u\Delta u = \int_{\partial\Omega} u\frac{du}{dn} dS\,. \tag{5}$$

From (5) we immediately obtain a uniqueness statement for Poisson's equation. Consider the following problems

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{6}$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{du}{dn} = g & \text{on } \partial\Omega \end{cases} \tag{7}$$

for prescribed (say smooth for the moment) functions f and g. Problem (6) is called the Dirichlet problem, problem (7) the Neumann problem for Poisson's equation.

Suppose we have two solutions $\Delta u_1 = f$ and $\Delta u_2 = f$ of the Dirichlet problem with $u_1, u_2 \in C^2\left(\bar{\Omega}\right)$, then their difference satisfies Laplace's equation, $\Delta\left(u_1 - u_2\right) = 0$ with $u_1 - u_2 = 0$ on the boundary. By (5)

$$\sum_i \left(\partial_i \left[u_1 - u_2\right]\right)^2 = 0\,,$$

and hence $u_1 - u_2 = c$ must be constant in Ω. But since $u_1 - u_2 = 0$ on $\partial\Omega$, we can conclude $u_1 = u_2$ and hence the uniqueness of $C^2\left(\bar{\Omega}\right)$ solutions to the Dirichlet problem (6). For the Neumann problem (7) there remains ambiguity up to a constant. Note, that there is an obvious constraint for *existence* of solutions to the Neumann problem given by (4) which produces a non-trivial relation between f and g.

1.2 The fundamental solution

The Laplacian is *spherically symmetric*. By this we mean that if $u(x)$ is a solution of $\Delta u = 0$ then $v(x) = u(Rx)$, with R a rotation in $\mathbb{R}^d$, satisfies $\Delta v = 0$ (Exercise). There is also invariance under translations and dilations. In two-dimension, the Laplacian is also invariant under spherical inversion:

Exercise 1.1. *Prove that $\Delta u = 0$ for $u = u(x_1, \ldots, x_d)$ implies that*

$$\Delta \left[|x|^{2-d} u\left(\frac{x}{|x|^2}\right)\right] = 0$$

in the domain of u. Conclude that in two dimensions the Laplacian is invariant under conformal transformations.

In view of the spherical symmetry of the Laplace operator, one may try to find *spherically symmetric solutions* of $\Delta u = 0$. Setting $u = \psi(r)$ and expressing the Laplacian in spherical coordinates one obtains the ODE

$$\psi''(r) + \frac{d-1}{r}\psi'(r) = 0$$

which is readily solved as $\psi' = Cr^{1-d}$ for some constant C and

$$\psi(r) = \begin{cases} \frac{C}{2-d} r^{2-d} & \text{for } d > 2 \\ C \log r & \text{for } d = 2 \end{cases} \tag{8}$$

We can still add a non-trivial constant to ψ (which we suppress). The function (8) solves $\Delta \psi = 0$ for $r \neq 0$ by construction but has a singularity for $r = 0$. Note that $\psi(r)$ and also $\psi'(r)$ are still in $L^1_{loc}(\mathbb{R}^d)$ but that $\psi''(r)$ fails (barely). Note also that in view of the translation invariance we could have introduced polar coordinates around any point. We will now prove

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^d$ open, bounded and connected and $\xi \in \Omega$. Then for any $u \in C^2(\bar{\Omega})$ and $\xi \in \Omega$ we have the identity*

$$u(\xi) = \int_\Omega K(x,\xi)\,\Delta u dx - \int_{\partial\Omega}\left[K(x,\xi)\frac{du}{dn_x}(x) - u(x)\frac{dK(x,\xi)}{dn_x}\right] dS_x \tag{9}$$

with $K(x,\xi) = \psi(|x-\xi|)$ and the constant C in ψ chosen such that $C^{-1} = \omega_d$ equals the surface area of the unit-sphere in $d-1$ dimensions.

Corollary 1.3. *The function $K(x,\xi)$ is a fundamental solution with pole ξ for the Laplace operator in Ω in that*

$$\Delta v = \delta_\xi$$

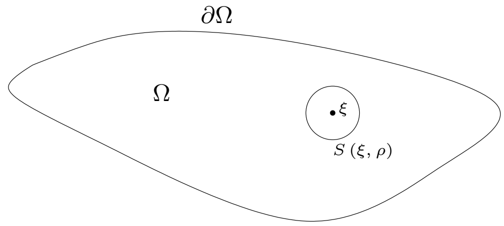
holds in Ω the sense of distributions.

Proof. Indeed, by the identity of the Lemma,

$$\int_\Omega K(x,\xi)\,\Delta u dx = u(\xi)$$

for all test functions $u \in C_0^\infty(\Omega) \subset C^2(\bar{\Omega})$. $\square$

Proof of Lemma 1.2. Consider the region $\Omega_\rho = \Omega \setminus B(\xi,\rho)$ obtained by cutting out a small ball from Ω around ξ.



Then, for $u \in C^2(\bar{\Omega})$ we have from (3) applied in Ω_ρ (where v is harmonic) the identity

$$\int_{\Omega_\rho} v\Delta u = \int_{\partial\Omega}\left(v\frac{du}{dn} - u\frac{dv}{dn}\right)dS + \int_{S(\xi,\rho)}\left(v\frac{du}{dn} - u\frac{dv}{dn}\right)dS \tag{10}$$

where $\frac{d}{dn}$ denotes the *inward* normal on $S(\xi,\rho)$ and the *outward* normal on $\partial\Omega$. We claim that the boundary term on $S(\xi,\rho)$ converges to $u(\xi)$ in the limit as $\rho \to 0$. To see this, note that since v is radial around ξ we have for $d > 2$

$$\begin{aligned}\Big|\int_{S(\xi,\rho)} v\frac{du}{dn}\Big| &= \Big|v(\rho)\int_{S(\xi,\rho)}\frac{du}{dn}\Big| = \Big|v(\rho)\int_{B(\xi,\rho)}\Delta u\Big| \\ &\le \frac{C}{2-d}\rho^{2-d}\rho^{d-1}\omega_d|\max_{B(\xi,\rho)}\Delta u(\xi)|\end{aligned} \tag{11}$$

where we have used (2), the explicit form of v and recall the notation ω_d for the surface area of the $d-1$-dimensional unit-sphere. (For $d = 2$ we would obtain $\log\rho\cdot\rho$ instead of ρ.) Clearly, the right hand side goes to zero as $\rho \to 0$. On the other hand, by a similar computation, we have

$$\int_{S(\xi,\rho)} u\frac{dv}{dn} = C\rho^{1-d}\int_{S(\xi,\rho)} u = u\left(\tilde{\xi}\right)C\omega_d \tag{12}$$

for some $\tilde{\xi} \in S(\xi,\rho)$ by the mean value theorem. By the continuity of u, the expression converges to $C\omega_d u(\xi)$ in the limit as $\rho \to 0$. Choosing $C = \omega_d^{-1}$ produces the identity claimed in the Lemma. $\square$

The formula (9) is extremely useful, as it expresses the value of u at any point in Ω by the value of Δu in Ω (which is prescribed in Poission's equation) and the values of u and its normal derivative on the boundary. However, we already know that the solution of Poisson's equation is already unique by prescribing *either* u on the boundary *or* its normal derivative. In other words, once u is prescribed on the boundary, there can be at most one $C^2\left(\bar{\Omega}\right)$ solution, whose normal derivative would then also be uniquely determined and can no longer be freely specified. This tells us that the Cauchy-problem for Laplace's equation $\Delta u = 0$ (i.e. specifying the function and its normal derivative on $\partial\Omega$) is not solvable in general. Of course we know it is solvable for analytic data in a small neighborhood of a point on the boundary by the Cauchy Kovalevskaya theorem. However, this does not mean that one can solve the problem in the analytic class in a full neighborhood of all of $\partial\Omega$ (and indeed one cannot in general!).

1.3 Regularity of harmonic functions

The formula (9) provides important insights into the regularity of harmonic functions. The latter satisfy (recall this formula was derived for $u \in C^2\left(\bar{\Omega}\right)$)

$$u(\xi) = -\int_{\partial\Omega}\left[K(x,\xi)\frac{du}{dn_x}(x) - u(x)\frac{dK(x,\xi)}{dn_x}\right]dS_x\,. \tag{13}$$

We can easily show that u is C^∞ at any $\xi \in \Omega$. Namely, given ξ we find a small ball $B(\xi,\rho)$ around ξ such that $dist\left(\tilde{\xi},\partial\Omega\right) \geq \delta > 0$ for all $\tilde{\xi} \in B(\xi,\rho)$. Then we apply formula (13) in that ball $B(\xi,\rho)$. Differentiating $u(\xi)$ we can interchange the derivative with the integral as the integrand and all its derivatives are uniformly bounded on $\partial\Omega$. It follows that u is C^1 in the ball $B(\xi,\rho)$ and hence $u \in C^1(\Omega)$. By the same argument one sees $u \in C^n(\Omega)$ for any n. For this argument, we have only used the formula (13) in small balls inside Ω, on which u is always $C^2\left(\overline{B(\xi,\rho)}\right)$ provided u is $C^2(\Omega)$. We summarize this as

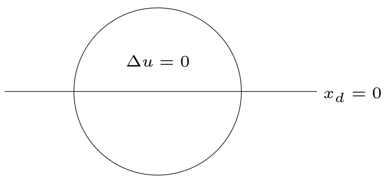
Lemma 1.4. *Any $u \in C^2(\Omega)$ which solves $\Delta u = 0$ in Ω is actually $C^\infty(\Omega)$.*

In fact, one can show something stronger:

Lemma 1.5. *Any u like in the previous Lemma is actually analytic in Ω.*

The proof is an exercise. It only requires suitably extending the formula (13) into a complex neighborhood. Fixing a ξ it is not hard to see that (13) defines a complex analytic function one the disc $|z-\xi| < \delta$ for suitable small δ. Alternatively, you can construct the power series for u directly and show it converges using the estimates obtained from the mean value theorem (see the exercises below).

The two previous Lemmata give yet another proof of the fact that the Cauchy-problem for the Laplace equation cannot be solved in general. Assume that you specified Cauchy data for u on $x_d = 0$ in $\mathbb{R}^d$



and that you were able to solve Laplace's equation in a small ball around the origin. By the Lemmas, you know that the solution has to be C^∞ (in fact, analytic) inside that ball. But this implies that your Cauchy-data would have to have been analytic in the first place! It follows that you cannot solve the Cauchy problem for Laplace's equation except if the data are analytic. This is very similar to the equation $u_t + iu_x = 0$ we discussed previously!

1.4 Mean value formulas

Let us return once more to the formula (9). It is clear that the function

$$G(x,\xi) = K(x,\xi) + w(x) \tag{14}$$

for $w \in C^2\left(\bar{\Omega}\right)$ with $\Delta w = 0$ in Ω is another fundamental solution with pole ξ and that the formula

$$u(\xi) = \int_\Omega G(x,\xi)\,\Delta u dx - \int_{\partial\Omega}\left[G(x,\xi)\frac{du}{dn_x}(x) - u(x)\frac{dG(x,\xi)}{dn_x}\right]dS_x \tag{15}$$

is valid. (Simply observe that the right hand side is zero if G is replaced by the harmonic $w \in C^2\left(\bar{\Omega}\right)$ in view of (3).)

For instance, taking for Ω a ball $B(\xi,\rho)$ centered around ξ and

$$G(x,\xi) = K(x,\xi) - \psi(\rho) = \psi(|x-\xi|) - \psi(\rho)\ , \tag{16}$$

we have that $G(x,\xi) = 0$ for $x \in \partial\Omega$ and, in view of $\frac{dG}{dn_x}|_{\partial\Omega} = \frac{1}{\omega_d}\rho^{1-d}$, the identity

$$u(\xi) = \int_{|x-\xi|<\rho}\left(\psi(|x-\xi|) - \psi(\rho)\right)\Delta u(x)\,dx + \frac{1}{\omega_d\rho^{d-1}}\int_{|x-\xi|=\rho} u(x)\,dS_x \tag{17}$$

expressing the values of u at the center of a ball in terms of its values on the boundary and a volume integral involving Δu (which is typically prescribed). Noting that $\psi(|x-\xi|) - \psi(\rho) < 0$ (recall $\psi(|x-\xi|)$ is monotone and gets large and negative for small arguments), we immediately produce

Lemma 1.6. *If $u \in C^2$ in $\overline{B(\xi,\rho)}$ and $\Delta u \geq 0$ in the ball, then*

$$u(\xi) \leq \frac{1}{\omega_d\rho^{d-1}}\int_{|x-\xi|=\rho} u(x)\,dS_x\,. \tag{18}$$

Note that for harmonic functions this is the well-known mean value property: The value of a harmonic function at a point equals its average value over a sphere surrounding that point.

Definition 1.7. *We call u subharmonic in Ω if (18) holds for any point $\xi \in \Omega$ and sufficiently small balls $B(\xi,\rho)$.*

By Lemma 1.6, functions with $\Delta u \geq 0$ in Ω are subharmonic. Conversely, a subharmonic function which is also C^2 satisfies $\Delta u \geq 0$. This follows directly from (17). It particular, a C^2 function which has the mean value property is harmonic.[2]

1.5 Poisson's formula

The goal of this subsection is to prove Poisson's formula,

$$u(\xi) = \Delta_\xi \int_\Omega K(x,\xi)\, u(x)\, dx \tag{19}$$

valid for $u \in C^2\left(\bar{\Omega}\right)$ and $\xi \in \Omega$. The "quick and dirty" prove of (19) would be to pull the Laplacian through the integral and use that $\Delta_\xi K = \delta_\xi$.

To do it carefully, we first let $u \in C_0^2(\Omega)$. Then formula (9) reduces to

$$u(\xi) = \int_\Omega K(x,\xi)\, \Delta_x u(x)\; dx = \int_{\mathbb{R}^d} K(x,\xi)\, \Delta_x u(x)\; dx \tag{20}$$

since the boundary terms vanish by the assumption of compact support. The domain can be increased to all of $\mathbb{R}^d$ for the same reason. Then

$$\begin{aligned}\int_{\mathbb{R}^d} K(x,\xi)\, \Delta_x u(x)\, dx = \int_{\mathbb{R}^d} \psi(|x-\xi|)\, \Delta_x u(x)\, dx &= \int_{\mathbb{R}^d} \psi(|y|)\, \Delta_y u(y+\xi)\, dy \\ &= \int_{\mathbb{R}^d} \psi(|y|)\, \Delta_\xi u(y+\xi)\, dy = \Delta_\xi \int_{\mathbb{R}^d} \psi(|y|)\, u(y+\xi)\, dy \\ &= \Delta_\xi \int_{\mathbb{R}^d} \psi(|x-\xi|)\, u(x)\, dx = \Delta_\xi \int_\Omega K(x,\xi)\, u(x)\, dx\,,\end{aligned}$$

which proves the desired formula for $u \in C_0^2(\Omega)$ and any $\xi \in \Omega$. Now for a given ξ, a general $u \in C^2\left(\bar{\Omega}\right)$ can be decomposed into a part which is supported away from the boundary and a part which is supported away from a small ball around ξ. More precisely, let $\xi \in \Omega$ be given and choose a small ball $b \subset \Omega$ around ξ. Choose also a larger ball B such that $\bar{b} \subset B \subset \Omega$ and a cut-off function $\chi \in C_0^2(\Omega)$ which is equal to 1 in B. Then $u \in C^2\left(\bar{\Omega}\right)$ can be written as

$$u = u_1 + u_2 = \chi \cdot u + (1-\chi)\, u$$

with $u_1 \in C_0^2(\Omega)$ and u_2 supported away from B. For u_2 we have

$$\Delta_\xi \int_\Omega K(x,\xi)\, u_2(x)\, dx = \Delta_\xi \int_{\Omega\setminus B} K(x,\xi)\, u_2(x)\, dx = 0$$

since $K(x,\xi)$ is in $C^2\left(\overline{\Omega \setminus B}\right)$ and harmonic for any $\xi \in b$. This concludes the proof.

[2]This remains true assuming only that u is C^0.

The relevance of Poisson's formula is that for a given $u \in C^2\left(\bar{\Omega}\right)$ (for instance, a charge distribution), the expression

$$w(\xi) = \int_\Omega K(x,\xi)\, u(x)\, dx$$

provides a special solution to Poisson's equation $\Delta_\xi w = u$ (whose negative gradient is the electric field). We can in fact consider the above $w(x)$ as a function on all of $\mathbb{R}^d$. We know that w is C^2 inside Ω and[3] satisfies Poisson's equation. It's not hard to see that w is harmonic (hence analytic) at points $\xi \notin \bar{\Omega}$, and that for points on $\partial\Omega$, w may fail to be C^2.

1.6 Maximum Principles

The previous considerations relied heavily on the exact form of the Laplacian. The ideas considered in this section, on the other hand, carry over to more general (even non-linear!) elliptic equations. More about this in the exercises.

Proposition 1.8. *(Weak maximum principle) Let $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ and $\Delta u \geq 0$ in Ω. Then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

Proof. Since $u \in C^0\left(\bar{\Omega}\right)$ the expressions are well-defined. For $\Delta u > 0$ the conclusion would follow immediately since at an interior maximum we would need both $\partial_x^2 u \leq 0$ and $\partial_y^2 u \leq 0$ and hence $\Delta u \leq 0$. For the general case, we consider the auxiliary function

$$v(x) = u(x) + \epsilon |x|^2$$

which satisfies $\Delta v = \Delta u + 2\epsilon n > 0$ in Ω for any $\epsilon > 0$. Therefore

$$\max_{\bar{\Omega}} \left(u + \epsilon |x|^2\right) = \max_{\partial\Omega} \left(u + \epsilon |x|^2\right)$$

$$\max_{\bar{\Omega}} u + \epsilon \min_{\bar{\Omega}} |x|^2 \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} |x|^2$$

and since this holds for any $\epsilon > 0$, $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$. Since the reverse inequality is trivial, we obtain the desired equality. $\square$

For u being *harmonic*, the same argument leads to the equality for the minimum and in view of $|u| = \max(u, -u)$ we also have

Corollary 1.9. *Let $u \in C^2(\Omega) \cap C^0\left(\bar{\Omega}\right)$ and $\Delta u = 0$ in Ω. Then*

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$$

[3] This is clear for u supported away from ξ. On the other hand, for u of compact support away from the boundary $\partial\Omega$ one has, similar to the above computation, $w(\xi) = \int_\Omega K(x,\xi)\, u(x)\, dx = \int_{\mathbb{R}^d} \psi(|x-\xi|)\, u(x)\, dx = \int_{\mathbb{R}^d} \psi(|y|)\, u(y+\xi)\, dy$. Since ψ is in L^1_{loc}, w can be differentiated (at least) as many times as u.

Corollary 1.10. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $\Delta u = 0$. Then $u = 0$ on $\partial\Omega$ implies $u = 0$ in all of Ω.*

Corollary 1.10 improves our uniqueness statement about the Dirichlet problem for Poisson's equation (6) from Section 1.1, which required $u \in C^2(\bar{\Omega})$. Now we know there can only be one solution to this problem in $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

Proposition 1.8 still allows the maximum of u to be attained in the interior (of course *in addition* to it being attained on the boundary). The following "strong" maximum principle asserts that this can only happen if u is actually constant:

Proposition 1.11. *Let $u \in C^2(\Omega)$ and $\Delta u \geq 0$ in Ω. Then either u is constant or*

$$u(\xi) < \sup_{\Omega} u$$

for all $\xi \in \Omega$.

Corollary 1.12. *If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is non-constant, then*

$$u(\xi) < \max_{\partial\Omega} u$$

Proof. The continuity implies $\sup_{\Omega} u = \max_{\bar{\Omega}} u = \max_{\partial\Omega} u$, the last step being Proposition 1.8. □

Proof of Proposition 1.11. Set $M = \sup u$ and decompose $\Omega = \Omega_1 \cup \Omega_2$ as

$$\Omega_1 = \{\, \xi \in \Omega \mid u(\xi) = M \} \quad \text{and} \quad \Omega_2 = \{\, \xi \in \Omega \mid u(\xi) < M \} \tag{21}$$

These sets are obviously disjoint. If we can show they are both open, then one of them must be empty (a connected set cannot be written as the union of two open sets, by definition). Now Ω_2 is open by the continuity of u in Ω. To see that Ω_1 is open, fix $\xi \in \Omega_1$ and consider the mean value inequality (18) in a small ball around ξ contained in Ω

$$0 \leq \int_{|x-\xi|=\rho} u(x)\, dS_x - \omega_d \rho^{n-1} u(\xi) = \int_{|x-\xi|=\rho} [u(x) - u(\xi)]\, dS_x$$

Since $u(\xi) = M$ and $u(x) \leq M$ for all x, the integrand is non-positive. This is only consistent if $u(x) = M$ holds in the ball, establishing that Ω_1 is open. □

1.7 Existence of solutions: Green's function for a ball

So far we have been talking about the uniqueness of solutions and about their properties but not about whether they actually exist! Let us return to equation (15) which we know holds for any $u \in C^2(\bar{\Omega})$. The claim is the following: If we would be able to find a $G(x,\xi)$ (and hence a $w(x)$) such that $G(x,\xi) = 0$ holds for all of $x \in \partial\Omega$ (independently of $\xi \in \Omega$), then we would have a formal solution of the Dirichlet problem (6) expressed purely in terms of the "data" f and g.

We say formal because formula (15) hinges on $u \in C^2(\bar{\Omega})$ and it is not clear a-priori whether a solution of (6) has to be C^2 up to the boundary. However, once one has found the desired $G(x,\xi)$, one can directly check whether the $u(\xi)$ thus defined satisfies (6) and what regularity properties it has.

We will carry out this program for Ω being a ball of radius a, which wlog (translation invariance) we put at the origin. Given a $\xi \in B(0,a)$ we define it's dual point (cf. Exercise 1.1) to be

$$\xi^\star = \frac{a^2}{|\xi|^2}\xi$$

The point of this definition is that the quotient $\frac{r^\star}{r}$ of $r = |x-\xi|$, the distance from a point x to ξ, and $r^\star = |x - \xi^\star|$, the distance from a point x to $\xi^\star$, is constant for x being any point on the boundary of the ball $S(0,a)$:

$$\frac{r^\star}{r} = \frac{a}{|\xi|}$$

for $x \in \partial\Omega$. This is verified by a quick computation. It follows that, for $d > 2$, if we define

$$G(x,\xi) = K(x,\xi) - \left(\frac{|\xi|}{a}\right)^{2-d} K(x,\xi^\star) \ ,$$

then $G(x,\xi) = 0$ for $x \in \partial\Omega$. Moreover, for any *fixed* $\xi \in \Omega$, we have $\xi^\star \notin \bar{\Omega}$ and hence $K(x,\xi^\star)$ is both $C^2(\bar{\Omega})$ in x and harmonic (recall this is required of the w in (14)!) Note also that for $\xi \to 0$ we recover the G used in the mean value formula (16).

Plugging this into (15) we obtain the formula

$$u(\xi) = \int_{|x|=a} H(x,\xi)\, g dS_x \qquad \text{where } H(x,\xi) = \frac{1}{a\omega_d}\frac{a^2 - |\xi|^2}{|x-\xi|^d} \tag{22}$$

valid for $|\xi| < a$ as a natural candidate to solve the Dirichlet problem "$\Delta u = 0$ in $B(0,a)$ and $u = g$ on $S(0,a)$". (In fact, we know that if $u \in C^2(\bar{\Omega})$ then this *has to be* the solution.) The following Proposition checks this directly.

Proposition 1.13. *Let g be continuous for $|x| = a$. Then the function*

$$u(\xi) = \begin{cases} g(\xi) & \text{for } |\xi| = a \\ \int_{|x|=a} H(x,\xi)\, g(x)\, dS_x & \text{for } |\xi| < a \end{cases} \tag{23}$$

is continuous in the closed ball $\overline{B(0,a)}$ and C^∞ and harmonic in the inside.

By our improved uniqueness theorem (Corollary 1.10) we know this is the unique solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$ of the desired Dirichlet problem.

Proof. We first check the following properties of the Poission kernel $H(x,\xi)$:

- $H(x,\xi) \in C^\infty$ for $|x| \leq a$, $|\xi| < a$, $x \neq \xi$

- $\Delta_\xi H(x,\xi) = 0$ for $|\xi| < a$, $|x| = a$
- $\int_{|x|=a} H(x,\xi)\, dS_x = 1$ for $|\xi| < a$ (use (23) with $g = 1$. (This problem has the unique solution $u = 1$, which is $C^2\left(\bar{\Omega}\right)$ and harmonic, hence the formula is valid.))
- $H(x,\xi) > 0$ for $|x| = a$, $|\xi| < a$.
- If $|y| = a$, then

$$\lim_{\xi \to y, |\xi| < a} H(x,\xi) = 0 \tag{24}$$

  uniformly in x for $|x-y| > \delta > 0$.

The first property justifies interchanging derivatives in ξ with the integral in the expression for u and hence yields that $u \in C^\infty$ in Ω. Doing this interchange for the Laplacian, the second property shows that u is harmonic in Ω.

The difficult part is to establish the continuity at the boundary. We fix a y with $|y| = a$. We need to show that $\lim_{\xi \to y} u(\xi) = g(y)$ that is

$$\lim_{\xi \to y} \int_{|x|=a} H(x,\xi)\left[g(x) - g(y)\right] dS_x = 0\,, \tag{25}$$

where we have used the third property of H above for the reformulation. Now g is continuous on the boundary at y. Hence, given $\epsilon > 0$ we can fix a $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ holds for all x with $|x-y| < \delta$. Therefore,

$$\begin{aligned} &\int_{|x|=a, |x-y|<\delta} H(x,\xi)\left[g(x) - g(y)\right] dS_x \\ &\leq \sup_{|x-y|<\delta} |g(x) - g(y)| \int_{|x|=a} H(x,\xi)\, dS_x \leq \epsilon \end{aligned} \tag{26}$$

Note that here we have used the fourth (positivity) property of H above.

On the other hand, using the uniformity in the last property of H above, we can find now a δ' such that with $M = \max_{\partial\Omega} g > 0$ (otherwise we know the unique solution is $u = 0$).

$$H(x,\xi) < \epsilon \cdot \frac{1}{2M\omega_d a^{d-1}} \quad \text{holds for all } |\xi - y| < \delta' \text{ and all } |x-y| > \delta.$$

Therefore,

$$\int_{|x|=a, |x-y|>\delta} H(x,\xi)\left[g(x) - g(y)\right] dS_x < \epsilon \quad \text{holds for} \quad |\xi - y| < \delta' \tag{27}$$

Adding (26) and (27) we have shown that given any ϵ we can find a δ' such that the integral in (25) is smaller than 2ϵ for all $|\xi - y| < \delta'$, which proves the continuity. □

We have successfully solved the Dirichlet problem for harmonic functions, giving existence and uniqueness. It is also not hard to see the continuous dependence of u on g from the explicit formulae we derived.

1.8 Applications

The fact that we can solve explicitly the Dirichlet problem for a ball is very useful. If we differentiate u at the center of the ball, a computation yields

$$\partial_{\xi_i} u\,(0) = \frac{d}{\omega_d a^{n+1}} \int_{|x|=a} x_i u\,(x)\, dS_x$$

expressing the derivative of u at the center in terms of the value of u on the boundary. Therefore, we have the estimate

$$|\partial_{\xi_i} u\,(0)\,| \leq \frac{d}{a} \max_{|x|=a} |u\,(x)\,|$$

and similarly for higher derivatives. Now for an open region $\Omega \subset \mathbb{R}^d$ and a point $\xi \in \Omega$, we can always find a small ball around ξ which is contained in Ω. By the property above, translated to ξ we have

$$|\partial_{\xi_i} u\,(\xi)\,| \leq \frac{d}{dist\,(\xi, \partial\Omega)} \sup_{\Omega} |u\,(x)\,| \tag{28}$$

Consequently, on any compact subset of Ω we will obtain uniform bounds for all derivatives up to a fixed order.

Completeness and compactness properties of harmonic functions

Suppose you have a sequence $u_k \in C^2\,(\Omega) \cap C^0\left(\bar{\Omega}\right)$ which is harmonic in Ω. Then if the u_k converge uniformly to f on the boundary, they converge uniformly to a continuous function u on the entire disk by the maximum principle. In any compact subset of Ω, all derivatives up to a fixed order converge uniformly. It follows that $u \in C^\infty\,(\Omega)$ and harmonic in Ω.

Exercise 1.14. *Suppose you only knew $|u_k| \leq C$ on the boundary for $u_k \in C^2\,(\Omega) \cap C^0\left(\bar{\Omega}\right)$ a sequence of harmonic functions in Ω. Can you extract a subsequence which converges? Hint: Arzela-Ascoli.*

2 Exercises

1. At various stages we have used the interchange of differentiation and integration. The following theorems cover these situations.

   (a) Suppose that $I \subset \mathbb{R}$ is open and that $f : I \times \Omega \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$ with $\Omega \subset \mathbb{R}^n$ is a function which is in $L^1(\Omega)$ for all t, differentiable in t for all x with the derivative satisfying $|\frac{\partial}{\partial t} f(t, x)| \leq g(x)$ for all (t, x) with g a function in $L^1(\Omega)$. Then

   $$\frac{d}{dt}\int_\Omega f(t, x)\, dx = \int_\Omega \frac{\partial f}{\partial t}(t, x)\, dx < \infty\,.$$

   Hint: Use the dominant convergence theorem.

   (b) Let $\mathcal{U} \subset \mathbb{R}^n$ be open and $\Omega \subset \mathbb{R}^k$ be compact. Let $f : \mathcal{U} \times \Omega \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ be continuous with continuous partial derivatives $\partial_{x_i} f : \mathcal{U} \times \Omega \to \mathbb{R}$. Then $F : \mathcal{U} \to \mathbb{R}$, $x \mapsto \int_\Omega f(x, y)dy$ is continuously differentiable in x_i and

   $$\frac{\partial}{\partial x_i}\int_\Omega f(x, y)dy = \int_\Omega \frac{\partial}{\partial x_i} f(x, y)dy$$

   for all $x \in \mathcal{U}$.
   Hint: Uniform continuity and exchange of limit and integral.

2. Let $\Delta u = 0$ for $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $g \geq 0$ on $\partial\Omega$. If $g > 0$ at *one point* of $\partial\Omega$, then $u > 0$ in Ω.

3. (Fritz John, 4.2 (1)) Let Ω denote the unbounded set $|x| > 1$. Let $u \in C^2(\bar{\Omega})$, $\Delta u = 0$ in Ω and $\lim_{|x|\to\infty} u(x) = 0$. Show that

   $$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|\,.$$

4. (Fritz John, 4.2 (2)) Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of

   $$\Delta u + \sum_{k=1}^{n} a_k(x)\, u_{x_k} + c(x)\, u = 0$$

   where $c(x) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω.

5. (Fritz John, 4.2 (3)) Prove the weak maximum principle for solutions of the two-dimensional elliptic equation

   $$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y = 0$$

   where a, b, c, d, e are continuous functions of x and y in $\bar{\Omega}$ and $ac - b^2 > 0$ (ellipticity) as well as $a > 0$ hold. HINT: Prove it first under the strict condition $Lu > 0$, then use $u + \epsilon v$ for $v = \exp\left[M(x - x_0)^2 + M(y - y_0)^2\right]$ with appropriate M, x_0, y_0.

6. (Harnack's inequality) Let $u \in C^2$ for $|x| < a$ and $u \in C^0$ for $|x| \le a$. Let also $u \ge 0$ and $\Delta u = 0$ hold for $|x| < a$ (in other words, u is a *non-negative* harmonic function). Show that for $|\xi| < a$

$$\frac{a^{n-2}\left(a-|\xi|\right)}{\left(a+|\xi|\right)^{n-1}} u\left(0\right) \le u\left(\xi\right) \le \frac{a^{n-2}\left(a+|\xi|\right)}{\left(a-|\xi|\right)^{n-1}} u\left(0\right)$$

Discuss the meaning of this estimate. What can you say for arbitrary regions?

7. *(Analyticity of harmonic functions.) In lectures we saw an outline on how to show that harmonic functions are analytic by extending the formula involving the fundamental solution to a* complex *analytic function. Here we give an alternative argument.*

   Let $u \in C^2\left(\Omega\right)$ be harmonic in $\Omega \subset \mathbb{R}^d$ (bounded, connected, open, as in lectures). In lectures, we proved that u is smooth in Ω and also that for any closed ball $\overline{B}(\xi, a) \subset \Omega$ we have the estimate

$$|\partial_{\xi_i} u\left(\xi\right)| \le \frac{d}{a} \max_{\overline{B}(\xi,a)} |u\left(x\right)|. \tag{29}$$

   (a) Let $\overline{B}(\xi, a) \subset \Omega$ and $|u(x)| \le M$ hold in $\overline{B}(\xi, a)$. Use (29) to prove that
$$|D^\alpha u(\xi)| \le \left(\frac{m}{a} d\right)^m M \quad \text{for } |\alpha| = m.$$
   Hint: Apply (29) successively to the k^{th} derivatives in the balls $|x - \xi| \le a\frac{m-k}{m}$ for $k = 0, 1, ..., m-1$.

   (b) Prove that u is analytic in Ω. Hint: Note $m^m \le m! e^m$.