Partial Differential Equations (Week 8+9)
Schrödinger and Heat Equation

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This week we begin with the study *evolution* equations. We start with a discussion of the Schroedinger equation $\partial_t u = i\Delta u$, followed by the heat equation $\partial_t u = \Delta u$. The wave equation, $\partial_t^2 u = \Delta u$, which is in fact the first PDE to be studied (by d'Alembert) will concern us in the last three weeks of the course along with some non-linear applications.

1 Preliminaries

The idea of this first section is to collect basic results about the Fourier transform and tempered distributions that will be needed in the analysis of the Schroedinger equation. Once we have those tools we can also prove the Sobolev embedding theorem and Rellich's theorem (which we used earlier in the analysis of elliptic equations).

1.1 Schwartz space

It will be convenient to work with the Schwartz space

$$\mathcal{S}\left(\mathbb{R}^d\right) = \{u \in C^\infty\left(\mathbb{R}^d\right) \ : \ \forall \alpha, \beta \in \mathbb{N}^d \ , \ \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta u\left(x\right)| < \infty\} \tag{1}$$

equipped with the countable family of semi-norms

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta u\left(x\right)|\,.$$

We recall the following facts:

- We say $g_n \to g$ in $\mathcal{S}$ if for all α, β we have $\|g_n - g\|_{\alpha,\beta} \to 0$
- We can define a metric on $\mathcal{S}$ inducing by defining

$$\rho\left(g, f\right) = \sum_{\alpha,\beta} 2^{-|\alpha|-|\beta|} \frac{\|g-f\|_{\alpha,\beta}}{1+\|g-f\|_{\alpha,\beta}}$$

  It is easy to see that $g_n \to g$ in $\mathcal{S}$ is equivalent to $\rho\left(g_n, g\right) \to 0$.

- $\mathcal{S}\left(\mathbb{R}^d\right)$ equipped with ρ is a complete metric space (Exercise).
- The functions of compact support C_0^∞ are dense it $\mathcal{S}\left(\mathbb{R}^d\right)$.

1.2 The Fourier transform $\mathcal{F}$

Definition 1.1. *For $u \in \mathcal{S}\left(\mathbb{R}^d\right)$ we define*

$$\mathcal{F}u\left(\xi\right) = \hat{u}\left(\xi\right) := \frac{1}{\left(2\pi\right)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u\left(x\right) dx$$

where $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$.

Exercise 1.2. *Show that $u\left(x\right) = e^{-\frac{x^2}{2}}$ has Fourier transform $\mathcal{F}u\left(\xi\right) = e^{-\frac{\xi^2}{2}}$.*

Using integration by parts it is not hard to see

$$\left(-i\partial_\xi\right)^\alpha \hat{u} = \mathcal{F}\left((-x)^\alpha u\right) \quad \textit{and} \quad \mathcal{F}\left(\left(-i\partial_x\right)^\alpha u\right) = \xi^\alpha \mathcal{F}\left(u\right) .$$

This already gives a hint how useful the Fourier transform can be in solving *constant coefficient* linear PDEs. Indeed, if

$$L = \sum_{|\alpha|\leq n} a_\alpha \partial_x^\alpha$$

then taking the Fourier transform of $Lu = f$ (assuming both u and f are in the Schwartz-space)

$$P\left(\xi\right)\hat{u}\left(\xi\right) = \hat{f}\left(\xi\right) \qquad \text{with } P\left(\xi\right) = \sum_{|\alpha|\leq n} a_\alpha \left(i\xi\right)^\alpha$$

The $P\left(\xi\right)$ is called the characteristic polynomial of the operator L. This leads to the topic of Fourier integral operators and singular integrals.

Exercise 1.3. *Show that $\mathcal{F}$ is a continuous map from $\mathcal{S}\left(\mathbb{R}^d\right)$ to itself.*

Exercise 1.4. *Show that*

- $\mathcal{F}\left(u\left(x-h\right)\right) = \exp\left(-ih\xi\right)\hat{u}\left(\xi\right)$
- $\mathcal{F}\left(u\left(\lambda x\right)\right)\left(\xi\right) = |\lambda|^{-d}\left(\mathcal{F}u\right)\left(\frac{\xi}{\lambda}\right)$

To find the Fourier inversion formula we observe that with

$$< f, g > := \int fg dx \tag{2}$$

denoting the pairing of two Schwarz functions we have

$$< \mathcal{F}\phi, \psi > = < \phi, \mathcal{F}\psi > \tag{3}$$

which follows from Fubini and the symmetry of the Fourier kernel $(2\pi)^{-d/2}\exp(-ix\xi)$. Similarly, for the L^2 inner-product

$$(\phi,\psi)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \phi(x)\,\overline{\psi(x)}\,dx$$

we have

$$(\mathcal{F}\phi,\psi) = (\phi,\mathcal{F}^\star\psi) \tag{4}$$

where $\mathcal{F}^\star$ has a + in the exponent instead of a minus. The operator $\mathcal{F}^\star$ is called the inverse Fourier transform in view of the following

Proposition 1.5. *The map*

$$\mathcal{F} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right)$$

is a bijection. The inverse is $\mathcal{F}^\star$ with the inversion formula

$$u(x) = \mathcal{F}^\star\hat{u}(x) = (2\pi)^{-d/2}\int e^{+ix\cdot\xi}\hat{u}(\xi)\,d\xi \tag{5}$$

1.3 Extension of $\mathcal{F}$ to L^p

The basic idea is

Proposition 1.6. *Let X,Y be Banach spaces and $E\subset X$ a dense linear subspace of X. If $T:E\to Y$ is continuous, i.e.*

$$\|Te\|_Y \le c\|e\|_X$$

holds for some $c>0$ for all $e\in E$. Then T has a unique extension $T_{ext}:X\to Y$ with $T_{ext}\Big|_E = T$ satisfying $\|Tx\|_Y \le c\|x\|_X$ for all $x\in X$.

To extend the Fourier transform, the following two estimates are key, which hold for $f\in\mathcal{S}\left(\mathbb{R}^d\right)$.

$$\|\mathcal{F}f\|_{L^\infty} \le (2\pi)^{-d/2}\|f\|_{L^1} \tag{6}$$

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} = \|\mathcal{F}^\star f\|_{L^2} \tag{7}$$

The first is almost immediate from the definition and the second follows from (4) and $\mathcal{F}\mathcal{F}^\star = id = \mathcal{F}^\star\mathcal{F}$.

It is then not hard to show that the Fourier transform extends to a map from L^1 to L^∞ (Riemann Lebesgue Lemma) and to a unitary map from L^2 to L^2 (Plancherel). By interpolation between the two it can be extended to a map from L^p to L^q for $1\le p\le 2$ and $\frac{1}{p}+\frac{1}{q}=1$ (Hausdorff-Young).

1.4 Tempered Distributions and Extension of $\mathcal{F}$ to $\mathcal{S}'(\mathcal{R}^n)$

A tempered distribution is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$, which is obviously a larger space of test functions than $\mathcal{D}(\mathbb{R}^d)$ considered before. The space of tempered distributions is denoted $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$. When is a distribution in $\mathcal{D}'(\mathbb{R}^d)$ tempered? Well, if it extends to a continuous linear map $\mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. If it does, the extension is unique as $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

The notions of continuity, convergence etc are defined in analogy to what we did for $\mathcal{D}'$, cf. Week . Moreover, the operators ∂^α, τ_h (translation), σ_λ (scaling), the Fourier transform and convolution (denoted collectively by L) can be extended to act on elements of $\mathcal{S}'$ by the usual integration by parts formula:

$$< LT, \phi > := < T, L'\phi > \quad \text{for all } T \in \mathcal{S}' \text{ and } \phi \in \mathcal{S}$$

where L' is the transpose of T. In particular, for the Fourier transform we have

$$< \mathcal{F}T, \phi > := < T, \mathcal{F}(\phi) >$$

So $\mathcal{F}$ lifts to a map from $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Example 1.7. *The (distributional) Fourier transform of the δ-distribution is the constant $(2\pi)^{n/2}$:*

$$< \mathcal{F}\delta, \phi > = < \delta, \hat{\phi} > = \hat{\phi}(0) = \int \frac{1}{(2\pi)^{-n/2}} \phi(x)\, dx$$

Let us look at a special case of tempered distributions. Suppose we have a $u : \mathbb{R}^n \to \mathbb{C}$ measurable and such that $\left(1 + |x|^2\right)^{n/2}$ is in $L^2(\mathbb{R}^n, \mathbb{C})$ (note this implies in particular $u \in L^1_{loc}(\mathbb{R}^n)$). Then we can define an associated tempered distribution $U \in \mathcal{S}'(\mathbb{R}^n)$ by the formula

$$U(\phi) = \int_{\mathbb{R}^n} \phi(x)\, u(x)\, dx$$

This representation of U is unique (why?) and we will identify U with u.

If we specialise further to $u \in \mathcal{S}(\mathbb{R}^n)$ then there are now two interpretations of the Fourier transform of u, the old $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$ and the distributional one. It is easy to check they are compatible in that

$$< \hat{u}, \phi > = < u, \hat{\phi} > = \int_{\mathbb{R}^n} u(k)\, \hat{\phi}(k)\, dk = \int_{\mathbb{R}^n} \hat{u}(k)\, \phi(k)\, dk$$

where (3) has been used.

We now define the Fourier-version of the Sobolev spaces defined earlier

Definition 1.8. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $s \in \mathbb{R}$. We say that $u \in H_{(s)}(\mathbb{R}^n)$ if $\hat{u}$ is measurable and $\hat{u}(\xi)\left(1 + |\xi|^2\right)^{s/2}$ is in $L^2(\mathbb{R}^n)$. If $u \in H_{(s)}(\mathbb{R}^n)$ we define the norm*

$$\|u\|_{(s)} = \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |u(\xi)|^2 \left(1 + |\xi|^2\right)^s d\xi \right)^{\frac{1}{2}}$$

Note that $H_{(s)}(\mathbb{R}^n)$ is a complex Hilbert space (what is the inner product?). Note also that $H_{(s)}(\mathbb{R}^n) \subset H_{(t)}(\mathbb{R}^n)$ for $s \leq t$ and $H_{(0)}(\mathbb{R}^n) = L^2(\mathbb{R}^n, \mathbb{C})$.

Exercise 1.9. *Show that $\delta \in H^s(\mathbb{R}^n)$ iff $s < -\frac{n}{2}$ (see Example 1.7).*

There is a canonical way of relating the spaces $H_{(s)}$ and $H_{(t)}$ and it will help to show that the Schwartz functions are dense in $H_{(s)}$ (more generally, they are dense in $\mathcal{S}'$).

Definition 1.10. *Let $u \in H_{(s)}(\mathbb{R}^n)$ and t be a real number. We define*

$$(1-\Delta)^t u \in H_{(s-2t)}(\mathbb{R}^n) := \{\textit{temp. dist. whose FT is given by } \left(1+|\xi|^2\right)^t \hat{u}(\xi)\}$$

Note that

$$(1-\Delta)^t u \in H_{(s-2t)}(\mathbb{R}^n)$$

and

$$\|(1-\Delta)^t u\|_{(s-2t)} = \|u\|_{(s)} \tag{8}$$

from which we conclude that $(1-\Delta)^t$ is a bounded linear map from $H_{(s)}$ to $H_{(s-2t)}$ with bounded inverse $(1-\Delta)^{-t}$. In particular $H_{(s)}(\mathbb{R}^n)$ is the image of $L^2(\mathbb{R}^n)$ under the map $(1-\Delta)^{s/2}$. This implies that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_{(s)}(\mathbb{R}^n)$: Indeed, pick a $u \in H_{(s)}(\mathbb{R}^n)$. Take the unique $v \in L^2(\mathbb{R}^n)$ with $u = (1-\Delta)^{s/2} v$. The Schwarz space is dense in $L^2(\mathbb{R}^n)$ (as $C_0^\infty(\mathbb{R}^n)$ is) and we can hence approximate $v_n \to v$ in L^2 with v_n Schwartz. Then $(1-\Delta)^{s/2} v_n$ is also Schwartz and by the identity (8) converges to $u \in H_{(s)}(\mathbb{R}^n)$.

Exercise 1.11. *Check that if u is Schwartz and k is an integer, then the "usual" definition of $(1-\Delta)^k u$ is compatible with the Fourier definition given above.*

Exercise 1.12. *Show that $H_{(-s)}$ is the dual of $H_{(s)}$ in that for every bounded linear functional f on $H_{(s)}$ there is a $\phi \in H_{(-s)}$ with*

$$f(u) = \int_{\mathbb{R}^n} u\phi$$

Solution: Given f, the functional $g(v) = f\left((1-\Delta)^{-\frac{s}{2}} v\right)$ for $v \in L^2$ defines a functional on $v \in L^2$. Since L^2 is dual to L^2 we find a $w \in L^2$ such that

$$g(v) = \int_{\mathbb{R}^n} v\overline{w} d\xi = \int_{\mathbb{R}^n} \hat{v}\overline{\hat{w}} d\xi \quad \textit{for all } v \in L^2$$

using (4) for the last equality. Therefore, for any $u \in H_{(s)}$ we can find $w \in L^2$ so that

$$f(u) = g\left((1-\Delta)^{\frac{s}{2}} u\right) = \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}\overline{\hat{w}} d\xi = \int_{\mathbb{R}^n} \hat{u}\overline{(1+|\xi|^2)^{\frac{s}{2}} \hat{w}} d\xi$$

Defining $W = (1-\Delta)^{\frac{s}{2}} w$ which is in $H_{(-s)}$, f is represented as $f(u) = \int_{\mathbb{R}^n} uW d\xi$ by Parseval.

Finally, let us relate the definition of the $H_{(k)}(\mathbb{R}^n)$-spaces to the $H^k(\mathbb{R}^n)$-spaces defined earlier when k is a non-negative integer.

From Parseval we have for ϕ, ψ Schwartz

$$\|\phi\|_{H^k} = \sum_{|\alpha|<k} \int_{\mathbb{R}^n} \partial^\alpha \phi(x) \partial^\alpha \bar{\phi}(x) \, dx = \sum_{|\alpha|<k} \int_{\mathbb{R}^n} \xi^{2\alpha} \hat{\phi}(\xi) \hat{\phi}(\xi) \, d\xi$$

and since there are constants $c_{1,k}$, $c_{2,k}$ with

$$c_{1,k} \left(1+|\xi|^2\right)^k \le \sum_{|\alpha|<k} \xi^{2\alpha} \le c_{2,k} \left(1+|\xi|^2\right)^k$$

we find that the norms are equivalent on a dense subspace ($\mathcal{S}$ being dense in both H^k and $H_{(k)}$) hence we can identify an elements of H^k and $H_{(k)}$ and vice versa.

1.5 Sobolev embedding

We can finally prove the Sobolev embedding theorem mentioned earlier.

Theorem 1.13. *Let k be a non-negative integer and $s > \frac{k}{+}\frac{n}{2}$. Then there exists a constant C depending on k, n and s such that for all $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \le C\|f\|_{(s)} \, . \tag{9}$$

Here $C_b^k(\mathbb{R}^n, \mathbb{C})$ is the Banach space of C^k functions whose derivatives up to order k are bounded equipped with the usual sup-norm (sum of the sup's of all derivatives). We are only going to prove the theorem for $k = 0$ and leave higher k as an exercise. We already note

Corollary 1.14. *Let $f \in H_{(s)}(\mathbb{R}^n)$ for $s > \frac{n}{2}$. Then after possibly redefining f on a set of measure zero, f is a continuous function.*

Proof. Approximate $\phi_k \to u$ in $H_{(s)}(\mathbb{R}^n)$ with $\phi_k \in \mathcal{S}(\mathbb{R}^n)$. By the estimate (9), the sequence ϕ_k is Cauchy in $C_b^0(\mathbb{R}^n, \mathbb{C})$ and hence converges pointwise to a continuous $\tilde{u}$. On the other hand, we can extract from $\phi_k \in H_{(s)}(\mathbb{R}^n)$ a subsequence which converges *pointwise almost everywhere* to u (why?[1]). Hence up to a set of measure zero $u = \tilde{u}$. □

Proof of Theorem 1.13 for $k = 0$.

$$\begin{aligned} |f(x)| &\le (2\pi)^{n/2} \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi \\ &= (2\pi)^{n/2} \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-s/2} \left(1+|\xi|^2\right)^{s/2} |\hat{f}(\xi)| d\xi \\ &\le (2\pi)^{n/2} \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-s} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{s} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\le C\|f\|_{(s)} \, . \end{aligned} \tag{10}$$

□

[1] Recall the proof of the completeness of L^2.

1.6 Rellich's theorem

We are now in a position to prove Rellich's theorem which was used before.

Theorem 1.15. *Let $s \in \mathbb{R}$, K a compact subset of $\mathbb{R}^n$ and $\{u_n\}$ be a sequence in $H_{(s)}(\mathbb{R}^n)$ which is*

- *uniformly bounded in $H_{(s)}(\mathbb{R}^n)$: $\|u_n\|_{(s)} \leq C$ for a constant C and all n.*
- *supported in K: supp $(u_n) \subset K$ for all n.*

Then for $t < s$ there is a subsequence $\{u_{n_k}\}$ which converges strongly in $H_{(t)}(\mathbb{R}^n)$.

Proof. Step 1: By Banach Alaoglu we first extract a subsequence u_{n_k} of u_n which converges *weakly* in $H_{(s)}$ to some $u \in H_{(s)}$, i.e.

$$< u_{n_k}, \phi >= \int dx \; u_{n_k}(x) \phi(x) \to \int dx \; u(x) \phi(x) =< u, \phi >$$

for all $\phi \in H_{(-s)}$.

Step 2: Next, choosing a φ, smooth, compactly supported in a neighbourhood of K and equal to 1 on K we have $\varphi u_{n_k} = u_{n_k}$ and

$$\mathcal{F} u_{n_k}(\xi) =< \varphi u_{n_k}, (2\pi)^{-n/2} e^{-ix\xi} >=< u_{n_k}, (2\pi)^{-n/2} \varphi e^{-ix\xi} >$$

Since for any ξ the function $(2\pi)^{-n/2} \varphi(x) e^{-ix\xi}$ is in $H_{(-s)}$ (simply check that $\|\varphi e^{-ix\xi}\|_{(-s)} \leq C_\xi$ holds for a constant depending on ξ) and u_{n_k} converges weakly to u we obtain

$$\hat{u}_{n_k}(\xi) \to \hat{u}(\xi) \quad \text{pointwise for every } \xi.$$

Step 3: We need to show that given $\epsilon > 0$ we can choose a k such that

$$\int_{\mathbb{R}^n} |\hat{u}_{n_k}(\xi) - \hat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^t d\xi < \epsilon \,,$$

as this is precisely the statement $u_{n_k} \to u$ strongly in $H_{(-t)}$. To achieve this, we fix R so large that

$$\begin{aligned} &\int_{|\xi| \geq R} |\hat{u}_{n_k}(\xi) - \hat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^t d\xi \\ \leq &\left(1 + |R|^2\right)^{t-s} \int_{|\xi| \geq R} |\hat{u}_{n_k}(\xi) - \hat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^s d\xi \\ &\leq 2\left(1 + |R|^2\right)^{t-s} \left(\|u_{n_k}\|^2_{(s)} + \|u\|^2_{(s)}\right) < \frac{\epsilon}{2} \,. \end{aligned} \tag{11}$$

This is possible because the weak limit is bounded in $H_{(s)}$. Note that the choice of R does not depend on n_k! Finally, we choose n_k so large that

$$\int_{|\xi| \leq R} |\hat{u}_{n_k}(\xi) - \hat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^t d\xi < \frac{\epsilon}{2} \,.$$

This is possible because as a consequence of the dominant convergence theorem the left hand side converges to zero as $n_k \to \infty$. □

2 The Schroedinger equation

The Schroedinger equation for $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ and $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$

$$u_t = i\Delta u \tag{12}$$

describes the evolution of the wave function of a free particle in quantum mechanics. More often you will see

$$u_t = i\Delta u + iV(x)\,u$$

describing the evolution in the presence of a potential. This is already a much harder problem for which (except for some very special potentials) no explicit solution can be obtained. Another popular topic are non-linear Schroedinger equations

$$u_t = i\Delta u + |u|^{p-1}u$$

which we will discuss in the next section. We already noted that

$$\partial_t \int_{\mathbb{R}^d} \|u\|^2 = 0 \tag{13}$$

is constant in time. Physically, this is interpreted as the conservation of probability and one normalizes $\int_{\mathbb{R}^d} \|u\|^2 = 1$.

Exercise 2.1. *Establish the conservation of energy:*

$$\partial_t \int_{\mathbb{R}^d} |\nabla u|^2 = 0$$

The above conservation laws suggest to pose and initial value problem with data on $t = 0$. However, we already know that $t = 0$ surfaces are characteristic. Hence, for instance, we cannot expect the Cauchy-Kovalevskaya theorem to be applicable here (cf. the problems of Week 2; for polynomial data, CK would actually work but such solutions are forbidden if we make the physically natural requirement that $\|u\|_{L^2} < \infty$!).

Nevertheless it turns out that $t = 0$ is natural to specify data $u(0,x) = u_0(x)$. In order to ensure uniqueness, we will need to impose suitable decay at infinity $|x| \to \infty$. Let us derive a simple solution formula assuming that $u(t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ is continuous in t with values in the Schwartz space and that $\frac{\partial}{\partial t}u(t,x)$ is also continuous function with values in the Schwartz space. The idea is to first do formal computations to derive a solution formula and then prove that the latter actually provides the correct solution.

Taking the Fourier transform in space of (12) we find

$$\hat{u}_t = -i|\xi|^2\hat{u} \quad \text{with data} \quad \hat{u}(t=0,\xi) = \hat{f}(\xi).$$

Solving this ODE produces

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{f}(\xi)$$

and hence the solution formula

$$u\left(t,x\right)=\left(2\pi\right)^{-d/2}\int e^{-it|\xi|^{2}}e^{ix\xi}\mathcal{F}f\left(\xi\right)d\xi\ . \tag{14}$$

Theorem 2.2. *For any $f\in\mathcal{S}\left(\mathbb{R}^{d}\right)$, there is a unique $u\in C^{\infty}\left(\mathbb{R},\mathcal{S}\left(\mathbb{R}^{d}\right)\right)$ satisfying (12). It is given by the formula (14).*

Proof. (Sketch. Consult the book of Rauch for more details.) It is clear that the u given by (14) is infinitely differentiable in both t and x. The differentiation under the integral sign is justified by the rapid decay of $\mathcal{F}f$ in ξ: All we are only going to see is a finite sum of terms of the form

$$polynomial(\xi)\exp\left(-it|\xi|^{2}\right)\exp\left(ix\xi\right)\mathcal{F}f\left(\xi\right)$$

under the integral and this expression is integrable by the rapid decay of $\mathcal{F}f\left(\xi\right)$ in ξ. This implies in particular that (12) holds. Next, for fixed t, $D^{\alpha}_{t,x}u$ is seen to be in the Schwartz space (simply check that its Fourier transform is Schwartz) so that the map $t\to u\left(t\right)$ is a map $\mathbb{R}\to\mathcal{S}\left(\mathbb{R}^{d}\right)$. To show that this map is continuous we need to show that

$$\sup_{x}|x^{\alpha}\partial_{x}^{\beta}\left(u\left(t+h,x\right)-u\left(t,x\right)\right)|$$

goes to zero as $h\to 0$. Plugging in the formula for u we see that

$$\sup_{x}|x^{\alpha}\partial_{x}^{\beta}\int\left(e^{-i(t+h)|\xi|^{2}}-e^{-it|\xi|^{2}}\right)e^{ix\xi}\mathcal{F}f\left(\xi\right)d\xi|$$

Pulling through the ∂_{x}^{β} and replacing $x^{\alpha}e^{ix\xi}$ by $\left(-i\partial_{\xi}\right)^{\alpha}e^{ix\xi}$ and integrating by parts we see that we obtain a finite sum of terms of the form

$$\left[p\left(t+h,\xi\right)e^{-i(t+h)|\xi|^{2}}-p\left(t,\xi\right)e^{-it|\xi|^{2}}\right]\partial_{\xi}^{\mu}\mathcal{F}\left(f\right)$$

By the mean value theorem we can bound the square bracket by $h\left(1+|\xi|+|t|\right)^{N}$ for some N. The rapid decay of $\partial_{\xi}^{\mu}\mathcal{F}\left(f\right)$ lets the integral converge and we can let $h\to 0$ to obtain the result. Higher regularity can be proven similarly.

Once one has that $u\in C^{1}\left(\mathbb{R},\mathcal{S}\left(\mathbb{R}^{d}\right)\right)$, the uniqueness follows by repeating the (previously purely formal) computation above that a solution of (12) with this regularity has to be represented by the formula (14). □

2.1 Generalized Solutions

The idea is to construct solutions to the initial value problem

$$\begin{cases} u_{t}=i\Delta u \\ u\left(0,\cdot\right)=f\left(\cdot\right)\in H_{(s)}\left(\mathbb{R}^{d}\right)\end{cases} \tag{15}$$

by approximation with data in $\mathcal{S}$. Obviously, the extension will only work if we have $H_{(s)}$ estimates for the solution. We let $S\left(t\right):\mathcal{S}\left(\mathbb{R}^{d}\right)\to\mathcal{S}\left(\mathbb{R}^{d}\right)$ be the map sending the data f to the solution $u\left(t\right)$.

Proposition 2.3. *For any $s \in \mathbb{R}, t \in \mathbb{R}$, $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ one has*

$$\|S(t) f\|_{(s)} = \|f\|_{(s)} \,. \tag{16}$$

Proof.

$$\|S(t) f\|_{(s)} = \|e^{-it|\xi|^2} \left(1+|\xi|^2\right)^{\frac{s}{2}} \mathcal{F}f\|_{L^2(\mathbb{R}^d)} = \|f\|_{(s)} \,.$$

□

Corollary 2.4. *For any $s \in \mathbb{R}$, $t \in \mathbb{R}$, the operator $S(t)$ extends uniquely to a unitary map of $H_{(s)}\left(\mathbb{R}^d\right)$ to itself. The extended operator satisfies*

$$(S(t) f) = \mathcal{F}^\star \exp\left(-it|\xi|^2\right) \mathcal{F}f \tag{17}$$

for any $f \in H_{(s)}\left(\mathbb{R}^d\right)$.

Proof. By Proposition 1.6 the extension is well-defined and unique and preserves the H^s norm. We have $S(t_1+t_2) = S(t_1) S(t_2)$ when acting on elements of $\mathcal{S}$ and since both sides are also continuous, the formula also extends to $H_{(s)}$. But then $S(-t)$ is the inverse of $S(t)$ proving that S is unitary. The formula follows from observing that both sides are bounded on $H_{(s)}$ and agree on the dense set $\mathcal{S}$. □

This already tells us that given $f \in H_{(s)}\left(\mathbb{R}^d\right)$ the solution $u(t)$ is bounded in $H_{(s)}$. Of course we would like this extended map to be continuous in time. This follows almost immediately from the estimate (16). Indeed, let $f_n \in \mathcal{S}$ be a Cauchy sequence approximating a given $f \in H_{(s)}$. Then by (16)

$$\sup_{t\in[-T,T]} \|u_m(t) - u_n(t)\|_{(s)} = \|f_m - f_n\|_{(s)}$$

which shows that $u_m(t)$ is Cauchy in the Banach space $C\left([-T,T], H_{(s)}\right)$.

We formalize this as follows:

Definition 2.5. *Suppose $T>0$ and $f \in H_{(s)}$ for $s \in \mathbb{R}$. A function $C\left([0,T], H_{(s)}\right)$ is called a generalized solution of (12) if there exists a sequence of Schwartz solutions $u_n : [0,T] \to \mathcal{S}$ such that $u_n \to u$ in $C\left([0,T], H_{(s)}\left(\mathbb{R}^d\right)\right)$.*

Theorem 2.6. *For $f \in H_{(s)}\left(\mathbb{R}^d\right)$ with $s \in \mathbb{R}$, there exists a unique generalized solution of (12) living in $C\left([0,T], H_{(s)}\right)$.*

2.2 Physical space picture, decay

Let us try to rewrite our solution formula (17) in "physical space", i.e. as an integral of the initial data against some kernel. We first note for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} u(t,x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^d} e^{ix\xi} e^{-it|\xi|^2} \hat{f}(\xi)\, d\xi \\ &= (2\pi)^{-n/2} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} e^{-\epsilon|\xi|^2} e^{ix\xi} e^{-it|\xi|^2} \hat{f}(\xi)\, d\xi \\ &= (2\pi)^{-n} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} e^{-\epsilon|\xi|^2} e^{ix\xi} e^{-it|\xi|^2} \left(\int_{\mathbb{R}^d} f(y)\, e^{-i\xi y} dy\right) d\xi \,. \end{aligned} \tag{18}$$

The reason for introducing the regularising factor is that now (for $\epsilon > 0$) we can exchange the order of integration using Fubini to find

$$u(t,x) = \int_{\mathbb{R}^d} f(y) K(x-y,t)\, dy\,. \tag{19}$$

with the kernel

$$K(z,t) = (2\pi)^{-n} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} e^{iz\cdot\xi} e^{-it|\xi|^2} e^{-\epsilon|\xi|^2} d\xi\,. \tag{20}$$

Exercise 2.7. *By evaluating (20) show that (19) can be written as*

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y)\, dy \tag{21}$$

for $t \neq 0$.

Two things can be read-off from the representation (21) of the solution to (12). The first is smoothing: If $u_0(y)$ has compact support, then the solution u is smooth for times $t > 0$. The equation immediately trades localization for regularity. The other estimate that immediately follows from (21) is

$$|u(t,x)| \leq \frac{C}{|t|^{\frac{d}{2}}} \|u_0\|_{L^1}\,,$$

which is another manifestation of the dispersion.

On the other hand, in the representation (21) it is hard to see that u agrees with f at $t = 0$ and also that the L^2 norm is conserved, for instance. Luckily these things are almost immediate from our representation (17).

3 Non-linear Schroedinger Equations

Let us study the PDE

$$i\partial_t u + \Delta u = u|u|^2 \tag{22}$$

We would like to establish local well-posedness for initial data $u(x,0) = \phi(x)$ living in $H_{(s)}$ for suitable s.

The idea is to first understand the inhomogenous problem

$$(i\partial_t + \Delta) u = f$$

for a prescribed $f(x,t)$ which is at least L^2 in space. Taking the Fourier transform yields

$$\left(i\partial_t - |\xi|^2\right) \hat{u}(\xi,t) = \hat{f}(\xi,t)$$

Defining

$$\hat{v}(\xi,t) := e^{it|\xi|^2} \hat{u}(\xi,t)$$

we derive

$$i\partial_t \hat{v}(\xi,t) = e^{it|\xi|^2}\hat{f}(\xi,t)$$

and hence

$$\hat{v}(t,\xi) = (-i)\int_0^t e^{is|\xi|^2}\hat{f}(\xi,s)\,ds + \hat{\phi}(\xi)\ . \tag{23}$$

Going back to $u(t,\xi)$, we find

$$\hat{u}(\xi,t) = e^{-it|\xi|^2}\hat{\phi}(\xi) - i\int_0^t e^{-it|\xi|^2}e^{is|\xi|^2}\hat{f}(\xi,s)\,ds \tag{24}$$

which after taking the Fourier transform results in

$$u(x,t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}f(s)\,ds \tag{25}$$

where $S(t) = e^{-it\Delta}$ denotes the Schroedinger propagator mapping initial data $\phi \in H_{(s)}$ to the solution $u(t)$ of the homogenous equation at time t.[2] Formula (25) is called the Duhamel formula.

For our non-linear problem we would like to set up an iteration based on the Duhamel formula. In other words, we have $f(t) = u(t)|u(t)|^2$. In Step 1, we would like to solve the Duhamel formula with $u_1(t) = e^{it\Delta}\phi$ (=the homogenous solution) constituting the inhomogeneity $f_1(t) = u_1(t)|u_1(t)|^2$. The Duhamel formula would provide at u_2 which then constitutes the next homogeneity f_2, etc. We would hope that this procedure converges:

$$\begin{aligned} u_1(t) &= e^{it\Delta}\phi \\ u_m(t) &= e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta}\left(u_{m-1}(s)|u_{m-1}(s)|^2\right)ds \end{aligned} \tag{26}$$

Ideally, the $u_m(t)$ should live in a complete metric space $X(I)$ and form a Cauchy sequence there.

Theorem 3.1. *Let $\sigma \geq \sigma_0(d)$, $\phi \in H_{(\sigma)}$, then there exists a unique solution u with $u \in C\left([-T,T], H_{(\sigma)}\right)$ of the NLS equation (22). The number $T > 0$ depends on the size of the $H_{(\sigma)}$ norm of ϕ, $R := \|\phi\|_{H_{(\sigma)}}$.*

Proof. Define

$$X^\sigma(I) = \{f \in C\left(I, H_{(\sigma)}\right)\ ,\ d(f,g) = \sup_t \|f(t) - g(t)\|_{H_{(\sigma)}} \text{ with } d(f,0) \leq 2\|\phi\|_{H_{(\sigma)}}\}$$

a closed ball in a Banach space. Clearly $u_1(X^\sigma(I)) =: X^\sigma$. We claim that if $u_n \in X^\sigma$, then also $u_{n+1} \in X^\sigma$, at least if T is sufficiently small. Indeed,

$$\begin{aligned} \|u_{n+1}\|_{H_{(\sigma)}} &\leq \|e^{it\Delta}\phi\|_{H_{(\sigma)}} + \|\int_0^t e^{i(t-s)\Delta}f_n(s)\|_{H_{(\sigma)}} \\ &\leq \|\phi\|_{H_{(\sigma)}} + \|\int_0^t \|f_n(s)\|_{H_{(\sigma)}} \end{aligned} \tag{27}$$

[2]Recall that we already showed this map preserves all $H_{(s)}$ norms in Proposition 2.3.

Now we know that

$$\|f_n(s)\|_{H_{(\sigma)}} = \|u \cdot u \cdot \bar{u}\|_{H_{(\sigma)}} \leq C\|u\|^3_{H_{(\sigma)}}$$

holds for sufficiently large σ by Sobolev embedding ($\sigma > \frac{d}{2}$). Hence

$$\|u_{n+1}\|_{H_{(\sigma)}} \leq \|\phi\|_{H_{(\sigma)}} + C\int_0^t \|u\|^3_{H_{(\sigma)}} ds \leq \|\phi\|_{H_{(\sigma)}} + C|T| \cdot R^3 < \frac{3}{2}R < 2R \tag{28}$$

for sufficiently small T.

To establish the contraction property, we will need to estimate differences.

$$\begin{aligned} \|u_{n+1}(t) - u_n(t)\|_{H_{(\sigma)}} &= \|\int_0^t e^{i(t-s)\Delta}(f_n(s) - f_{n-1}(s))\|_{H_{(\sigma)}} ds \\ &\leq \int_0^t \|f_n(s) - f_{n-1}(s)\|_{H_{(\sigma)}} ds \end{aligned} \tag{29}$$

We have

$$\begin{aligned} & u_n u_n \bar{u}_n - u_{n-1} u_{n-1} \bar{u}_{n-1} \\ = & (u_n - u_{n-1}) u_n \bar{u}_n + u_{n-1}(u_n - u_{n-1})\bar{u}_n + u_{n-1}u_{n-1}(\bar{u}_n - \bar{u}_{n-1}) \end{aligned} \tag{30}$$

and therefore

$$\|f_n(s) - f_{n-1}(s)\|_{H_{(\sigma)}} \leq CR^2\|u_n - u_{n-1}\| \tag{31}$$

so upon integration and choosing T small (depending only on R and Sobolev constants) we establish

$$\|u_{n+1} - u_n\|_{X^\sigma} \leq \frac{1}{2}\|u_n - u_{n-1}\|_{X^\sigma} \tag{32}$$

the contraction property. By the completeness we have the existence of a limit $u = \lim u_n$ in X^σ. This limit solves the integral equation

$$u(x,t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta} u|u|^2 ds$$

as follows by taking the limit of this for the u_n. A posteriori we can establish more regularity, e.g. $\partial_t u \in C\left(I, H^{\sigma-2}\right)$ (Exercise). Finally, for the uniqueness, we consider two solutions $u(t)$ and $v(t)$ arising from the same data. Their difference satisfies

$$w(t) = -i\int_0^t e^{i(t-s)\Delta}\left(u(s)|u(s)|^2 - v(s)|v(s)|^2\right) ds,.$$

The difference can be treated as previously in (30) to establish the estimate

$$\|w(t)\|_{H_{(\sigma)}} \leq t \cdot R^2\|w(t)\|_{H_{(\sigma)}}$$

which implies $w(t)$ for a small time interval. Iteration yields $w(t) = 0$ on all of $[-T, T]$. □

More interesting is of course the issue of global existence. Note that the mass and the energy are conserved in time (Exercise)

$$M = \int |u|^2 dx \quad , \quad E = \frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{4}\int |u|^4 dx \tag{33}$$

In particular, we have a global uniform bound on the H^1-norm. IF that norm controlled the norm for which one has well-posedness, global existence would be trivial.

Studying local and global well-posedness properties of non-linear Schroedinger equations is a very popular research topic (see the monograph of Terry Tao on non-linear dispersive equations). Here we simply mention that there are much more refined estimates to deal with the non-linearities (for instance, Strichartz estimates).

4 The heat equation

4.1 Fourier synthesis and generalized solutions

At the formal level the analysis of the heat equation proceeds exactly analogously as that for the Schroedinger equation. Hence we will be brief in the presentation and emphasize the main differences when carrying out the strategy of the previous section.

We consider the initial value problem for the heat equation

$$\begin{aligned} u_t &= \Delta u \\ u(0,\cdot) &= f(\cdot) \end{aligned} \tag{34}$$

Note that again $t = 0$ is characteristic for the heat equation: Just as for the Schroedinger equation, it turns out that we will need to impose a (very weak) condition near infinity (preventing the solution from growing too fast there) to ensure uniqueness of solutions of this IVP. In particular, if the data are in the Schwartz space, we will obtain existence and uniqueness of Schwartz solutions.

To prove this, we proceed as in the Schroedinger case. We first assume that we already have a solution in $\mathcal{S}\left(\mathbb{R}^d\right)$, take the Fourier transform and find

$$\hat{u}_t = -|\xi|^2 \hat{u} \tag{35}$$

which is solved by

$$\hat{u} = \hat{f}(\xi)\exp\left(-t|\xi|^2\right) \tag{36}$$

Now for $t > 0$ this is obviously in $\mathcal{S}\left(\mathbb{R}^d\right)$ while for $t < 0$ it is not even in $\mathcal{S}'\left(\mathcal{R}^d\right)$. Therefore we expect the theory developed for the Schroedinger equation to work only for $t > 0$. Indeed, just as before, setting

$$u(t,x) = \mathcal{F}^{-1} \circ \exp\left(-t|\xi|^2\right) \circ \mathcal{F}(f) \tag{37}$$

one shows that given $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ the u defined above has $u(t,\cdot) \in \mathcal{S}\left(\mathbb{R}^d\right)$ for $t > 0$ and that the solution map is continuous and in fact infinitely differentiable:

Theorem 4.1. *If $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ then there exists one and only one $u \in C^\infty\left([0,\infty), \mathcal{S}\left(\mathbb{R}^d\right)\right)$ such that (34) holds. The solution is given by the formula (37).*

As mentioned, the proof is almost identical to the one for Schroedinger. The next step is to find generalized solutions, i.e. to extend to solutions in H^s using appropriate estimates. If we denote the solution map $u(0,\cdot) \mapsto u(t,\cdot)$ by $S_H(t)$ we have

$$\|u(t)\|_{H^s} = \|\exp\left(-t|\xi|^2\right) < \xi >^s \mathcal{F}(f)\|_{L^2} \leq \| < \xi >^s \mathcal{F}(f)\|_{L^2} = \|u(0)\|_{H^s}.$$

Clearly one can also obtain this with $\|u(0)\|_{L^2}$ on the right hand side (how?). Using the same ideas as those leading to Theorem 2.6 one proves

Theorem 4.2. *If $f \in H^s\left(\mathbb{R}^d\right)$ then the generalized solution (defined as in Definition 2.5) satisfies*

$$\partial_t^j u \in C\left([0,\infty), H^{s-2j}\left(\mathbb{R}^d\right)\right).$$

4.2 The physical space picture

In the previous section we saw that we could only establish well-posedness for (34) in the future direction. Let us understand a bit better why this is the case.[3] Let us suppose we have a solution

$$u \in C^1\left(\mathbb{R}^n \times [0,\infty)\right) \cap C^2\left(\mathbb{R}^n \times (0,\infty)\right)$$

satisfying $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$ and $u = u_0$ on $\mathbb{R}^n \times \{0\}$. Then, integrating the expression

$$u\partial_t u = u\Delta u$$

by parts over a spacetime slab $[0,T] \times \mathbb{R}^n$ yields the identity

$$\frac{1}{2}\int_{t=T} u^2\, dx + \int_0^T dt \int_{\mathbb{R}^n} dx\left[\nabla u \nabla u - \nabla\left(u\nabla u\right)\right] = \frac{1}{2}\int_{t=0} u^2 dx \tag{38}$$

Now if we assume that u decays near infinity sufficiently rapidly, the boundary term $\nabla\left(u\nabla u\right)$ is going to vanish and we get a nice identity (this is obviously the case for the Schwartz function considered earlier). We summarize

Proposition 4.3. *Let $u \in C^1\left(\mathbb{R}^n \times [0,\infty)\right) \cap C^2\left(\mathbb{R}^n \times (0,\infty)\right)$ satisfy $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$ and $u = u_0$ on $\mathbb{R}^n \times \{0\}$. Then*

$$\int_{t=T} u^2 dx \leq \int_{t=0} u^2 dx$$

provided that u decays sufficiently rapidly near infinity.

[3]Note also the obvious fact that the PDE (34) is not invariant under $t \to -t$.

Of course, we have obtained the above already at the Fourier level. The above Proposition immediately yields a uniqueness theorem for solutions to the heat equation within the class of rapidly decaying solutions (Exercise: State and prove it.)

We can make the following remarks:

- Note that the estimate (38) gives another explanation of why we cannot expect to solve the heat equation backwards. The spacetime term $\int\int \nabla u \nabla u$ needs to have the "right sign " in order to control the solution at later times from the data. For negative times, the estimate (38) is hopeless, as the derivative term cannot be absorbed and one cannot control the solution in terms of the data. The backwards problem is indeed ill-posed both, in the sense that there doesn't exist a solution for *all* Schwartz data and in the sense that there is a loss of continuous dependence in the data. See the exercises.
- *Some* condition at infinity is necessary to ensure uniqueness of solutions (although the "decaying sufficiently rapidly at infinity" condition is much stronger than what is needed). You will study the example of Tychonoff in the exercises to illustrate this. One may think of the condition as imposing "boundary" conditions at infinity.

The heat kernel

Similarly to the Schroedinger equation, we can write the solution for the heat equation as

$$u(x,t) = \int K(t, x-y)\, u_0(y)\ dy \tag{39}$$

for the so-called heat kernel

$$K(t,z) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-|z|^2/(4t)\right) > 0 \tag{40}$$

which is the inverse Fourier transform of $\exp\left(-|\xi|^2 t\right)$ for $t > 0$. We observe the following

- Smoothing: If $f \in L^p$ it follows that $u \in C^\infty$ for $t > 0$. (Exercise)
- Positivity is preserved. In fact, if $u \geq 0$ at $t = 0$ (but not identically zero), then for any $t > 0$ we have $u(x,t) > 0$ at *all* points. This illustrates that the heat equation exhibits infinite speed of propagation.
- Decay in t. We have $\|u(x,t)\|_{L^\infty} \leq Ct^{-\frac{n}{2}}$ a for the Schroedinger equation. Note, however, that for the latter the L^2-norm is conserved while here the L^2-norm decreases!

Exercise 4.4. *Show that $K = K(t,z)$ satisfies $K_t = \Delta K$ for $t > 0$ and $K(\cdot,t) \rightharpoonup \delta$ in $\mathcal{S}'$ as $t \to 0^+$.*

Exercise 4.5. *Show using physical space techniques only that if* $u_0 = f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *then the* u *given by (39) satisfies the IVP (34). In particular,*

$$\lim_{(x,t)\to(x_0,0), x\in\mathbb{R}^n, t>0} u(x,t) = f\left(x^0\right) \tag{41}$$

for each $x^0 \in \mathbb{R}^n$*.*

4.3 Initial boundary value problems

Prove the following Proposition

Proposition 4.6. *Let* $\mathcal{U}$ *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary* $\partial\mathcal{U}$*. If* $u \in C^1\left(\overline{\mathcal{U}} \times [0,\infty)\right) \cap C^2(\mathcal{U} \times [0,\infty))$ *satisfies* $u_t = \Delta u$ *in* $\mathcal{U} \times (0,\infty)$ *with* $u = 0$ *in* $\partial U \times [0,\infty)$ *("Dirichlet boundary conditions") and* $u = u_0$ *on* $\mathcal{U} \times \{0\}$*, then*

$$\int_{\mathcal{U}} u^2(t,x)\,dx + \int_0^T \int_{\mathcal{U}} |\nabla_x u|^2(x,t)\,dxdt = \int_{\mathcal{U}} u_0^2 dx\,.$$

Show that this implies that u *decays exponentially in time (both in energy and pointwise). (Hint: Use the Poincare inequality and a version of Gronwall's inequality proven in Week 1)*

4.4 Inhomogeneous problems

To illustrate the similarity between the heat equation and the Laplace equation, let us recall a question we spent a lot of time on when discussing Poisson's equation. We noted that given

$$\Delta u = f$$

the equation actually "gained" two derivatives if measured in Sobolev spaces, in the sense that if f was in H^k then u was seen to be in H^{k+2}. A similar regularity estimate can be derived for the heat equation. We will focus here on the crucial estimate (and not bother with the technical details of mollifying/ using finite differences, which we have discussed at length for Poisson's equation). The crucial estimate follows by squaring the equation

$$\partial_t u - \Delta u = F \tag{42}$$

and integrating by parts:

$$\int_0^T \int dx \left[(\partial_t u)^2 - 2\partial_t u \Delta u + (\Delta u)^2\right] = \int_0^T \int dx\ F^2 \tag{43}$$

yields – provided u vanishes sufficiently rapidly at infinity – the estimate

$$\int_{t=T} |\nabla u|^2 + \int_0^T \int \left[(\partial_t u)^2 + (\Delta u)^2\right] = \int_{t=0} |\nabla u|^2 + \int_0^T \int dx\ F^2 \tag{44}$$

and using elliptic regularity

$$\|u\|_{H^2([0,T]\times\mathbb{R}^n)} \leq \|F\|_{L^2([0,T]\times\mathbb{R}^n)} + \|u(0,\cdot)\|_{H^1(\mathbb{R}^n)} \tag{45}$$

which illustrates that we expect the heat equation to "gain" two derivatives as well. To derive (45) rigorously (i.e. without assuming smoothness of u a priori to justify the above computations) requires mollification or difference quotient as seen for the Laplace equation.

4.5 A maximum principle

Let us give a basic version of the maximum principle for the heat equation. Let $\mathcal{U} \subset \mathbb{R}^n$ be an open bounded region of $\mathbb{R}^n$. For fixed $T > 0$ form the cylinder

$$\Omega = \{ \ (x,t) \ \ | \ \ x \in \mathcal{U} \ , \ 0 < t < T\} \tag{46}$$

and define the boundary components

$$\begin{aligned} \partial'\Omega &= \{ \ (x,t) \ \ | \ \ \textit{either } x \in \partial\mathcal{U} \ , \ 0 \leq t \leq T \ \ \textit{or} \ \ x \in \mathcal{U} \ , \ t = 0\} \\ \partial''\Omega &= \{ \ (x,t) \ \ | \ \ x \in \partial\mathcal{U} \ , t = T\} \end{aligned} \tag{47}$$

Theorem 4.7. *Let u be continuous in $\overline{\Omega}$ and u_t, $u_{x_i x_k}$ exist and be continuous in Ω and satisfy $u_t - \Delta u \leq 0$. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial'\Omega} u$$

Note that we do not assume that the first derivatives extend continuously to $\overline{\Omega}$. Compare this with the maximum principle seen for elliptic equations.

Proof. Assume first the strict inequality $u_t - \Delta u < 0$ in Ω. For any $0 < \epsilon < T$ let Ω_ϵ denote the set

$$\Omega_\epsilon = \{(x,t) \ | \ x \in \mathcal{U} \ , \ 0 < t < T - \epsilon\}$$

Since $u \in C^0\left(\overline{\Omega_\epsilon}\right)$ there exists a point $(x,t) \in \overline{\Omega_\epsilon}$ with

$$u(x,t) = \max_{\overline{\Omega_\epsilon}} u$$

Now if that point where the maximum is assumed is in the interior, $(x,t) \in \Omega_\epsilon$, then $u_t = 0$ and $\Delta u \leq 0$ which is in contradiction with $u_t - \Delta u \leq 0$. Similarly if $(x,t) \in \partial''\Omega_\epsilon$, then $u_t \geq 0$, $\Delta u \leq 0$ yields a similar contradiction. Therefore $(x,t) \in \partial'\Omega_\epsilon$ and

$$\max_{\overline{\Omega_\epsilon}} u = \max_{\partial'\Omega_\epsilon} u \leq \max_{\partial'\Omega} u$$

Since every point of $\overline{\Omega}$ with $t < T$ belong so some $\overline{\Omega}_\epsilon$ and u is moreover continuous, the statement (46) follows under the assumption that $u_t - \Delta u < 0$ in Ω. To treat the equality case, we introduce

$$v(x,t) = u(x,t) - \delta \cdot t$$

which satisfies

$$v_t - \Delta u = u_t - \Delta u - \delta < 0 \, .$$

By the previous part, we have

$$\max_{\overline{\Omega}} u = \max_{\overline{\Omega}} (v + \delta t) \leq \max_{\overline{\Omega}} v + \delta T = \max_{\partial' \Omega} v + \delta T \leq \max_{\partial' \Omega} u + \delta T$$

and the limit $\delta \to 0$ establishes the result. □

The maximum principle yields another uniqueness theorem:

Proposition 4.8. *Let u be continuous in $\overline{\Omega}$ and u_t, $u_{x_i x_k}$ exist and be continuous in Ω. Then u is uniquely determined in $\overline{\Omega}$ by the value of $u_t - \Delta u$ in Ω and of u on $\partial' \Omega$.*

Proof. Exercise. □

The above maximum principle can also be used to prove a uniqueness theorem for (34) on the *unbounded* domain $\mathbb{R}^n \times [0, T]$ provided a mild growth condition is imposed at infinity ($u\left(x,t\right) \leq A \exp\left(a |x|^2\right)$ holds for constants A and $a > 0$). See Chapter 2.3 (Theorem 6) of Evans.

4.6 More general equations

Similar to the elliptic equations, we can consider more general operators of the form

$$Pu := \partial_t u + Lu = F \tag{48}$$

where $L = \partial_i \left(a^{ij} \partial_j u\right) + b^i \partial_i u + cu$ is uniformly elliptic with coefficients a, b, c being functions of t and the spatial variables. If these coefficients are not constant, the approach via the Fourier transform is not immediately available. However, one can develop a similar theory of weak solutions as we did for the elliptic problem, based on Galerkin approximations. See the book of Evans.

5 Exercises

1. Do the Exercises in the text. Proving Proposition 4.6 and doing Exercises 1.3, 4.4 and 4.5 is particularly recommended.

2. This exercise illustrates the problem of (non)-uniqueness of the IVP (34). We will construct non-trivial solutions of (34) with zero initial data in $1+1$ dimensions (this is due to Tychonoff). These solutions grow very fast near infinity. You should compare and contrast with Exercise 2 of Week 3+4. See Fritz John's book, Chapter 7.1 for more details.

   The construction starts by prescribing the following Cauchy data for $u_t = u_{xx}$ on the t-axis:

   $$u = g\left(t\right) \quad , \quad u_x = 0 \tag{49}$$

Step 1: Show that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{2k!} x^{2k} \tag{50}$$

constitutes a *formal* power series solution.

We now choose for a real number $\alpha > 1$ the data

$$g(t) = \begin{cases} \exp(-t^{-\alpha}) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \tag{51}$$

Step 2: Show that there exists a $\theta = \theta(\alpha)$ with $\theta > 0$ such that for all $t > 0$ we have

$$|g^{(k)}(t)| < \frac{k!}{(\theta t)^k} \exp\left(-\frac{1}{2} t^{-\alpha}\right)$$

Hint: This follows from the Cauchy integral formula. Consider $g(t + is)$ as a function in the complex t-plane. To estimate $g^{(k)}(t)$ choose a small ball around $(t, 0)$ of radius $\theta \cdot t$ for sufficiently small θ...

Step 3: Show that (50) is majorized by the power series for

$$U(x,t) = \begin{cases} \exp\left(\frac{1}{t}\left[\frac{x^2}{\theta} - \frac{1}{2} t^{1-\alpha}\right)\right] & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \tag{52}$$

Conclude that $\lim_{t \to 0} u(x,t) = 0$ uniformly in x for bounded x, that $u \in C^\infty\left(\mathbb{R}^{1+1}\right)$ and that it satisfies the heat equation $u_t = u_{xx}$.

Remark: Uniqueness of the heat equation *can* be proven in the class of solutions satisfying the growth condition $|u(t,x)| \leq A \exp\left(a|x|^2\right)$ with constants A, a. Cf. Evans, Theorem 7 in Chapter 2.3. Compare with the above!

3. We have seen that the backwards evolution for the heat equation is ill-posed in the sense that there is no solution for all Schwartz initial data. Here is an illustration of the fact that there is also loss of continuous dependence. For simplicity, we again restrict to the $1+1$ dimensional case.

   Check that

   $$u_n(t,x) = e^{-n} \sin(nx) e^{n^2 \cdot t}$$

   satisfies the backwards heat equation for any $n \in \mathbb{N}$ and explain how this is related to the loss of continuous dependence on the data at $t = 0$.

4. Even if we cannot solve the heat equation backwards for *all* Schwartz data prescribed at $t = 0$ (more precisely there is generally no solution in $\mathcal{S}'(\mathbb{R}^n)$ for all $t < 0$), for *some* data there is obviously a solution (just solve the forward problem from $t = -1$ and take what you obtain at $t = 0$ as your

data). It turns out that *if* a solution of the backwards problem exists, then it is unique. You can read more about this in Evans, Theorem 11 in Chapter 2.3.

5. (Fritz John, Chapter 7.1, Problem 12)

   (a) Let $n = 1$ and $\mu > 0$. Let $u(t, x)$ be a positive solution of class C^2 of

   $$u_t = \mu u_{xx} \quad \text{for } t > 0.$$

   Show that $\theta = -2\mu \frac{u_x}{u}$ satisfies the viscous Burger's equation

   $$\theta_t + \theta\theta_x = \mu\theta_{xx} \quad \text{for } t > 0\,. \tag{53}$$

   (b) For $\phi \in C_0^2(\mathbb{R})$ find a solution of (53) with initial values $\theta(x, 0) = \phi(x)$, for which

   $$\lim_{t\to\infty} \theta(x, t) = 0\,.$$

   Compare and contrast this with the inviscous Burger's equation and the formation of shocks!

   (c) Find a solution of (53) which is independent of t. Interpretation?