Partial Differential Equations (Week 9+10)
The wave equation

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March 21, 2019

1 Introduction

This week we begin the study of the wave equation in $\mathbb{R}^{1+d}$:

$$\Box u = -\partial_t^2 u + \partial_{x_1}^2 u + \partial_{x_2}^2 u + \ldots + \partial_{x_d}^2 u = 0 \tag{1}$$

Just as the Laplace equation is the prototype of and elliptic equation, the wave equation is the prototype of a hyperbolic[1] equation. We already discussed the case $d = 1$ of one spatial dimension in Week 4, when we derived *d'Alembert's formula* and introduced the notions of *domain of dependence* and *domain of influence*. In the general case we computed the characteristic hypersurfaces and saw that they were given as solutions of the *eikonal equation*. We also observed that $t = 0$ is non-characteristic, which means that the Cauchy Kovalevskaya theorem will apply for Cauchy data (specifying u and the normal derivative) prescribed on $t = 0$.

Our plan of action is the following

1. Derive and understand geometrically the basic energy estimate for the wave equation and the domain of dependence property from this point of view (cf. our discussion of Fritz John's global Holmgren's theorem discussed previously); applications

2. Derive an explicit solution formula for (1) in physical space in the physically most interesting case $d = 3$, illustrating the strong Huygen's principle and the loss of regularity of solutions if measured at the level of C^k; decay of solutions

3. Derive an explicit solution formula for (1) in Fourier space giving yet another perspective on the conservation of energy and regularity at the L^2 level

[1] There a various definitions of this. We use it here to mean that the Cauchy problem is well-posed on a non-characteristic hypersurface. While an elliptic equation has no-characteristic hypersurfaces, hyperbolic equations in some sense have as many as possible. You should compare with Exercise 2.21 of Week 3.

4. Prove well-posedness for general linear equations $\Box u = b^i(t,x)\,\partial_i u + c \cdot u$

After this we will hopefully be in a position to discuss some non-linear applications in the last week.

2 The energy estimate

Let us assume that we have a classical C^2 solution of $\Box u = 0$ on $[0,T] \times \mathbb{R}^d$ with "data" $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$. Multiplying the wave equation by $-\partial_t u$ yields

$$0 = -\Box u \cdot \partial_t u = \frac{1}{2}\partial_t (\partial_t u)^2 - \nabla_x (\partial_t u \nabla_x u) + \nabla_x \partial_t u \cdot \nabla_x u = 0$$

or

$$= \frac{1}{2}\partial_t \left[(\partial_t u)^2 + |\nabla_x u|^2 \right] - \nabla_x (\partial_t u \nabla_x u) \tag{2}$$

If we integrate this over the spacetime slab $[0,T] \times \mathbb{R}^d$, then assuming that u decays sufficiently rapidly near infinity (more on this below!) we would obtain the energy conservation law

$$\int_{t=T} d^d x \left[(\partial_t u)^2 + |\nabla_x u|^2 \right] = \int_{t=0} d^d x \left[(\partial_t u)^2 + |\nabla_x u|^2 \right]$$

As this works for any $\tau \leq T$ we obtain

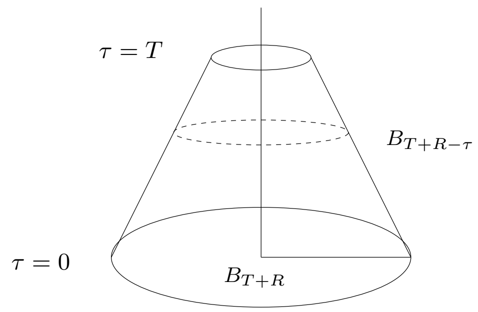
$$\| \partial_t u(\tau, \cdot) \|_{L^2_x} + \| u(\tau, \cdot) \|_{\mathring{H}^1_x} = \| u_1 \|_{L^2} + \| u_0 \|_{\mathring{H}^1} \tag{3}$$

In order to derive this identity we have assumed that u is C^2 and that it vanishes sufficiently rapidly near spatial infinity in order to make the boundary term arising from $\nabla_x (\partial_t u \nabla_x u)$ vanish. We will now see that we can do much better if we suitably localize the estimate.

Fix $T > 0$, $R > 0$ and consider a region

$$K = \bigcup_{\tau \in [0,T]} \{\tau\} \times B_{R+T-\tau} \tag{4}$$

where $B_{R+T-\tau}$ is the ball of radius $B + T - \tau$ centered at the origin.



You may think of this region as a cut-off (at $t = 0$ and $t = T$) past light cone with tip at $\left(T + R, \vec{0}\right)$. We will denote the boundary of $B_{R+T-\tau}$ in $\mathbb{R}^3$ by $S_{R+T-\tau}$ and the unit outward normal to this boundary by N.

Integrating (2) over the region K then yields

$$\begin{aligned}
&\frac{1}{2}\int_{\{t=T\}\times B_R} d^dx \left[(\partial_t u)^2 + |\nabla_x u|^2\right] \\
+ &\int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} \left[\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla_x u|^2 - \partial_t u \cdot Nu\right] d\sigma_{S_{R+T-\tau}} \\
&= \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^dx \left[(\partial_t u)^2 + |\nabla_x u|^2\right] \qquad (5)
\end{aligned}$$

It is not hard to see using Cauchy-Schwarz that the integrand in the second line is non-negative. We can actually obtain something more quantitative. Let us denote the induced gradient on the spheres $S_{R+T-\tau}$ by $\nabla\!\!\!/$ (i.e. the derivatives *tangent* to these $d-2$ dimensional spheres). We may decompose

$$\partial_t = N + V$$

where V is a derivative tangent to the wall of the cone[2] Then, from the easily verified identities

$$\begin{aligned}
-\partial_t u N u &= -(Nu)^2 - Nu \cdot Vu \\
\frac{1}{2}\partial_t u \partial_t u &= \frac{1}{2}(Nu)^2 + Nu \cdot Vu + \frac{1}{2}(Vu)^2 \\
\frac{1}{2}|\nabla_x u|^2 &= \frac{1}{2}(Nu)^2 + \frac{1}{2}|\nabla\!\!\!/ u|^2
\end{aligned}$$

we see that (5) becomes

$$\begin{aligned}
&\frac{1}{2}\int_{\{t=T\}\times B_R} d^dx \left[(\partial_t u)^2 + |\nabla_x u|^2\right] \\
+ &\int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} \left[\frac{1}{2}(Vu)^2 + \frac{1}{2}|\nabla\!\!\!/ u|^2\right] d\sigma_{S_{R+T-\tau}} \\
&= \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^dx \left[(\partial_t u)^2 + |\nabla_x u|^2\right] \qquad (6)
\end{aligned}$$

This identity is truly remarkable and illustrates the domain of dependence property of the wave equation. Indeed, we certainly have

$$\int_{\{t=T\}\times B_R} d^dx \left[(\partial_t u)^2 + |\nabla_x u|^2\right] \leq \int_{\{t=0\}\times B_{R+T}} d^dx \left[(\partial_t u)^2 + |\nabla_x u|^2\right] \qquad (7)$$

and hence

[2] In polar coordinates $\partial_t = \partial_r + (\partial_t - \partial_r)$ since the wall of the cone is given by zero set of $H(t, x_1, \ldots, x_d) = t + \sqrt{x_1^2 + \ldots x_d^2} - R - T = t - T + r - R$, so that indeed $(\partial_t - \partial_r) H = 0$.

Corollary 2.1. *Suppose $u = 0$ in $\{t = 0\} \times B_{R+T}$. Then $u = 0$ in $\bigcup_{\tau \in [0,T]} \{\tau\} \times B_{R+T-\tau}$.*

Corollary 2.2. *Two C^2 solutions u and v in $K = \bigcup_{\tau \in [0,T]} \{\tau\} \times B_{R+T-\tau}$ that satisfy $u = v$ and $\partial_t u = \partial_t v$ on $\{t = 0\} \times B_{R+T}$ have to agree in all of K.*

Exercise 2.3. *Can you generalize this domain of dependence/ uniqueness properties to more general wave equations? Hint: Gronwall's inequality. You should also compare and contrast with the statement we obtained from Holmgren's theorem.*

Let us understand a bit better the underlying geometry of this computation. The expression (2) is apparently a boundary term and it will induce different expressions dependent on the geometry of the boundary hypersurfaces. What is useful in the estimates is if the expressions induced are non-negative, as it was the case for the hypersurfaces of constant t and the characteristic hypersurfaces discussed above. More generally, we define

Definition 2.4. *We will call a hypersurface spacelike if it can be represented locally as $f = 0$ for f satisfying*

$$-(\partial_t f)^2 + |\nabla_x f|^2 < 0 \tag{8}$$

We call it timelike if "$<$" above is replaced by "$>$" and null if "$<$" is replaced by "$=$".

Remark 2.5. *For those familiar with Minkowski geometry and its inner-product $< x, y >= -x_0 y_0 + \sum_{i=1}^d x_i y_i$ (giving rise to a notion of timelike, spacelike and null vectors depending on the sign of $< x, x >$), note that the expression (8) is precisely the Minkowski norm of the gradient of f. Hence a hypersurface is spacelike if its normal vector is timelike etc.*

Exercise 2.6. *Obtain the energy estimate for two homologous spacelike hypersurfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, i.e. spacelike hypersurfaces with common boundary $\partial\mathcal{S}_1 = \partial\mathcal{S}_2$ bounding a region. Hint: Observe that integrating (2) over a spacelike hypersurface produces after using (8) a non-negative expression controlling all derivatives.*

Note that given the domain of dependence property we can combine the Cauchy-Kovalevskaya theorem with the energy estimate to obtain a notion of generalized solution (how?).

It is remarkable that the energy estimate (which is at the level of L^2) does not lose regularity: It relates the $H^1 \times L^2$ norm for data to the same norm for the solution at any later time. We will see that this property does not hold at the C^k level.

Finally, it's important to realize the flexibility of the energy method. In principle you can add lower order terms and apply Gronwall. Unlike in Holmgren's theorem (which you can now revisit and interpret the one-parameter family of non-characteristic hypersurfaces as foliation of spacelike slices for which you could apply the energy estimate), the method does not require analyticity of the coefficients.

2.1 Pointwise bounds

We can obtain pointwise bounds for solutions to the wave equation using Sobolev embedding. We give two different ways of doing this:

Method 1: Commute the equation with a basis of the tangent space of $\mathbb{R}^n$, i.e. with $\partial_{x_1}, \ldots, \partial_{x_d}$. We obviously have

$$[\Box, \partial_{x_i}] = 0$$

The underlying is the translation invariance of the wave operator. Applying the energy estimate to each of the commuted equations we immediately obtain

$$\|u(\tau,\cdot)\|_{\mathring{H}^1} + \|u(\tau,\cdot)\|_{\mathring{H}^2} \leq \|u_1\|_{H^1} + \|u_0\|_{\mathring{H}^1} + \|u_0\|_{\mathring{H}^2} \tag{9}$$

Now in $\mathbb{R}^3$ we have the Sobolev embedding

$$\sup|u| \leq C\left(\|u(\tau,\cdot)\|_{L^6} + \|Du(\tau,\cdot)\|_{L^6}\right) \leq C\left(\|u(\tau,\cdot)\|_{\mathring{H}^1} + \|u(\tau,\cdot)\|_{\mathring{H}^2}\right)$$

for u of compact support, which implies

$$\sup|u| \leq C\left(\|u_1\|_{H^1} + \|u_0\|_{\mathring{H}^1} + \|u_0\|_{\mathring{H}^2}\right)$$

global pointwise control on the solution.

Method 2: Commute with ∂_t. We obviously have

$$[\Box, \partial_t] = 0$$

The underlying is the translation time translation invariance of the wave operator. Applying the energy estimate to the ∂_t-commuted equation we immediately obtain

$$\begin{aligned} \int_{t=\tau} |\Delta u|^2 = \int_{t=\tau} |\partial_t^2 u|^2 \leq \int_{t=0} |\partial_t^2 u|^2 + |\nabla_x \partial_t u|^2 = \int_{t=0} |\Delta u|^2 + |\nabla_x u_1|^2 \\ \leq |u_0|^2_{\mathring{H}^2} + |u_1|^2_{\mathring{H}^1} \end{aligned} \tag{10}$$

Now using elliptic theory we know that for u of compact support the right hand side controls all second derivatives. Combining this with the energy estimate as the uncommuted level yields the same pointwise bound as in Method 1 after applying the Sobolev embedding.

3 Kirchhoff's formula for the wave equation

We now switch gears a bit and leave the estimates behind to obtain an explicit representation formula for solution to (1) in terms of initial data. The formula will be useful to illustrate the Huygen's principle and also the loss of regularity at the level of C^k regularity.

We consider the Cauchy problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= 0 \\ u|_{t=0} &= f \\ u_t|_{t=0} &= g \end{aligned} \tag{11}$$

For $h(x) = h(x_1, \ldots, x_d)$ continuous from $\mathbb{R}^d \to \mathbb{R}$ we define

$$M_h(x,r) = \frac{1}{\omega_d r^{n-1}} \int_{|y-x|=r} h(y)\, dS_y \tag{12}$$

the average over a sphere of radius r around x (ω_d denoting the area element of the unit-sphere in d dimensions). Writing $y = x + r\xi$ with $|\xi| = 1$ we have

$$M_h(x,r) = \frac{1}{\omega_d} \int_{|\xi|=1} h(x + r\xi)\, dS_\xi \, .$$

We can extend $M_h(x,r)$ to an *even* function defined for all real r (change $\xi \to -\xi$). Observe also that $h \in C^k(\mathbb{R}^d)$ implies $M_h \in C^k(\mathbb{R}^{d+1})$.

Now, for $h \in C^2(\mathbb{R}^d)$ we find using the divergence theorem the identity

$$\left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right] M_h(x,r) = \Delta_x M_h(x,r) \tag{13}$$

This is known as the *Darboux* equation. Note the "initial conditions", $M_h(x,0) = h(x)$ and $\partial_r M_h(x,r)\Big|_{r=0} = 0$ since M_h is even in r.

To derive (13), note

$$\begin{aligned} \partial_r M_h(x,r) = \frac{1}{\omega_d}\int_{|\xi|=1} \xi \cdot \nabla_y h(x+r\xi)\, dS_\xi &= \frac{1}{\omega_d r}\int_{|\xi|=1} \xi \cdot \nabla_\xi h(x+r\xi)\, dS_\xi \\ = \frac{1}{\omega_d r}\int_{|\xi|<1} \Delta_\xi h(x+r\xi)\, d\xi &= \frac{r}{\omega_d}\int_{|\xi|<1} \Delta_x h(x+r\xi)\, d\xi \\ = \frac{r}{\omega_d}\Delta_x \int_0^1 d\rho \rho^{d-1} \int_{|\xi|=1} h(x+r\rho\xi)\, dS_\xi &= r\Delta_x \int_0^1 d\rho\rho^{d-1}\; M_h(x,r\rho) \end{aligned}$$

and after a further change of variables

$$\partial_r M_h(x,r) = r^{-d+1}\Delta_x \int_0^r d\rho\; \rho^{d-1} M_h(x,\rho) \ \ , \tag{14}$$

which yields

$$\partial_r\left(r^{d-1}\partial_r M_h(x,r)\right) = \Delta_x\left(r^{d-1} M_h(x,r)\right) \tag{15}$$

and hence (13).

The idea to solve (1) is to write down an equation for the spherical means of u. This will be a $1+1$ dimensional PDE which we can solve explicitly. Conversely, we shall be able to recover the solution from its spherical means.

We define

$$M_u(x,r,t) = \frac{1}{\omega_d} \int_{|\xi|=1} u(x + r\xi, t)\, dS_\xi \tag{16}$$

Note that $u(x,t) = M_u(x,0,t)$, recovering u from its means. Now by the Wave equation and by Darboux we know that we have

$$\partial_t^2 M_u = \Delta_x M_u = \left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right] M_u\,.$$

Let us restrict to $d=3$. In this case we have

$$\frac{\partial^2}{\partial t^2}(rM_u) = c^2 \frac{\partial^2}{\partial r^2}(rM_u)\ .$$

Hence we have that rM_u satisfies the $1+1$ dimensional wave equation with initial values:

$$rM_u = rM_f(x,r) \quad \textit{and} \quad \partial_t(rM_u) = rM_g(x,r) \quad \text{at } t=0 \tag{17}$$

and hence d'Alembert's formula applies. Therefore,

$$\begin{aligned} rM_u(x,r,t) = \frac{1}{2}\left[(r+ct)\, M_f(x, r+ct) + (r-ct)\, M_f(x, r-ct)\right] \\ +\frac{1}{2c}\int_{r-ct}^{r+ct} \xi M_g(x,\xi)\, d\xi \end{aligned} \tag{18}$$

Dividing by r and taking the limit $r \to 0$ (Exercise – use that $M_g(x,r)$ is even!) we finally find

$$\begin{aligned} u(x,t) = tM_g(x,ct) + \partial_t\left(tM_f(x,ct)\right) \\ = \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} g(y)\, dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} f(y)\, dS_y\right) \end{aligned} \tag{19}$$

We conclude:

Proposition 3.1. *Any solution u of the initial value problem (11) which is C^2 for $t \geq 0$ in $n=3$ space dimensions is given by formula (19) and is hence unique. Conversely, given $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ the $u(x,t)$ defined by the above formula is C^2 and satisfies (11).*

Note the loss of regularity! To make this loss more manifest, we compute

$$\begin{aligned} \partial_t\left(tM_f(x,ct)\right) &= M_f(x,ct) + t\partial_t\left(\frac{1}{\omega_d}\int_{|\xi|=1} f(x+ct\xi)\, dS_\xi\right) \\ &= M_f(x,ct) + t\frac{1}{\omega_d}\int_{|\xi|=1} Df(x+ct\xi)\cdot\frac{y-x}{ct}\, dS_\xi \end{aligned} \tag{20}$$

and obtain

$$u(x,t) = \frac{1}{4\pi c^2 t^2} \int_{|y-x|=ct} \left(tg(y) + f(y) + \sum_i f_{y_i}(y_i - x_i) \right) dS_y \, . \tag{21}$$

Besides the aforementioned loss of regularity, many more things can be read off from the above formula (19). One is the *strong Huygen's principle.* The solution at a point $(x, t > 0)$ depends only on the "data" of the surface $S(x, ct)$ the intersection of the past light cone with the hypersurface $t = 0$. It does not depend on the values inside the ball $B(x, ct)$. This sharp propagation of signals is special for the wave equation in odd spatial dimensions (with the exception $d = 1$). In even dimensions one only has the weak Huygen's principle (value at (t, x) depends on the values in the entire ball $B(x, ct)$) and for most hyperbolic equations one also only has the weak form. Conversely, the data near a point y on the initial hyperplane $t = 0$ only influence the solution at points (t, x) near the cone $|x - y| = ct$ emanating from y.

Finally, the formula (19) allows us to show that the solution decays in time (we already know it spreads over larger and larger regions of space). Suppose you have initial data of compact support at $t = 0$, the support being contained in a large ball $B(0, \rho)$. Now consider a point (t, x) for some large t. It is clear that the past light cone from (t, x) can only intersect the ball $B(0, \rho)$ is a set of area $4\pi\rho^2$. Thus the solution satisfies $|u(t, x)| \leq Ct^{-1}$ with the constant depending on the size of the support.

An important goal is to establish such decay estimates for more general (possibly non-linear) wave equations. This will require more stable methods than the explicit solution formula derived above and we'll mention some in the last week of the course.

3.1 General dimensions

A similar formula to (19) can be derived for general d via Hadamard's method of decent. As mentioned, in even dimensions the solution at (t, x) will depend on the values in the entire ball $|x - y| \leq ct$ (not only the sphere $|x - y| = ct$) of the data. See the books of Evans or John.

3.2 Duhamels principle

Consider the inhomogeneous wave equation with trivial data

$$\begin{aligned} u_{tt} - \Delta u = f(t, x) \qquad &\text{in } \mathbb{R}^d \times (0, \infty) \\ u = 0 \quad , \quad u_t = 0 \qquad &\text{for } \mathbb{R}^d \times \{0\} \end{aligned} \tag{22}$$

We define $\tilde{u} = \tilde{u}(x, t; s)$ to be the solution of

$$\begin{aligned} \tilde{u}_{tt}(\cdot, s) - \Delta \tilde{u}(\cdot; s) = 0 \qquad &\text{in } \mathbb{R}^d \times (s, \infty) \\ \tilde{u}(\cdot, s) = 0 \quad , \quad \tilde{u}_t(\cdot, s) = f(\cdot, s) \qquad &\text{for } \mathbb{R}^d \times \{t = s\} \end{aligned} \tag{23}$$

Now set

$$u(x,t) := \int_0^t \tilde{u}(x,t;s)\, ds \quad for \quad x \in \mathbb{R}^d \;\; and \;\; t \geq 0 \tag{24}$$

This is a solution of the problem (22):

Proposition 3.2. *Let $d = 3$ and $f \in C^2\left(\mathbb{R}^d \times (0,\infty)\right)$. Then u defined by (24) solves (22).*

Proof. Note first that by our well-posedness result for the homogeneous wave equation, the $\tilde{u}(x,t;s)$ are well-defined and C^2 in all its arguments for $0 \leq s \leq t$ (why?). Hence $u(x,t)$ is $C^2\left(\mathbb{R}^d \times [0,\infty)\right)$. Clearly, the trivial initial conditions of (22) are also satisfied. Clearly, the trivial initial conditions of (22) are also satisfied. To see that it also solves the inhomogenous wave equation ,we compute

$$u_t(x,t) = \tilde{u}(x,t;t) + \int_0^t \tilde{u}_t(x,t;s)\, ds = \int_0^t \tilde{u}_t(x,t;s)\, ds\,, \tag{25}$$

$$u_{tt}(x,t) = \tilde{u}_t(x,t;t) + \int_0^t \tilde{u}_{tt}(x,t;s)\, ds = f(x,t) + \int_0^t u_{tt}(x,t;s)\, ds\,. \tag{26}$$

Combining this with

$$\Delta u(x,t) = \int_0^t \Delta\tilde{u}(x,t;s)\, ds \tag{27}$$

yields the result. $\square$

4 Fourier synthesis

As for the Schroedinger and heat equation, we can follow a Fourier based approach. Assuming we have a solution of the problem (11) in the Schwartz space, we obtain the second order ODE

$$\hat{u}_{tt} = -c^2|\xi|^2\hat{u} \tag{28}$$

which is solved by

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos(c|\xi|t) + \hat{g}(\xi)\,\frac{\sin(c|\xi|t)}{c|\xi|} \tag{29}$$

Theorem 4.1. *If $f, g \in \mathcal{S}\left(\mathbb{R}^d\right)$, then there exists a unique $u \in C^\infty\left(\mathbb{R}, \mathcal{S}\left(\mathbb{R}^d\right)\right)$ solving the initial value problem (11).*

The proof of this theorem is as in the case of the Schroedinger and the heat equation. Note that taking a time derivative only leads to polynomial growth in ξ which is easily compensated for by the Schwartz property of $\hat{f}$ and $\hat{g}$.

It is a straightforward exercise to derive the identity

$$|\hat{u}_t|^2 + c^2|\xi|^2|\hat{u}|^2 = |\hat{g}|^2 + c^2|\xi|^2|\hat{f}(\xi)|^2 \tag{30}$$

which when integrated over al ξ yields the familiar energy conservation (in particular, the Schwartz space property ensures the "vanishing sufficiently fast near infinity property" stated loosely in the first section above).

Of course more is true in the sense that the identity (30) holds for any frequency ξ in itself!

Exercise 4.2. *Show that for $s \geq 1$ real the expression*

$$\|u_t\|^2_{H_{(s-1)}} + c^2\|\nabla_x u\|^2_{H_{(s-1)}} \tag{31}$$

is independent of t. Relate this to the conservation laws derived in physical space (for integer s).

Using these estimates one can – again as for the Schroedinger and heat equation – derive a notion of generalized solutions associated with data $f, g \in H_{(s)} \times H_{(s-1)}$....

Exercise 4.3. *Use the domain of dependence property to remove the assumption that the data are Schwartz from Theorem 4.1 to obtain existence and uniqueness of a solution $u \in C^\infty\left(\mathbb{R}^{d+1}\right)$ from data $f, g \in C^\infty(\mathbb{R})$.*

5 Back to the estimates

Our final goal is to prove the following well-posedness theorem, which we will state in the smooth category. The proof will suggest appropriate Sobolev versions.

Theorem 5.1. *Let $u_0 \in C^\infty\left(\mathbb{R}^d\right), u_1 \in C^\infty\left(\mathbb{R}^d\right)$ and $b^\alpha,\ c,\ F \in C^\infty\left(\mathbb{R}^{d+1}\right)$. Then there exists a unique solution $u \in C^\infty\left(\mathbb{R}^{d+1}\right)$ of the wave equation*

$$\begin{gathered} \Box u + \sum b^\mu(x,t)\,\partial_\mu u + c(x,t)\,u + F(x,t) = 0 \\ u(0,\cdot) = u_0 \quad , \quad \partial_t u(0,\cdot) = u_1\,. \end{gathered} \tag{32}$$

Moreover, any solution $v \in C^2\left(\mathbb{R}^{d+1}\right)$ of (32) satisfying $v(0,x) = u_0(x)$, $\partial_t v(0,x) = u_1(x)$ for all $x \in X \subset \mathbb{R}^d$ satisfies $u = v$ in the domain of dependence of X.

Note that we already proved the domain of dependence property.

5.1 The inhomogenous wave equation

Recall the Duhamel formula (24). Let us apply the energy estimate in the inhomogeneous case,

$$\Box u = F$$

for F a given function of x and t. Multiplying again by $-\partial_t u$ and integrating over the cut-off cone K (4) now produces (Exercise) in view of

$$\Big|\int_K \partial_t u\cdot F\Big| \le \int_K |\partial_t u|^2+|F|^2 \le \int_0^T dt\int_{\{t=\tau\}\times B_{R+T-\tau}} |\partial_t u|^2+|F|^2 \tag{33}$$

the estimate

$$E(\tau)\le\int_0^\tau d\tilde{\tau}E(\tilde{\tau})+(E(0)+G(\tau)) \tag{34}$$

where

$$E(\tau)=\int_{\{t=\tau\}\times B_{R+T-\tau}}\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla_x u|^2 \tag{35}$$

and

$$G(\tau)=\int_0^\tau\int_{\{t=\tau\}\times B_{R+T-\tau}}|F|^2\,. \tag{36}$$

Gronwall's inequality yields the estimate

$$E(\tau)\le[E(0)+G(\tau)]\,e^\tau\,. \tag{37}$$

Note that for trivial data there is a gain of *one* derivative compared with the inhomogeneity measured here at the level of L^2. Again, at the C^k level there is no gain (we needed $F\in C^2$ to get $u\in C^2$ when deriving Duhamel's formula). You should compare with the case of elliptic regularity, where one could gain *two* derivatives (at Sobolev or Hoelder level).

5.2 Lower order perturbations

Let us now try to prove an energy estimate for the operator (32). The main part of the estimate will be the same as for the box, except that now we have to control an additional term of the form

$$\begin{aligned}
&\int_K(\partial_t u)\left[\sum b^\mu(x,t)\,\partial_\mu u+c(x,t)\,u+F(x,t)\right]\\
&\le C\int_K\left[(\partial_t u)^2+|\nabla_x u|^2\right]+C\int_K u^2+C\int_K F^2\\
&\le C\int_K\left[(\partial_t u)^2+|\nabla_x u|^2\right]+C\int_K F^2+\int_{\{t=0\}\times B_{R+T}}u^2
\end{aligned}\tag{38}$$

Here the last step follows from a Poincare inequality (Exercise). The constants depend on the T of the region K. With the usual definitions of energy (35) and the inhomogeneous term (36) we find

$$E(\tau)=E(0)+\int_{\{t=0\}\times B_{R+T}}u^2+G(\tau)+\int_0^\tau d\tilde{\tau}E(\tilde{\tau}) \tag{39}$$

for which Gronwall's inequality yields

$$E(\tau) \leq \left(E(0) + \int_{\{t=0\}\times B_{R+T}} u^2 + G(\tau) \right) e^{C\tau} \tag{40}$$

We actually have (Exercise – this is a byproduct of the previous Exercise on the Poincare inequality)

$$\int_{\{t=\tau\}\times B_{R+T-\tau}} u^2 + E(\tau) \leq \left(E(0) + \int_{\{t=0\}\times B_{R+T}} u^2 + G(\tau) \right) e^{C\tau} \tag{41}$$

which yields

$$\| u(\tau,\cdot) \|_{H^1(B_{R+T-\tau})} \leq C \left[\| u(0,\cdot) \|_{H^1(B_{R+T})} + \sqrt{G(\tau)} \right] e^{C\tau} \tag{42}$$

Exercise 5.2. *Obtain higher order Sobolev estimates as well as pointwise bounds for u! Hint: Use commutation as in Section 2.1.*

5.3 The proof of Theorem 5.1

With these estimates at hand, we can prove Theorem 5.1. The idea is to use the fact that we know how to solve the inhomogeneous equation

$$\Box u = G(x,t)$$

for a given $G(x,t)$ and trivial data using the Duhamel formula.[3] We let

$$\begin{aligned} \Phi : C^\infty \left([0,T] \times \mathbb{R}^d \right) &\to C^\infty \left([0,T] \times \mathbb{R}^d \right) \\ G &\mapsto u \end{aligned} \tag{43}$$

be the map which maps the smooth inhomogeneity to a smooth solution of the inhomogeneous wave equation with trivial data. In view of the energy estimate for any integer $k \geq 1$

$$\| u(\tau,\cdot) \|_{H^k(\mathbb{R}^d)} \leq C \|G\|_{H^{k-1}([0,T]\times\mathbb{R}^d)}$$

this map extends to a bounded map (also denoted by Φ)

$$\Phi : H^k \left([0,T] \times \mathbb{R}^d \right) \to H^{k+1} \left([0,T] \times \mathbb{R}^d \right)$$

We can write the wave equation as

$$u = \Phi \left(\sum b^\mu (x,t)\, \partial_\mu u + c(x,t)\, u + F(x,t) \right) \tag{44}$$

We now show the following (Exercise)

[3] Note that the case of general initial data can be reduced to the case of trivial data by considering the equation for $v = u - f - g \cdot t$.

1. The map $v \mapsto \Phi\left(\sum b^{\mu}(x,t)\,\partial_{\mu} v + c(x,t)\,v + F(x,t)\right)$ is a bounded map from $H^1\left([0,T]\times\mathbb{R}^d\right)$ to $H^1\left([0,T]\times\mathbb{R}^d\right)$
2. The map is a contraction (for sufficiently small T).

By Banach's fixed point theorem there exists a unique fixed point $u \in H^1$ which is our solution to the wave equation. The fact that it is more regular (in fact smooth) is obtained a posteriori from commutation.

5.4 Cauchy Kovalevskaya

Find an alternative proof of Theorem 5.1 using Cauchy Kovalevkaya, Holmgren's theorem and the energy estimate. Hint: Approximate first the data and then the coefficients in the equation by analytic functions.

6 Exercises

1. (Fritz John, Problem 1, Chapter 5.1(c))
   Let S denote a space like hyperplane with equation $t = \gamma x_1$ in xt-space. Show that the Cauchy problem for $\Box u = 0$ with data on S can be reduced to the initial value problem (i.e. posed at $t' = 0$) for the same equation by introducing new independent variables x', t' by the *Lorentz transformation*
   $$x_1' = \frac{x_1 - \gamma c^2 t}{\sqrt{1-\gamma^2 c^2}} \quad , \quad x_2' = x_2 \quad , \quad x_3' = x_3 \quad , \quad t' = \frac{t - \gamma x_1}{\sqrt{1-\gamma^2 c^2}} \, .$$
2. (Fritz John, Problem 1, Chapter 5.1(a))
   (a) Show that for $n = 3$ the general solution of (11) with spherical symmetry about the origin has the form ($r = |x|$)
   $$u(t,r) = \frac{F(r+ct) + G(r-ct)}{r}$$
   with suitable F, G. (Hint: See Week 4!)
   (b) Show that the solution with initial data of the form
   $$u = 0 \quad , \quad u_t = g(r)$$
   with g an even function of r is given by
   $$u(t,r) = \frac{1}{2cr}\int_{r-ct}^{r+ct} \rho g(\rho)\, d\rho \tag{45}$$
   (c) For
   $$g(r) = \begin{cases} 1 & \text{for } 0 < r < a \\ 0 & \text{for } r > a \end{cases} \tag{46}$$
   find u explicitly from (45) in the different regions bounded by the cone $r = a \pm ct$ in $\mathbb{R}^4$. Show that u is discontinuous at $\left(\vec{0}, t = \frac{a}{c}\right)$.

Note that the above exercise illustrates the loss of regularity at the C^k level due to focussing: u is perfectly smooth (in fact trivial) initially.

3. This exercise illustrates that the L^2 energy estimate for the wave equation is very special in the sense that it is generally not possible to control the L^p norm at some later time from the L^p norm of the data if $p \neq 2$. To establish this, we will use Exercise 2, i.e. work with spherically symmetric solutions and initial data $f = 0$ and g supported in a small ball around the origin. Then use that the support spreads to a (large) annulus to establish Littman's theorem: For $d = 3$, $p \neq 2$ and $t \neq 0$ we have

$$\sup_{g \in \mathcal{S}(\mathbb{R}^d) \backslash 0} \frac{\| u_t\,(t)\,\|_{L^p}}{\| u_t\,(0)\,\|_{L^p}} = \infty\,.$$

You can find more on this in Rauch's book, end of Section 4.7.