Partial Differential Equations
Basic Distribution theory

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1 Introduction

We have touched the idea of distributional solutions when we discussed Burger's equation, and also when we discussed Holmgren's theorem. The key in proving the latter was the observation that for a linear partial differential operator $P = \sum_{|\alpha|\leq m} a_\alpha (x) D^\alpha$ and a classical solution of $Pu = w$ in Ω (open subset of $\mathbb{R}^n$) with $D^\beta u = 0$ on $\partial\Omega$ for $|\beta| \leq m-1$, we have the integration by parts formula

$$\int_\Omega w \cdot v \, dx = \int_\Omega \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha (x) v) \cdot u \, dx \tag{1}$$

valid for all $v \in C_0^\infty (\Omega)$. The formula (1) makes sense for u merely continuous or even $u \in L^1_{loc} (\Omega)$ and it is natural to declare $u \in L^1_{loc} (\Omega)$ a "weak" or a "distributional" solution of $Pu = w$ if it satisfies (1) for all $v \in C_0^\infty (\Omega)$.

Generally, with any (real- or) complex valued function $f : \Omega \to \mathbb{C}$, continuous on $\Omega \subset \mathbb{R}^n$ open (or even $u \in L^1_{loc} (\Omega)$), we can associate its integrals against test functions by defining

$$f [\phi] := \int_\Omega f (x) \phi (x) \, dx \tag{2}$$

for $\phi \in \mathcal{D} := C_0^\infty (\Omega)$. Note that

- $f [\phi]$ is a linear functional on the space of test functions $\mathcal{D}$
- $f [\phi]$ is well defined for $f \in L^1_{loc} (\Omega)$
- the definition supports the idea of "smeared averages" from physics: If f is an observable like a velocity or a temperature, you will never be able to determine its value at a point but only averaged over a small interval (finite detector size).
- If f is continuous, then $f [\phi]$ determines f uniquely (so in some sense we don't lose anything by considering $f [\phi]$ instead of f)

- One can differentiate f in the sense of distributions by defining

$$D_k f[\phi] := -f[D_k \phi]$$

which agrees with the usual derivative if $f \in C^1$ by the standard integration by parts formula. Therefore, linear partial differential operators act naturally on the functionals $f[\phi]$.

2 The space $\mathcal{D}'(\Omega)$

We broaden our view further and consider general linear functionals on the space $\mathcal{D}$ of test-functions (of which those arising by integration against an L^1 function, the $f[\phi]$ above, are a particular example). We shall introduce a notion of continuity of such functionals below (which is desirable if we would like to keep the interpretation as physical observables). This notion of continuity ist most easily formulated via sequential continuity.

Definition 2.1. *We say that $\underline{\phi_n \in \mathcal{D} \text{ converges to } \phi \text{ in } \mathcal{D}}$ if*

- *there is a compact set K such that all ϕ_n vanish outside K*
- *there is a $\phi \in \mathcal{D}$ such that for all $\alpha \in \mathbb{N}^d$ we have $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly in x.*

Definition 2.2. *A distribution is a linear functional $\ell : \mathcal{D}(\Omega) \to \mathbb{C}$, which is continuous in the sense that if ϕ_n converges to ϕ in $\mathcal{D}$, the $\ell(\phi_n) \to \ell(\phi)$. The vectorspace of distributions in denoted $\mathcal{D}'(\Omega)$.*

Example 2.3. *Each continuous (or L^1_{loc}) function generates a distribution via (2). Such distributions are called regular distributions. Not every distribution is regular, as the next example shows.*

Example 2.4. *The distribution $\delta_\xi[\phi] = \phi(\xi)$ is not regular. Indeed, the formula $\int dx\, g(x)\,\phi(x) = \phi(\xi)$ would imply that $g \in L^1_{loc}$ vanishes everywhere (modulo a set of measure 0). This example also makes it intuitive to talk about the support of a distribution: If $f[\phi] = g[\phi]$ for all ϕ with support in $\omega \subset \Omega$, we'll say that the two distributions agree in ω.*

The above notion of continuity may be cumbersome to check in practical applications. However, we have the following

Proposition 2.5. *The function $\ell : \mathcal{D}(\Omega) \to \mathbb{C}$ belong to $\mathcal{D}'(\Omega)$ if and only if for every compact subset $K \subset \Omega$ there is an integer $n(K, \ell)$ and a $c \in \mathbb{R}$ such that for all $\phi \in \mathcal{D}(\Omega)$ with support in K we have*

$$|\,\ell[\phi]\,| \leq c\|\phi\|_{C^n} \qquad \text{with } \|\phi\|_{C^n} = \sum_{|\alpha| \leq n} \max_x |\partial^\alpha \phi| \tag{3}$$

Proof. The "if" follows immediately from the estimate (3). For "only if" suppose that (3) was violated for some compact set K. Then we can find for this K a sequence ϕ_n with $\|\phi_n\|_{C^n} = 1$ and $|\ell[\phi_n]| \geq n$ (otherwise the estimate (3) would hold with $c = N$). But then $\psi_n = n^{-1/2}\phi_n$ is a sequence converging to zero in $\mathcal{D}$, while $|\ell[\psi_n]| \geq n^{1/2}$ does not go to zero. Contradiction. □

If there is a c such that (3) holds, ℓ is said to be <u>of order n on K</u>. IF ℓ is of order n on every compact subset $K \subset \Omega$, the ℓ is <u>of order n on Ω</u>.

Example 2.6. *Any regular distribution (Example 2.3) is of order* 0*. The Dirac delta of Example 2.4 is also of order zero.*

Example 2.7. *The principal value distribution*

$$\ell[\phi] := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\phi(x)}{x} = P.V. \int \frac{\phi(x)}{x}$$

is a distribution of order 1 (near 0 at least; away from zero it is order 0*). The proof is an exercise. Hint: Taylor-expand ϕ near* 0 *and use the symmetry of the integral.*

3 Distributional Derivatives

We can define the distributional derivative $D_k f$ as the distribution

$$D_k f[\phi] = -f[D_k\phi] \qquad \text{or more generally} \quad D^\alpha f[\phi] = (-1)^{|\alpha|} f[D^\alpha\phi] \ . \tag{4}$$

You should check that this indeed defines a distribution.

Exercise 3.1. *Compute $D_k\delta_\xi[\phi]$.*

Therefore, we can apply a linear operator P of order m to a distribution $u[\phi]$ via

$$Pu[\phi] = u\left[P^t\phi\right] \ .$$

Below we will be particularly interested in distributional solutions of

$$Pu = \delta_\xi \ . \tag{5}$$

A distribution u satisfying (5) is called a <u>fundamental solution with pole ξ</u> for the operator P.

4 Relation with weak derivatives

If f is a *regular* distribution, i.e.

$$f[\phi] = \int_\Omega \phi(x) f(x)\, dx$$

for some $f \in L^1_{loc}(\Omega)$ then it may be that the distributional derivative is again a regular distribution. In other words, there could be a $g \in L^1_{loc}$ such that

$$D^k f[\phi] := -\int D^k \phi(x) f(x)\,dx = \int \phi(x) g(x)\,dx$$

holds for any ϕ in $\mathcal{D}$. In this case, we say that f has $g = D^k f$ as its weak derivative. Using an argument similar to one already used above, it is easy to show that the weak derivative, if it exists, is unique.

To see that not every function ($\equiv$ regular distribution) has a weak derivative consider the example of the step function $H : \mathbb{R} \to \mathbb{R}$ defined as

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0\,. \end{cases} \tag{6}$$

This is clearly in L^1_{loc} but the distributional derivative is easily seen to be the delta distribution in view of the following computation:

$$D_x H[\phi] = -\int_{-\infty}^{\infty} D_x \phi(x) H(x)\,dx = \int_0^{\infty} -D_x \phi(x) H(x) = \phi(0)\ .$$

Using the notion of a weak derivatives one can define various notions of “weak solutions” to a PDE, which will typically require some number of weak derivatives to exist.

5 Exercises

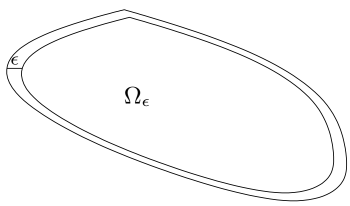
1. Show that the distributional derivative of $\log|x|$ is $P.V.\frac{1}{x}$.
2. (Fritz John, 3.6 (3)) Show that the function

$$u(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 > \xi_1\ ,\ x_2 > \xi_2 \\ 0 & \text{for all other } x_1, x_2 \end{cases} \tag{7}$$

   defines a fundamental solution with pole (ξ_1, ξ_2) of the operator $L = \frac{\partial^2}{\partial x_1 \partial x_2}$ in the $x_1 x_2$-plane.

6 Mollification (see Appendix of Evans)

Let $\Omega \subset \mathbb{R}^n$ be open. The goal is to approximate functions in $L^1_{loc}(\Omega)$ or $C^0(\Omega)$ by functions which are smooth. (Obviously, this can be very convenient in arguments involving lots of integration by parts.)



We write $\Omega_\epsilon \subset \Omega$ for

$$\Omega_\epsilon := \{x \in \Omega \mid dist\,(x, \partial\Omega) > \epsilon\}$$

Let $\eta \in C^\infty(\mathbb{R}^n)$ be the function ("standard mollifier")

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{8}$$

with C chosen such that $\int_{\mathbb{R}^n} \eta \, dx = 1$. For $\epsilon > 0$ we define

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right) \tag{9}$$

Note that η_ϵ is smooth, supported on the closed unit ball $B(0, \epsilon)$ and that $\int \eta_\epsilon(x)\, dx = 1$.

Definition 6.1. *The mollification of an $L^1_{loc}(\Omega)$ function $f : \Omega \to \mathbb{R}$ is defined as*

$$f^\epsilon := \eta_\epsilon \star f \qquad \textit{in } \Omega_\epsilon\,. \tag{10}$$

We claim that on Ω_ϵ, the function f^ϵ is smooth and approximates f in an appropriate sense as $\epsilon \to 0$:

Exercise 6.2. *Show that*

1. $f^\epsilon \in C^\infty(\Omega_\epsilon)$
2. $f^\epsilon \to f$ *almost everywhere as* $\epsilon \to 0$
3. *If $f \in C^0(\Omega)$ (=continuous), then $f^\epsilon \to f$ uniformly on compact subsets of Ω*
4. *If $f \in L^p_{loc}(\Omega)$ with $1 \leq p < \infty$, then $f^\epsilon \to f$ in $L^p_{loc}(\Omega)$*

Here is a guideline:

1. *Write out the difference quotients*

$$\frac{f^\epsilon(x + he_i) + f^\epsilon(x)}{h}$$

with h sufficiently small so that $x + he_i$ is still in Ω_ϵ. Then use uniform convergence to interchange limit and integral.

2. *Start with Lebesgue's Differentiation theorem*

$$\lim_{r \to 0} \frac{1}{vol\,(B(x, r))} \int_{B(x,r)} |f(x) - f(y)|\, dy = 0\,,$$

which holds at almost every x for f in L^1, to estimate $|f^\epsilon(x) - f(x)|$.

3. *Follows from 2. using uniform continuity on compact subsets.*

4. *Apply Hölder to the definition of* f^ϵ *to estalish*

$$\|f^\epsilon\|_{L^p(U)} \leq \|f\|_{L^p(V)} \tag{11}$$

for $U \subset\subset V \subset\subset \Omega$. *To show that* $f^\epsilon \to f$ *in* $U \subset\subset V \subset\subset \Omega$, *suppose* $\delta > 0$ *is given. Choose a continuous* $g \in C^0(V)$ *such that*

$$\|f - g\|_{L^p(V)} < \delta$$

which is possible by density. Then show that $\|f^\epsilon - f\|_{L^p(U)} \leq 3\delta$ *for* ϵ *sufficiently small using the triangle inequality and (11).*

7 More on Distributions (non examinable)

This material is for background only.

7.1 Convergence of Distributions

Definition 7.1. *A sequence of distributions* $\ell_n \in \mathcal{D}'(\Omega)$ *converges to* $\ell \in \mathcal{D}'(\Omega)$ *if and only if for every test function* $\phi \in \mathcal{D}(\Omega)$ *we have*

$$\ell_n[\phi] \to \ell[\phi]$$

with the usual notion of convergence in $\mathbb{C}$. *We will write* $\ell_n \rightharpoonup \ell$ *to denote this convergence and say that* ℓ_n *convergences "weakly" or "in the sense of distributions" to* ℓ.

Exercise 7.2. *Show that the sequence of (regular) distributions* $n^2 e^{inx}$ *converges weakly to zero as* $n \to \infty$.

Exercise 7.3. *Let* $j \in \mathcal{D}(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} j(x)\, d^d x = 1$. *Define* $j_\epsilon(x) = \epsilon^{-d} j\left(\frac{x}{\epsilon}\right)$. *Show that* $j_\epsilon \rightharpoonup \delta_0$.

The previous example is remarkable as it shows that the non-regular delta distribution can be approximated by function in $\mathcal{D}$. In fact, any element in $\mathcal{D}'$ can be approximated in this way: The space $\mathcal{D}$ is dense in $\mathcal{D}'$. This can be used to extend (uniquely) the usual operations of calculus (differentiation, translation, convolution) to $\mathcal{D}'$, which is very useful for PDE.

7.2 Extending Calculus from $\mathcal{D}(\Omega)$ to $\mathcal{D}'(\Omega)$

We will run in a rather informal way through the main ideas of extending various operations of calculus from test functions to distributions. The key is Proposition 2 in the Appendix of Rauch's book, which we repeat below. Remember we already have a notion of convergence in $\mathcal{D}$, Definition 2.1.

Proposition 7.4. *Suppose $L : \mathcal{D}(\Omega_1) \to \mathcal{D}(\Omega_2)$ is a linear, sequentially continuous map. Suppose in addition that there is a linear, sequentially continuous map $L^t : \mathcal{D}(\Omega_2) \to \mathcal{D}(\Omega_1)$ which is the transpose of L in the sense that*

$$\int_{\Omega_2} L\phi \cdot \psi = \int_{\Omega_1} \phi \cdot L^t \psi \quad \textit{holds for all} \quad \phi \in \mathcal{D}(\Omega_1)\,, \ \psi \in \mathcal{D}(\Omega_2)\ .$$

Then the operator L extends to a sequentially continuous map of $\mathcal{D}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ given by

$$L(\ell)[\psi] = \ell\left[L^t\psi\right] \quad \textit{for all} \quad \ell \in \mathcal{D}'(\Omega_1)\,, \ \psi \in \mathcal{D}(\Omega_2)\ .$$

You can find the (very easy) proof in Rauch's book or do it yourself. This simple proposition allows us to define the following operations on distributions

- multiplication of a distribution with a C^∞ function $f \in C^\infty(\Omega)$. Since on test functions the transpose is itself (multiplication by f), we have

$$(f \cdot \ell)[\psi] = \ell[f \cdot \psi]$$

- translation of a distribution (say $\Omega = \mathbb{R}^d$ so that we don't have to keep track of domains). Since $(\tau_y f)(x) = f(x-y)$ on test functions has transpose τ_{-y} we have $\tau_y(\ell)[\psi] = \ell[\tau_{-y}\psi]$ on distributions.
- reflection of a distribution: $\mathfrak{R}(\ell)[\psi] = \ell[\mathfrak{R}\psi]$, as the transpose of $(\mathfrak{R}f)(x) = f(-x)$ on test functions is itself.
- derivative of a distribution (we already did that!)
- convolution of a distribution with a C^∞ function (see below)

The remarkable point of convolving a distribution with a smooth function is that the result is actually a smooth function. You will prove this below. This is very useful and can be used to show that we can approximate any element in $\mathcal{D}'\left(\mathbb{R}^d\right)$ by elements in $\mathcal{D}\left(\mathbb{R}^d\right)$.

Let $\Omega = \mathbb{R}^d$ and $f \in \mathcal{D}\left(\mathbb{R}^d\right)$. For $g \in \mathcal{D}(\mathbb{R}^n)$, the convolution of g with f is defined as

$$(f \star g)(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy = (g \star f)(x)\ . \tag{12}$$

Exercise 7.5. *Show that (on test functions) the transpose of convolution with f is convolution with $\mathfrak{R}f$*

Therefore we define

$$(f \star \ell)[\psi] = \ell[\mathfrak{R}f \star \psi] \tag{13}$$

Exercise 7.6. *Compute $f \star \delta_0$.*

We can now show that the convolution (13) is actually smooth. This is suggested interpreting the convolution (12) itself as the regular distribution $\tau_{-x}\mathfrak{R}f$ acting on a test function g:

$$(f \star g)(x) = \tau_{-x}\mathfrak{R}f\,[g] = g\,[\tau_{-x}\mathfrak{R}f]\ .$$

Now the right hand side (which is a complex-valued function of x) actually makes sense for distributional g providing an alternative definition of the convolution of a distribution with a smooth function. Fortunately, the two "definitions" agree:

Exercise 7.7. *With $\ell \in \mathcal{D}'\left(\mathbb{R}^d\right)$ and $f \in \mathcal{D}\left(\mathbb{R}^d\right)$ given, show that*

- *the function $x \mapsto \ell\,[\tau_{-x}\mathfrak{R}f]$ is $C^\infty\left(\mathbb{R}^d\right)$*
- *$\ell\,[\mathfrak{R}f \star \psi] = \ell\,[\tau_{-x}\mathfrak{R}f] \cdot \psi$ holds for all test functions (where the right hand side is to be understood as integrating the C^∞ function again the test function ψ,)*

The second statement is precisely the statement that the two definitions agree as distributions. You can prove it expressing the right hand side in terms of Riemann sums....

Exercise 7.8.