1 Some general things

1. Make sure you understand how to solve a first order \textit{quasilinear} PDE using the method of characteristics. Review carefully the theory we developed for Burger’s equation.

2. Make sure you know the statements of Cauchy Kovalevskaya (at least the basic case) and Holmgren’s theorem. How do you compute whether a hypersurface (e.g. a hyperplane) is characteristic for a linear partial differential operator? Have examples in mind.

3. Work carefully through chapter 2 of Week 5 again. Go through the maximum principle again and do the relevant exercises of Week 5 if you haven’t done so already.

4. General Elliptic operators: Make sure you are familiar with the spaces $H^1_0(\Omega)$, the notion of a weak solution, the statement of Lax-Milgram and with the statements of the elliptic regularity theorems discussed in class. What is elliptic regularity based on? Review the existence theory of weak solutions for the Laplacian. How does the Lax-Milgram theorem enter? Why is $\Delta^{-1}$ compact? What is the Fredholm alternative and why is it useful for PDE?

5. The wave equation: Work carefully through sections 1-3 of Week 9+10. What is the energy estimate for the wave equation? What is domain of dependence and domain of influence? How do you prove that zero initial data at $t = 0$ implies that the solutions is globally zero? (Do it with Holmgren and the energy estimate. How does global Holmgren work in this case?)
2 Some Mock Exam Problems

Warning: Some of these problems are longer and more difficult than the exam questions but very good practice and encouragement to consult the lecture notes!

1. Consider Burger’s equation
   \[ u_t + uu_x = 0 \]
   with \( C^1 \) initial data \( u(x, 0) = h(x) \) and \( h \) of compact support. First answer the following TRUE/ FALSE questions:
   
   (a) The spatial \( L^p (\mathbb{R}) \) norm of \( u \) is conserved in time for \( 1 \leq p \leq \infty \).
   (b) The solution remains smooth for all times if \( h(x) \) is smooth.
   (c) The solution remains compactly supported in space for all times.
   (d) If the solution blows up then \( u \) itself must blow up.

Finally, prove directly that a solution \( u \in C^1 ([0, T] \times \mathbb{R}) \) arising from initial data \( u(x, 0) = h(x) \) is unique. [Hint: Consider an equation for the difference \( u - v \) of the form \( (u - v)_t = a(t, x)(u - v)_x + b(t, x)(u - v) \). Multiply by \( u - v \), integrate by parts and apply Gronwall’s inequality.]

2. Consider the paraboloid \( P \subset \mathbb{R}^3 \), defined by the set
   \[ z = x^2 + y^2. \]
   
   (a) Is \( P \) characteristic for the (2 + 1)-dimensional wave operator? Is it characteristic for the 3-dimensional Laplacian?
   (b) Suppose a \( C^2 \) solution of the (2 + 1)-dimensional wave equation vanishes along \( P \) with all its first derivatives. Where in \( \mathbb{R}^3 \) is the solution equal to zero?
   (c) Can you write down a partial differential operator \( P \) such that \( P \) is characteristic for \( P \)?

3. Consider Laplace’s equation \( \Delta u = 0 \) in dimension \( d = 2 \).
   
   (a) Prove that \( u(x, y) = \log \left( \sqrt{x^2 + y^2} \right) \) is a fundamental solution with pole at the origin. [Hint: Follow the lecture notes excising an \( \epsilon \)-ball around the origin.]
   (b) Let \( \Omega \) be the half plane \( y > 0 \) with boundary the \( x \)-axis. Let \( u \in C^2 (\Omega) \cap C^0 (\overline{\Omega}) \) be harmonic and satisfy \( \sup_{\Omega} u \leq C \). Prove that
      \[ \sup_{\Omega} u = \sup_{\partial \Omega} u \]
      Hint: Consider \( v = u - \epsilon \log(r + 1) \) where \( r = \sqrt{x^2 + y^2} \). Establish the desired equality for \( v \) by applying the maximum principle on sufficiently large half-balls.

4. This question concerns the wave equation in dimension \( d + 1 \).
(a) Consider a global $C^2$-solution to the inhomogeneous wave equation $\Box \psi = F$ where $F$ is smooth. Fix $T > 0, R > 0$ and consider the past light cone through $(t = T + R, \mathbf{0})$, truncated at $t = 0$ and $t = T$ (as in lectures). Prove the following estimate on the truncated cone for $0 \leq \tau \leq T$:

$$E(\tau) \leq E(0) + C \int_0^\tau d\tilde{\tau}E(\tilde{\tau}) + G(\tau),$$

where

$$E(\tau) = \int_{\{t=\tau\} \times B_{R+T-\tau}} d^d x \left[ \frac{1}{2} (\partial_t \psi)^2 + \frac{1}{2} |\nabla_x \psi|^2 + \psi^2 \right]$$

and

$$G(\tau) = \int_0^\tau d\tilde{\tau} \int_{\{t=\tau\} \times B_{R+T-\tau}} d^d x |F|^2.$$  

Hint: First prove it without the $\psi^2$-term and $C = 1$. Then try to estimate the lower order term from the derivative terms.

(b) Mimic the proof of Gronwall’s inequality to establish the estimate

$$E(\tau) \leq [E(0) + G(\tau)] e^{C\tau}$$

for any $\tau \in [0, T]$.

(c) Let $L > 0$. Show that any solution $\psi \in C^2([0, L] \times \mathbb{R}^d)$ to the non-linear equation $\Box \psi = \psi^6$ arising from initial data $\psi(0, x) = f(x)$, $\psi_t(0, x) = g(x)$ with $f$ and $g$ smooth, is unique.