BSc and MSci EXAMINATIONS (MATHEMATICS)
May-June 2014

M4P41/M5P41
Analytic Methods in Partial Differential Equations
Mastery Question.

We consider the Cauchy problem for the massive wave equation in $n + 1$ dimensions, with data prescribed at $t = 0$, i.e.

$$
\begin{align*}
  u_{tt} - \Delta u + m^2 u &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^n \\
  u(0, x) &= g(x) \\
  u_t(0, x) &= h(x),
\end{align*}
$$

(1)

where $\Delta$ denotes the $n$-dimensional Laplacian. We will assume throughout that $g$ and $h$ are of compact support and smooth. We define the following quantities

$$
\begin{align*}
  E_{\text{kin}}[t] &= \frac{1}{2} \int d^n x \, (\partial_t u)^2 \\
  E_{\text{pot}}[t] &= \frac{1}{2} \int d^n x \, (|\nabla u|^2 + m^2 u^2) \\
  E[t] &= E_{\text{kin}}[t] + E_{\text{pot}}[t] = \frac{1}{2} \int d^n x \, [(\partial_t u)^2 + |\nabla u|^2 + m^2 u^2].
\end{align*}
$$

(2) (3) (4)

(i) For $u$ a $C^2(\mathbb{R} \times [0, \infty))$ solution of (1) prove the conservation of energy formula

$$
E[t] = E[0] \quad \text{for all } t \in [0, \infty).
$$

You may use statements about the domain of influence provided you state them correctly.

(ii) For $n = 1$ and $m = 0$ prove that any $C^2(\mathbb{R} \times [0, \infty))$ solution of (1) is given by d’Alembert’s formula

$$
  u(t, x) = \frac{1}{2} [g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} dy \, h(y).
$$

For $n = 1$ and $m = 0$ show that for all sufficiently large $t$ we must have $E_{\text{kin}}[t] = E_{\text{pot}}[t].$

(iii) Finally, for any $n \geq 1$ and $m > 0$ prove that

$$
\frac{1}{T} \int_0^T dt \, (E_{\text{kin}}[t] - E_{\text{pot}}[t]) \rightarrow 0
$$

HINT: Differentiate $\int d^n x u \partial_t u$ and use (i).