5. (Generalization of seen material, the \(c = 0\) case was discussed in class.) Let \(\Omega\) be a bounded open region of \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\).

(a) Starting from the divergence theorem, derive the following identity for \(u, v \in C^2(\overline{\Omega})\):

\[
\int_\Omega [v (\Delta u + \lambda \cdot u) - u (\Delta v + \lambda \cdot v)] \, d\text{vol} = \int_{\partial \Omega} \left[ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right] \, dS
\]

for any real constant \(\lambda\) and explain the notation \(\frac{\partial}{\partial n}, \, \text{dvol}, \, dS\) appearing above. (4 points)

It obviously suffices to show the identity for \(\lambda = 0\). The latter then follows from Stokes Theorem.

\[
\int_\Omega \nabla^a (\nabla_a u \cdot v) \, d\text{vol} = \int_{\partial \Omega} u^a \nabla_a u \cdot v \, dS
\]

Subtracting from this the same identity with \(v\) and \(u\) interchanged yields the desired identity. Here \(d\text{vol}\) is the volume element of \(\mathbb{R}^n\), \(dS\) the induced volume element on the boundary \(\partial \Omega\) and \(\frac{\partial}{\partial n} = n^a \partial_a\) denotes the unit outward normal to the boundary \(\partial \Omega\).

(b) Define the operator \(L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + c^2\) for a positive constant \(c\). Find all spherically symmetric solutions of \(Lu = 0\) which are smooth for \(r > 0\). (6 points)

Writing the Laplacian in spherical coordinates yields the ODE

\[
u'' + \frac{2}{r} u' + c^2 u = 0
\]

for radial solutions. Writing this as \((u \cdot r)'' + c^2 (u \cdot r) = 0\) we immediately obtain the general solution

\[
u(r) = A \frac{\sin (c \cdot r)}{r} + B \frac{\cos (c \cdot r)}{r}
\]

The \(B\)-branch is singular at the origin, while the \(A\)-branch can be extended to \(r = 0\) to define a smooth solution on all of \(\mathbb{R}^3\).

(c) Prove that

\[K(x, \xi) = - \frac{\cos (c \cdot r)}{4 \pi r}\]

is a fundamental solution of \(L\) with pole \(\xi\). Establish also the formula

\[
u(\xi) = - \int_{\partial \Omega} \left( K(x, \xi) \frac{du}{dn_x}(x) - u(x) \frac{dK(x, \xi)}{dn_x} \right) \, dS_x
\]

for \(u \in C^2(\overline{\Omega})\) a solution of \(Lu = 0\) in \(\Omega\) and \(\xi \in \Omega\). Hint: Use part (a). (10 points)

Consider the region \(\Omega_\rho := \Omega \setminus B(\xi, \rho)\) (\(\Omega\) with a small ball of radius \(\rho\) around \(\xi\) removed). Apply Green’s identity from a) with \(u \in C^2(\overline{\Omega})\) arbitrary and \(v = K(x, \xi)\). Since \(v\) is harmonic in \(\Omega_\rho\) we obtain

\[
\int_{\Omega_\rho} v \cdot (\nabla u ) \, dx = \int_{\partial \Omega} \left[ K(x, \xi) \frac{du}{dn_x}(x) - u(x) \frac{dK(x, \xi)}{dn_x} \right] \, dS_x
\]

\[
+ \int_{\partial B(\xi, \rho)} \left[ K(x, \xi) \frac{du}{dn_x}(x) - u(x) \frac{dK(x, \xi)}{dn_x} \right] \, dS_x
\]

(17)
where we take the outward normal in both cases. We claim that the last term in (17) converges to \( u(\xi) \) in the limit as \( \rho \to 0 \). Indeed, we have (using that the fundamental solution is radial)

\[
\int_{\partial B(\xi, \rho)} K(x, \xi) \frac{du}{dn_x} dS_x = -\frac{\cos(c \cdot \rho)}{4\pi \rho} \int_{\partial B(\xi, \rho)} \frac{du}{dn_x} dS_x
\]

\[
= -\frac{\cos(c \cdot \rho)}{4\pi \rho} \int_{B(\xi, \rho)} \Delta u \, d\text{vol} \to 0
\]

(18)

using that \( \Delta u \) is continuous. On the other hand,

\[
- \int_{\partial B(\xi, \rho)} \frac{dK}{dn_x}(x, \xi) dS_x = \frac{\rho \sin(c \cdot \rho)c + \cos(c \cdot \rho)}{4\pi \rho^2} \int_{\partial B(\xi, \rho)} u \, dS_x
\]

and the right hand side approaches \( u(\xi) \) again by the continuity of \( u \) and the fact that the induced surface element is \( 4\pi \rho^2 \).

We have established

\[
u(\xi) = \int_{\Omega} v \cdot L u \, dx - \int_{\partial \Omega} \left[ K(x, \xi) \frac{du}{dn_x} - u \frac{dK}{dn_x}(x, \xi) \right] dS_x
\]

(19)

which immediately proves the desired formula for the case that \( Lu = 0 \). If \( u \in C^\infty_0(\Omega) \) the formula yields

\[
u(\xi) = \int_{\Omega} v \cdot L u \, dx
\]

(20)

which is precisely the (distributional) formulation of the fact that \( v \) is a fundamental solution (integrate \( Lv = \delta_\xi \) against a test function \( u \in C^\infty_0(\Omega) \)).