Partial Differential Equations 2020
Solutions to CW 1

Gustav Holzegel

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1 Week 1, Problem 4

Suppose $f : [0, \infty) \to \mathbb{R}^+$ is C^1 and satisfies

$$f(t_2) + c \int_{t_1}^{t_2} d\bar{t} f(\bar{t}) \leq C \cdot f(t_1)$$

for all $0 \leq t_1 < t_2 < \infty$ and positive constants c, C. Show that f has to decay exponentially in time.

Applying the estimate with $t_1 = 0$ and any $t = t_2$ we see that $f(t) \leq Cf(0)$ for all $t \geq 0$ and that $\int_0^\infty f(s)ds \leq \frac{C}{c} f(0)$. We can then consider

$$\frac{d}{dt}\left(e^{\frac{c}{C}t}\int_t^\infty f(s)ds\right) = \frac{1}{C}e^{\frac{c}{C}t}\left(c\int_t^\infty f(s)ds - Cf(t)\right) \leq \frac{-e^{\frac{c}{C}t}}{C}\limsup_{t\to\infty} f(t) \leq 0$$

for all t. Since the derivative decreases we have

$$\left(e^{\frac{c}{C}t}\int_t^\infty f(s)ds\right) \leq \int_0^\infty f(s)ds \leq \frac{C}{c}f(0)$$

which is

$$\int_t^\infty f(s)ds \leq \frac{C}{c}e^{-\frac{c}{C}t}f(0)\,.$$

From this we can deduce exponential decay for f. Indeed, consider the sequence of times (t_n) with $t_n = n$. Then by the mean value theorem we find a sequence $(\tilde{t}_n)$ with $\tilde{t}_n \in [t_n, t_{n+1}]$ such that

$$f(\tilde{t}_n) \leq Ce^{-\frac{c}{C}t_n}f(0)\,.$$

Now for any $t \geq 1$ we find the closest $\tilde{t}_n$ with $\tilde{t}_n \leq t$ and apply the estimate between $\tilde{t}_n$ and t to produce

$$f(t) \leq \frac{C^2}{c}e^{-\frac{c}{C}t_n}f(0) \leq \frac{C^2}{c}e^{-\frac{c}{C}(t-2)}f(0) = \frac{C^2}{c}f(0)e^{2\frac{c}{C}}e^{-\frac{c}{C}t}$$

where we have used $t - t_n \leq 2$.

2 Week 1, Problem 7

Give an alternative proof of Gronwall's inequality using a bootstrap argument. HINT: Bootstrap the estimate $\phi(t) \leq (1+\epsilon)A \exp\left(\int_{t_0}^t (1+\epsilon)B(s)ds\right)$ for $\epsilon > 0$.

Let $\epsilon > 0$ and define the set

$$\Omega = \Big\{ t \in [t_0, t_1] \ \mid \phi(s) \leq (1+\epsilon)A \exp\left(\int_{t_0}^s (1+\epsilon)B(x)dx\right) \ \forall s \in [t_0, t] \Big\}.$$

The set Ω is a closed subset of $[t_0, t_1]$. It is non-empty because clearly $t_0 \in \Omega$. We show that Ω is also open in $[t_0, t_1]$. Indeed, if $t \in \Omega$, then $\phi(t)$ obeys

$$\begin{aligned}\phi(t) &\leq A + \int_{t_0}^t B(s)(1+\epsilon)A \exp\left(\int_{t_0}^s (1+\epsilon)B(x)dx\right) ds \\ &\leq A \exp\left(\int_{t_0}^t (1+\epsilon)B(x)dx\right) < (1+\epsilon)A \exp\left(\int_{t_0}^s (1+\epsilon)B(x)dx\right)\end{aligned}$$

and hence by continuity a small neighbourhood of t in $[t_0, t_1]$ is in Ω.

3 Week 2, Problem 1

Consider the PDE $u_t - iu_x = 0$ for $u(t,x) \in \mathbb{C}$. Identifying the (t,x) plane appropriately with $\mathbb{C}$, show that the solution u has to be holomorphic. Conclude that the initial value problem can only be solved for *analytic* data. Compare and contrast with the transport equation.

We identify $z = x + it$ and consider $f(x+it) = u(t,x)$ as a function $f : \mathbb{C} \supset \Omega \to \mathbb{C}$.[1] We claim that the PDE implies that the function f is holomorphic. To see this, note that $\partial_{\bar{z}} f = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial t}\right] u = 0$. This implies that the restriction to $t = 0$ is necessarily a real analytic function.

We now show that the problem can actually be solved for any real-analytic data $u(x, t=0) = h(x)$ with h real analytic for $|x| < R$. Since h is analytic in $|x| < R$, we can fix $r < R$ and conclude bounds on the derivatives: $|h^{(n)}(0)r^n| \leq Mn!$ for a constant M. Therefore, the Taylor series

$$u(z) = \sum_{n=0}^{\infty} h^{(n)}(0)\frac{z^n}{n!}$$

converges for $|z| < r$ and defines a holomorphic function whose restriction to the real axis ($t = 0$) agrees with the data (the power series for h). Since any holomorphic function satisfies the PDE (after the identification $z = x+it$ above), we are done.

[1] Of course identifying $z = t + ix$ is equally possible, $f(t+ix) = u(t,x)$ will then be *antiholomorphic*.

Alternatively, one may show directly the convergence of the formal power series (computed from the data) as follows:

$$\Big|\sum_{j,k}\frac{(-i)^j\,h^{(j+k)}\,(0)}{j!k!}t^jx^k\Big| \le \sum_{j,k}\frac{|h^{(j+k)}\,(0)\,|}{(j+k)!}R^{j+k}\ \frac{(j+k)!}{j!k!}\Big|\frac{t}{R}\Big|^j\Big|\frac{x}{R}\Big|^k .$$

However, we know that $\left|\frac{h^{(j+k)}(0)}{(j+k)!}R^{j+k}\right| < C$ as the function h has a convergent power series at the origin $(0,0)$. Hence

$$... \le C\sum_{m=0}^{\infty}\sum_{j+k=m}\frac{(j+k)!}{j!k!}\Big|\frac{t}{R}\Big|^j\Big|\frac{x}{R}\Big|^k \le C\sum_{m=0}^{\infty}\left(\frac{|t|}{R}+\frac{|x|}{R}\right)^m \le \frac{C}{1-\left(\frac{|t|+|x|}{R}\right)}$$

where we have used the multinomial identity. Convergence in $|t|+|x|<R$ follows. Differentiating the power series one easily checks that the series thus defined is a solution of the PDE.

4 Week 2, Problem 4

(John's PDE book) Show that the function $u\,(x,t)$ defined for $t \ge 0$ by

$$u = -\frac{2}{3}\left(t+\sqrt{3x+t^2}\right) \qquad \textbf{for } 4x+t^2>0$$

$$u = 0 \qquad \textbf{for } 4x+t^2<0$$

is an integral solution of Burger's equation.

We define

$$\Omega_1 = \{(x,t)\mid t\ge 0 \,\text{and}\, t > 2\sqrt{-x}\ \text{ if } x\le 0\ \}$$

$$\Omega_2 = \{(x,t)\mid t\ge 0\,,\ x\le 0 \text{ and}\, t < 2\sqrt{-x}\ \}$$

Note that $u=0$ identically in Ω_2 by definition and that $\Omega_1\cup\Omega_2$ is the upper half plane up to a measure zero set.

An easy computation shows that u is a classical solution both in Ω_1 and Ω_2.

To check it is an integral solution we can either verify the Rankine Hugoniot condition or do it from first principles. Let's do the latter here.

We need to show that

$$\int_0^\infty dt\int_{-\infty}^{\infty}dxuv_t + \int_0^\infty dt\int_{-\infty}^{\infty}dx\ \frac{u^2}{2}v_x + \int_{-\infty}^{\infty}u(x,0)vdx\Big|_{t=0} = 0 \qquad (1)$$

holds for any test function $v:\mathbb{R}\times[0,\infty)\to\mathbb{R}$.

By Fubini we can interchange the integration in t and x. We split the first integral as

$$\boxed{1} := \int_{-\infty}^{\infty}dx\int_0^\infty dt\ uv_t = \int_{-\infty}^{0}dx\int_{2\sqrt{-x}}^{\infty}dt\ uv_t + \int_0^\infty dx\int_0^\infty dt\ uv_t \qquad (2)$$

Integrating by parts produces (recalling $u(x,0) = 0$ for $x < 0$)

$$\boxed{1} = -\int_{\Omega_1} dtdx \; u_t v - \int_{-\infty}^{0} dx \; 2\sqrt{-x} \; v\left(x, 2\sqrt{-x}\right) - \int_{-\infty}^{\infty} u(x,0) v dx\Big|_{t=0}$$

Similarly for the second integral in (1) we have

$$\begin{aligned}\boxed{2} := \int_0^\infty dt \int_{-\infty}^{\infty} dx \; \frac{u^2}{2} v_x &= \int_0^\infty dt \int_{-t^2/4}^{\infty} dx \; \frac{u^2}{2} v_x \\ &= -\int_{\Omega_1} u u_x v - \int_0^\infty dt \frac{t^2}{2} v\left(-\frac{t^4}{2}, t\right) \end{aligned} \tag{3}$$

Adding $\boxed{1}$ and $\boxed{2}$ we find using that u is a classical solution in Ω_1

$$\begin{aligned}\boxed{1} + \boxed{2} = &- \int_{-\infty}^{0} dx \; 2\sqrt{-x} v\left(x, 2\sqrt{-x}\right) - \int_0^\infty dt \frac{t^2}{2} v\left(-\frac{t^2}{4}, t\right) \\ &- \int_{-\infty}^{\infty} u(x,0) v dx\Big|_{t=0} \end{aligned} \tag{4}$$

A simple change of variables $x = -\frac{t^2}{4}$, $dx = -\frac{1}{2} t dt$ shows that the first two integrals on the right hand side cancel and hence proves the result.

5 Week 3, Problem 5

Use the Cauchy-Kowalevskaya theorem to show that the initial value problem

$$u_t u_x = f(t,x,u) \quad , \quad u(0,x) = g(x) \tag{5}$$

has a real analytic solution on a neighbourhood of $(0,0)$ provided that f is real analytic on a neighbourhood of $(0,0,g(0))$ and g is real analytic on a neighbourhood of 0 and $g'(0) \neq 0$. Construct an example with $g'(0) = 0$, $g''(0) \neq 0$, g and f real analytic and such that the initial value problem does not even have a C^1 solution on a neighbourhood of $(0,0)$.

We let $F(t,x,u,u_t,u_x) = u_t u_x - f(t,x,u)$ which is analytic at $(0,0,g(0),a,b)$ for all $a,b \in \mathbb{R}$. To apply the Cauchy-Kovalevskaya Theorem in the form of Theorem 2.11 in the notes, we define $u_t(0,0) = \frac{f(0,0,g(0))}{g'(0)}$ and check that the hypersurface $t = 0$ is non-characteristic at $(0,0)$, i.e.

$$\frac{\partial}{\partial(\partial_t u)} (u_t u_x - f(t,x,u))\Big|_{t=0,x=0,u=g(0)} = \partial_x u(0,0) = g'(0) \neq 0 \, .$$

The CK theorem now guarantees an analytic solution near $(0,0)$.

For the (counter)example if $g'(0) \neq 0$ is not satisfied, consider

$$u_t u_x = e^u \quad u(0,x) = g(x) = x^2$$

which at $(0,0)$ yields the contradiction $0 = 1$ using that u_t is continuous.

6 Week 6, Problem 8

Consider a classical solution u to the one-dimensional heat equation

$$\partial_t u = \partial_{xx} u$$

defined on the half space $\mathbb{R}^2_{t>0}$. Suppose $u = u_x = 0$ holds along the line segment $\{x = 0\} \times (1, 2)$. On what region does u necessarily vanish?

The key is to apply Holmgren's global uniqueness theorem (due to F. John). We claim that u vanishes in the set $(-\infty, \infty) \times [1, 2]$. By continuity it suffices to show vanishing in $(-\infty, \infty) \times (1, 2)$. We furthermore focus on showing vanishing in $(0, \infty) \times (1, 2)$ (because vanishing in $(-\infty, 0) \times (1, 2)$ can be shown entirely analogously using a reflection). Fix a point $(\bar{x}, \bar{t})$ with $\bar{x} \in (0, \infty)$ and $\bar{t} \in (1, 2)$. Let $\delta > 0$ be such that $1 < \bar{t} - \delta < \bar{t} + \delta < 2$. Consider the map $\sigma : [0, 1] \times [\bar{t} - \delta, \bar{t} + \delta] \to \mathbb{R}^2$ given by

$$\sigma(\lambda, t) = \left(t, (\bar{x} + 1)\lambda \cos^2 \left(\frac{t - \bar{t}}{\delta} \frac{\pi}{2} \right) \right)$$

This defines a continuous one-parameter family of hypersurfaces (curves) Σ_λ (the image of $\sigma(\lambda, (\bar{t} - \delta, \bar{t} + \delta))$) as in the assumptions of the global Holmgren theorem. In particular, one easily checks that all these hypersurfaces are non-characteristic for the heat equation (compute $(\partial_t - \partial_{xx})H^2\Big|_{H=0} = -2$ with $H = x - (\bar{x} + 1)\lambda \cos^2 \left(\frac{t-\bar{t}}{\delta} \frac{\pi}{2} \right)$) and clearly $(\bar{t}, \bar{x})$ lies on Σ_λ for $\lambda = \frac{\bar{x}}{\bar{x}+1} < 1$ (choose $t = \bar{t}$). The global Holmgren theorem therefore implies vanishing at $(\bar{t}, \bar{x})$. It follows that u vanishes in $(0, \infty) \times (1, 2)$. Reflecting the procedure above yields vanishing in all of $(-\infty, \infty) \times (1, 2)$ and continuity vanishing in $(-\infty, \infty) \times [1, 2]$. In view of Exercise 6 the result cannot be improved.