Partial Differential Equations 2020
Solutions to CW 2

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1 Week 5, Problem 5

(Fritz John, 4.2 (3)) Prove the weak maximum principle for solutions of the two-dimensional elliptic equation

$$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y = 0$$

where a, b, c, d, e are continuous functions of x and y in $\bar{\Omega}$ and $ac - b^2 > 0$ (ellipticity) as well as $a > 0$ hold (on $\bar{\Omega}$). HINT: Prove it first under the strict condition $Lu > 0$, then use $u + \epsilon v$ for $v = \exp\left[M(x - x_0)^2 + M(y - y_0)^2\right]$ with appropriate M, x_0, y_0.

Assume first the strict inequality $Lu > 0$. Suppose there is an interior maximum at $(x_0, y_0) \in \Omega$. Then $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$ and the Hessian

$$\mathbf{Hess\, u}(\mathbf{x_0}, \mathbf{y_0}) = \begin{pmatrix} u_{xx}(x_0, y_0) & u_{xy}(x_0, y_0) \\ u_{xy}(x_0, y_0) & u_{yy}(x_0, y_0) \end{pmatrix}$$

is non-positive definite (which means its eigenvalues $\lambda_i \leq 0$), i.e.

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) \leq 0 \tag{1}$$

$$u_{xx}u_{yy}(x_0, y_0) - (u_{xy})^2(x_0, y_0) \geq 0 \tag{2}$$

On the other hand, the condition $Lu(x_0, y_0) > 0$ gives

$$au_{xx}(x_0, y_0) + 2bu_{xy}(x_0, y_0) + cu_{yy}(x_0, y_0) > 0\,. \tag{3}$$

We see that $u_{xx}(x_0, y_0) > 0$ is already inconsistent with (1) and (2) so $u_{xx}(x_0, y_0) \leq 0$ and similarly $u_{yy}(x_0, y_0) \leq 0$. But then we can write (3) as

$$a\left(\sqrt{-u_{xx}} - \frac{|b|}{a}\sqrt{-u_{yy}}\right)^2 + \left(c - \frac{b^2}{a}\right)(-u_{yy})$$
$$+2|b|\sqrt{u_{xx}u_{yy}(x_0, y_0)} + 2bu_{xy}(x_0, y_0) < 0\,, \tag{4}$$

which yields the desired contradiction using (2) and the fact that $ac - b^2 > 0$ as well as $a > 0$. So in case of the strict inequality $Lu > 0$ the maximum can only be assumed on the boundary. For the non-strict case look at (this is easier than the hint)

$$v_\epsilon (x, y) = u (x, y) + \epsilon \cdot e^{\lambda x}$$

and compute

$$L v_\epsilon \geq \epsilon e^{\lambda x} \left(a\lambda^2 + 2d\lambda \right) > 0$$

for sufficiently large λ (now fixed) and any $\epsilon > 0$.

Applying the maximum principle to v_ϵ yields

$$\max_{\overline{\Omega}} \left(u (x, y) + \epsilon \cdot e^{\lambda x} \right) = \max_{\partial\Omega} \left(u (x, y) + \epsilon \cdot e^{\lambda x} \right) \, .$$

We deduce

$$\max_{\overline{\Omega}} u (x, y) + \epsilon \min_{\overline{\Omega}} e^{\lambda x} \leq \max_{\partial\Omega} u (x, y) + \epsilon \max_{\partial\Omega} e^{\lambda x}$$

Since Ω is bounded $\max_{\overline{\Omega}} e^{\lambda x}$ is bounded and we can let $\epsilon \to 0$ to obtain

$$\max_{\overline{\Omega}} u (x, y) \leq \max_{\partial\Omega} u (x, y) \, .$$

The reverse inequality is trivial and hence the maximum principle has been established.

2 Week 5, Problem 6

(Harnack's inequality) Let $u \in C^2$ for $|x| < a$ and $u \in C^0$ for $|x| \leq a$. Let also $u \geq 0$ and $\Delta u = 0$ hold for $|x| < a$ (in other words, u is a *non-negative* harmonic function). Show that for $|\xi| < a$

$$\frac{a^{n-2} (a - |\xi|)}{(a + |\xi|)^{n-1}} u (0) \leq u (\xi) \leq \frac{a^{n-2} (a + |\xi|)}{(a - |\xi|)^{n-1}} u (0)$$

Discuss the meaning of this estimate. What can you say for arbitrary regions?

We use the formula proven in lectures

$$u(\xi) = \int_{|x|=a} \frac{a^{d-2}}{a^{d-1}\omega_d} \frac{a^2 - |\xi|^2}{|x - \xi|^d} u(x) dS_x$$

together with the easily established inequalities for x with $|x| = a$ (draw a picture!)

$$a^{d-2} \frac{a - |\xi|}{(a + |\xi|)^{d-1}} \leq a^{d-2} \frac{a^2 - |\xi|^2}{|x - \xi|^d} \leq a^{d-2} \frac{a + |\xi|}{(a - |\xi|)^{d-1}} \, .$$

For the upper bound this produces

$$u(\xi) = \int_{|x|=a} \frac{a^{d-2}}{a^{d-1}\omega_d} \frac{a^2 - |\xi|^2}{|x-\xi|^d} u(x) dS_x$$
$$\leq a^{d-2} \frac{a+|\xi|}{(a-|\xi|)^{d-1}} \int_{|x|=a} \frac{1}{a^{d-1}\omega_d} u(x) dS_x = a^{d-2} \frac{a+|\xi|}{(a-|\xi|)^{d-1}} u(0)\,,$$

with the last bound following from the mean value property of harmonic functions. The lower bound is of course entirely analogous.

Note that this implies that in $B_a(0)$ we have

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)\,.$$

for all $x, y \in B_a(0)$ with the constant depending only on how close x and y are to the boundary of $B_a(0)$. In particular, on any compact subset V of the open ball $B_a(0)$ we can estimate the maximum of u by the minimum of u in V. This is a manifestation of the averaging effects of the Laplacian.

See the revision sheet for arbitrary regions (the idea is of course to cover them with balls!)!

3 Week 6, Problem 3

(Best constant in Poincare's inequality; F. John Chapter 5) Show that if there exists a function $u \in C^2\left(\overline{\Omega}\right)$ vanishing for $\partial\Omega$ for which the quotient

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

reaches its smallest value λ, then u is an eigenfunction to the eigenvalue λ, i.e. $\Delta u + \lambda u = 0$ in Ω. In fact λ must be the smallest eigenvalue belonging to an eigenfunction in $C^2\left(\overline{\Omega}\right)$.

Fix a $\phi \in C_0^\infty(\Omega)$. The function

$$\Psi : (-\epsilon, \epsilon) \ni t \to \frac{\int_\Omega |\nabla(u+t\phi)|^2}{\int_\Omega (u+t\phi)^2}$$

with u the minimiser of the assumptions is well defined for sufficiently small $\epsilon > 0$ (as $u + t\phi$ is non-trivial) and by the general result of Week 5 (on interchanging the derivative and the integral) Ψ is also differentiable. The assumptions of the problem imply that Ψ has a minimum at $t = 0$ and $\Psi(0) = \lambda > 0$. Hence $\frac{d}{dt}\Psi|_{t=0} = 0$ and we compute

$$0 = \frac{d}{dt}\left(\frac{\int_\Omega |\nabla(u+t\phi)|^2}{\int_\Omega (u+t\phi)^2}\right)\Bigg|_{t=0} = 2\frac{\int_\Omega \langle \nabla u, \nabla\phi\rangle}{\int_\Omega u^2} - 2\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}\frac{\int_\Omega u\phi}{\int_\Omega u^2}$$
$$= -2\frac{\int_\Omega (\Delta u + \lambda u)\phi}{\int_\Omega u^2} \qquad (5)$$

where we have integrated by parts using that ϕ vanishes on the boundary of Ω. Since $\phi \in C_0^\infty(\Omega)$ was arbitrary, the right hand side has to be zero for any test function $\phi \in C_0^\infty(\Omega)$ and this implies (since $\Delta u + \lambda u$ is continuous) that

$$\Delta u + \lambda u = 0$$

If we had a smaller eigenvalue, i.e. $\Delta u + \mu u = 0$ for $\mu < \lambda$ and $u \in C^2\left(\bar{\Omega}\right)$ non-trivial, we would have (multiplying the equation by u and integrating by parts) that

$$\int_\Omega |\nabla u|^2 = \mu \int_\Omega u^2$$

which contradicts the fact that the minimum value of the quotient in the exercise is λ.

4 Week 6, Problem 4

Let $n = 3$ and Ω be the ball $|x| < \pi$. Show that a solution u of $\Delta u + u = w(x)$ with vanishing boundary values can only exist if

$$\int_\Omega w(x) \frac{\sin|x|}{|x|} dx = 0$$

An easy computation shows that the homogeneous adjoint problem is given by $\Delta v + v = 0$ and $v = 0$ on $\partial\Omega$. Going to polar coordinates we easily check that

$$\Delta \frac{\sin|x|}{|x|} = \frac{1}{r^2}\partial_r\left(r^2 \partial_r \frac{\sin r}{r}\right) = \frac{1}{r^2}\partial_r\left(-\sin r + r\cos r\right) = -\frac{\sin r}{r}$$

and hence that $\frac{\sin|x|}{|x|}$ is a solution to the homogeneous adjoint problem. (Note $\frac{\sin|x|}{|x|}$ is smooth at the origin and vanishes on the boundary $r = \pi$.) By the Fredholm alternative, the original inhomogeneous problem can only have a solution if the right hand side $w(x)$ is L^2-orthogonal to the kernel of the adjoint problem and this yields precisely the condition stated.

Remark: To see that $\tilde{u}(x) = \frac{\sin|x|}{|x|}$ is a legitimate solution to the adjoint problem, we need to check that $\tilde{u} \in H_0^1(\Omega)$. First note that $\frac{\sin|x|}{|x|}$ is smooth on $\mathbb{R}^3$, including the origin, so that certainly $\tilde{u} \in H^1(\Omega)$. Note that the function vanishes at the boundary $r = \pi$. Let $\tilde{\chi} : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function such that $\tilde{\chi}(r) = 0$ when $r \leq 1$ and $\tilde{\chi}(r) = r$ when $r \geq 2$. Define $\chi_\varepsilon(x) = \varepsilon\tilde{\chi}\left(\frac{|x|}{\varepsilon}\right)$ for $x \neq 0$ and $\chi_\varepsilon(0) = 0$ and note this is a smooth function on $\mathbb{R}^3$.

Set $\tilde{u}_\varepsilon = \chi_\varepsilon(\tilde{u})$. We claim that $\tilde{u}_\varepsilon \in C_c^\infty(\Omega)$ for each $\varepsilon > 0$. As a composition of smooth functions, $\tilde{u}_\varepsilon$ is smooth. Note that by definition $\tilde{u}_\varepsilon = 0$ whenever $|\tilde{u}| < \varepsilon$. Since $\tilde{u}$ vanishes on the boundary and Ω is pre-compact, this will be true in a neighbourhood of $\partial\Omega$. In fact, $\{x \in \Omega : |\tilde{u}| < \varepsilon\} \subset \{\pi - \varepsilon \leq |x| \leq \pi\}$

which directly implies that $\mathrm{supp}(\tilde{u}_\varepsilon) \subset B_{\pi-\varepsilon}(0)$ which is compactly contained in Ω. Finally note that in the region $\{x \in \Omega : \tilde{u}(x) > 2\varepsilon\}$, $\tilde{u}_\varepsilon = \tilde{u}$, so that $\{x \in \Omega : \tilde{u}_\varepsilon(x) \neq \tilde{u}(x)\} \subset \{\pi - 2\varepsilon \leq |x| \leq \pi\}$.

To show that indeed $\tilde{u} \in H_0^1(\Omega)$, it suffices to prove that $\tilde{u}_\varepsilon \to u$ in $H^1(\Omega)$. Using that $|\nabla \chi_\varepsilon| \leq C$ and the fact that $\tilde{u}_\varepsilon$ and $\tilde{u}$ agree everywhere except an annulus of size ε around the boundary, we find

$$\|\tilde{u} - \tilde{u}_\varepsilon\|_{H^1(\Omega)} \leq C \sup_{x \in \Omega \setminus B_{\pi-2\varepsilon}} (|\tilde{u}(x)| + |\nabla \tilde{u}(x)|)\, \mathcal{L}^n(\Omega \setminus B_{\pi-2\varepsilon}) \xrightarrow{\varepsilon \to 0} 0 \quad (6)$$

5 Week 6, Problem 6

Prove the following *basic version of the Banach Alaoglu theorem* (which we used in connection with the difference quotients): Let (u_k) be a bounded sequence in a separable Hilbert space H, i.e. $\|u_k\|_H \leq C$. Then there exists a subsequence which converges weakly in H. Hint: Use the following outline

1. **Pick an ONB (e_k) and use a diagonal argument to show that for a subsequence of the (u_k), denoted $(u_n^{(n)})$ (arising from a Cantor diagonal argument) we have that**

$$\langle u_n^{(n)}, e_k \rangle \to v_k \in \mathbb{R} \quad \textbf{holds for all } e_k.$$

2. **Show that $\sum_{k=1}^{\infty} |v_k|^2 < \infty$ and hence $v = \sum_k v_k e_k \in H$.**

3. **Show that $u_n^{(n)} \rightharpoonup v$.**

Step 1: We use a diagonal argument to show that for a subsequence of the (u_k), denoted $(u_n^{(n)})$ (arising from a Cantor diagonal argument) we have that

$$\langle u_n^{(n)}, e_k \rangle \to v_k \in \mathbb{R}$$

holds for all e_k. Indeed, by Bolzano-Weierstrass we can find a subsequence $(u_n^{(1)})$ of (u_n) such that $\langle u_n^{(1)}, e_1 \rangle \to v_1$ as $n \to \infty$ for some $v_1 \in \mathbb{R}$. Next we choose a subsequence of $(u_n^{(1)})$, denoted $(u_n^{(2)})$ which is such that $\langle u_n^{(2)}, e_2 \rangle \to v_2$ as $n \to \infty$ for some $v_2 \in \mathbb{R}$. Continuing in this way and then choosing finally the diagonal sequence $(u_n^{(n)})$ we have that

$$\langle u_n^{(n)}, e_k \rangle \to v_k$$

for all k.

Step 2: We next show that $\sum_{k=1}^{\infty} |v_k|^2 < \infty$. To do this we compute

$$\sum_{k=1}^{K} |v_k|^2 = \lim_{n\to\infty} \sum_{k=1}^{K} v_k \langle u_n^{(n)}, e_k \rangle = \lim_{n\to\infty} \langle u_n^{(n)}, \sum_{k=1}^{K} v_k e_k \rangle \leq \limsup_{n\to\infty} \|u_n^{(n)}\| \sqrt{\sum_{k=1}^{K} |v_k|^2}$$

and hence

$$\sum_{k=1}^{K} |v_k|^2 \leq C$$

for all K and therefore $v_k \in \ell^2$ and hence that $v = \sum_{k=1}^{\infty} v_k e_k \in H$ (recall the sum converges iff $v_k \in \ell^2$).

Step 3: We show that $u_n^{(n)} \rightharpoonup v$, i.e. that for any $\phi = \sum_k \phi_k e_k \in H$ we have that

$$\lim_{n\to\infty} \langle u_n^{(n)} - v, \phi \rangle = \lim_{n\to\infty} \langle u_n^{(n)} - v, \sum_{k=1}^{\infty} \phi_k e_k \rangle = 0 \,.$$

To see this, let $\epsilon > 0$ be prescribed. We first fix K large (independently of n) such that

$$|\langle u_n^{(n)} - v, \sum_{k=K+1}^{\infty} \phi_k e_k \rangle| \leq \|u_n^{(n)} - v\| \| \sum_{k=K+1}^{\infty} \phi_k e_k \| \leq 2C \sum_{k=K+1}^{\infty} |\phi_k|^2 < \frac{\epsilon}{2} \,.$$

Next we choose n large such that

$$\langle u_n^{(n)} - v, \sum_{k=1}^{K} \phi_k e_k \rangle = \sum_{k=1}^{K} \phi_k \langle u_n^{(n)} - v, e_k \rangle < \frac{\epsilon}{2}$$

This is possible because K has been fixed and every summand in this finite sum goes to zero in view of $\langle u_n^{(n)} - v, e_k \rangle \to 0$ for all k. Adding the two terms finishes the proof.

Remark. The above proof is for a Hilbert space over $\mathbb{R}$ (which covers the spaces used in lectures). The proof for $\mathbb{C}$ is entirely analogous.