Revision Sheet

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1 Some general things

1. Make sure you understand how to solve a first order *quasilinear* PDE using the method of characteristics. Review carefully the theory we developed for Burger's equation.

2. Make sure you know the statements of Cauchy Kovalevskaya (at least the basic case) and Holmgren's theorem. How do you compute whether a hypersurface (e.g. a hyperplane) is characteristic for a linear partial differential operator? Have examples in mind.

3. Work carefully trough chapter 2 of Week 5 again. Go through the maximum principle again and do the relevant exercises of Week 5 if you haven't done so already.

4. Review the properties of mollifiers and the basic idea of the regularity via localisation and mollification.

5. General Elliptic operators: Make sure you are familiar with the spaces $H_0^1(\Omega)$, the notion of a weak solution, the statement of Lax-Milgram and with the statements of the elliptic regularity theorems discussed in class. What is elliptic regularity based on? Review the existence theory of weak solutions for the Laplacian. How does the Lax-Milgram theorem enter? Why is Δ^{-1} compact? What is the Fredholm alternative and why is it useful for PDE?

6. The wave equation: Work carefully through sections 1-2 of Week 9+10. What is the energy estimate for the wave equation? What is domain of dependence and domain of influence? How do you prove that zero initial data at $t = 0$ implies that the solutions is globally zero? (Do it with Holmgren and the energy estimate. How does global Holmgren work in this case?)

2 Some Mock Exam Problems

Warning: Some of these problems are more difficult than the exam questions but very good practice and encouragement to consult the lecture notes!

1. (a) Give an example (each) of a first order PDEs which is
   - linear
   - semi-linear
   - quasi-linear
   - fully non-linear

   (b) Let $\alpha \in \mathbb{R}$ be a constant. Use the method of characteristics to solve the PDE

$$xu_x + yu_y = \alpha u \tag{1}$$
$$u(x,1) = h(x) \tag{2}$$

   in the upper half-plane $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ where $h : \mathbb{R} \to \mathbb{R}$ is a smooth function. Show that the solution is a homogenous function of order α, i.e. $u(\lambda x, \lambda y) = \lambda^\alpha u(x,y)$.

   (c) Prove that for $\alpha < 0$ the only solution which is C^1 and defined in a neighbourhood of the origin is $u = 0$. HINT: Investigate the behaviour of solutions along rays through the origin $t \mapsto (x = at, y = bt)$.

2. Consider Laplace's equation $\Delta u = 0$ in dimension $d = 2$.

   (a) Prove that $u(x,y) = \frac{1}{2\pi} \log\left(\sqrt{x^2+y^2}\right)$ is a fundamental solution with pole at the origin. [Hint: Follow the lecture notes excising an ϵ-ball around the origin.]

   (b) Let Ω be the half plane $y > 0$ with boundary the x-axis. Let $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ be harmonic and satisfy $\sup_{\overline{\Omega}} u \leq C$. Prove that

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

   Hint: Consider $v = u - \epsilon \log(\sqrt{x^2 + (y+1)^2})$. Establish the desired equality for v by applying the maximum principle on sufficiently large half-balls.

   (c) Is the boundedness condition $\sup_{\overline{\Omega}} u \leq C$ in (b) necessary?

   (d) For aficionados (more difficult): What about higher dimensions? Hint: You may want to derive the Green's function for the half plane using the method of images (as we did in lectures for the ball).

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set with smooth boundary and $u \in C^2(\Omega)$ be harmonic.

   (a) Prove that for every $x \in \Omega$ and balls $B(x,\rho) \subset \Omega$ we have

   $$u(x) = \frac{1}{vol(B(x,\rho))} \int_{B(x,\rho)} u(x) d^n x \,. \tag{3}$$

   (b) Let now u be also *non-negative* and some $U \subset\subset \Omega$ fixed. Consider $x, y \in U$, first with $|x - y| < \frac{r}{4}$ where $r := dist(U, \partial\Omega)$. Prove that

   $$\frac{1}{2^n} u(y) \leq u(x) \leq 2^n u(y) \,.$$

   Hint: Use (a).

   (c) With the assumptions as in (b) deduce that for any $x, y \in U$ we have

   $$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

   for a constant depending only on U. (This is Harnack's inequality, compare also Week 5, Problem 6.) Interpretation?

4. Let

   $$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a^{ij}(x) \partial_j u\right) + \sum_{i=1}^{n} b^i(x) \partial_i u + c(x) u$$

   with $a^{ij}, b^i, c \in C^1(\Omega) \cap L^\infty(\Omega)$ and L uniformly elliptic.

   (a) Write down the associated bilinear form for L.

   (b) Let $f \in L^2(\Omega)$. Define what it means for $u \in H^1(\Omega)$ to satisfy $Lu = f$ weakly in Ω.

   (c) Assume $u \in H^1(\Omega)$ satisfy $Lu = f$ weakly in Ω and let $\mathcal{U}$, $\mathcal{V}$ be open sets with $\mathcal{U} \subset\subset \mathcal{V} \subset\subset \Omega$. Prove the estimate

   $$\int_{\mathcal{U}} |\nabla u|^2 \leq C \int_{\mathcal{V}} |u|^2 + C \int_{\mathcal{V}} f^2 \tag{4}$$

   Hint: Choose a $v = \xi^2 u$ with ξ an appropriate cut-off function in the weak formulation.

5. Do the mastery questions from the previous two years!