Measure and Integration: Example Sheet 8

Fall 2016 [G. Holzegel]

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1 Absolute Continuity I

a) Prove that if $F : [a, b] \to \mathbb{R}$ is Lipschitz, then it is absolutely continuous.

b) Give an example of an absolutely continuous function $F : [a, b] \to \mathbb{R}$ that is not Lipschitz.

2 The Cantor-Lebesgue function revisited one more time

Recall that we constructed the Cantor-Lebesgue function in Question 1 of Sheet 1 as a surjective increasing function $F : \mathfrak{C} \to [0, 1]$. In Question 5 of Sheet 5 we extended it to a continuous function $f : [0, 1] \to [0, 1]$. Here we give an alternative way to construct it (cf. Stein-Shakarchi pp.126-127). We recall the notation

$$\mathfrak{C} = \bigcap_{k=0}^{\infty} C_k$$

and that each C_k consists of 2^k disjoint intervals of length $(1/3)^k$. We construct a sequence of functions (g_n) as follows. We let $g_1, g_2 : [0, 1] \to [0, 1]$ be the continuous increasing piecewise linear functions defined by

$$g_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/2 & \text{if } 1/3 \leq x \leq 2/3, \\ 1 & \text{if } x = 1, \\ \text{linear} & \text{on } C_1, \text{ i.e. in } (0, 1/3) \text{ and } (2/3, 1), \end{cases}$$

$$g_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/4 & \text{if } 1/9 \leq x \leq 2/9, \\ 1/2 & \text{if } 1/3 \leq x \leq 2/3, \\ 3/4 & \text{if } 7/9 \leq x \leq 8/9, \\ 1 & \text{if } x = 1, \\ \text{linear} & \text{on } C_2, \text{ i.e. in } (0, 1/9) \text{ and } (2/9, 1/3) \text{ and } (2/3, 7/9) \text{ and } (8/9, 1). \end{cases} \tag{1}$$

In general, to obtain g_n from g_{n-1} one replaces each strictly increasing linear piece on the 2^{n-1} intervals of C_{n-1} by a piecewise linear function, which assumes the constant average value on the mid-third interval of C_n and is again linear on the remaining pieces (which are precisely the disjoint intervals of C_n).

a) Write down an expression for the general $g_n(x)$. Show that the sequence of functions converges to a continuous limit and that this limit is precisely the Cantor-Lebesgue function $f : [0, 1] \to [0, 1]$ constructed on Example Sheet 3.

b) Verify directly from the definition of absolute continuity that the Cantor-Lebesgue function is not absolutely continuous. (Recall that we have already shown this indirectly since the identity $\int_a^b F'(x)dx = F(b) - F(a)$ fails for the Cantor-Lebesgue function.)

3 Absolute Continuity II (Problem 19 from [SS])

Show that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then

a) f maps sets of measure zero to sets of measure 0.

b) f maps measurable sets to measurable sets.

4 Absolute Continuity III

If F and G are absolutely continuous on $[a, b]$, then so is the product FG and the integration by parts formula

$$\int_a^b (FG' + GF')(x)dx = F(b)G(b) - F(a)G(a)$$

holds.