1 Absolute Continuity I

a) Prove that if \( F : [a, b] \to \mathbb{R} \) is Lipschitz, then it is absolutely continuous.

b) Give an example of an absolutely continuous function \( F : [a, b] \to \mathbb{R} \) that is not Lipschitz.

2 The Cantor-Lebesgue function revisited one more time

Recall that we constructed the Cantor-Lebesgue function in Question 1 of Sheet 1 as a surjective increasing function \( F : \mathcal{C} \to [0, 1] \). In Question 5 of Sheet 5 we extended it to a continuous function \( f : [0, 1] \to [0, 1] \). Here we give an alternative way to construct it (cf. Stein-Shakarchi pp.126-127). We recall the notation

\[
\mathcal{C} = \bigcap_{k=0}^{\infty} C_k
\]

and that each \( C_k \) consists of \( 2^k \) disjoint intervals of length \((1/3)^k\). We construct a sequence of functions \((g_n)\) as follows. We let \( g_1, g_2 : [0, 1] \to [0, 1] \) be the continuous increasing piecewise linear functions defined by

\[
g_1(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1/2 & \text{if } 1/3 \leq x \leq 2/3, \\
1 & \text{if } x = 1, \\
\text{linear on } C_1, \text{ i.e. in } (0, 1/3) \text{ and } (2/3, 1),
\end{cases}
\]

\[
g_2(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1/4 & \text{if } 1/9 \leq x \leq 2/9, \\
1/2 & \text{if } 1/3 \leq x \leq 2/3, \\
3/4 & \text{if } 7/9 \leq x \leq 8/9, \\
1 & \text{if } x = 1, \\
\text{linear on } C_2, \text{ i.e. in } (0, 1/9) \text{ and } (2/9, 1/3) \text{ and } (2/3, 7/9) \text{ and } (8/9, 1).
\end{cases}
\]

(1)

In general, to obtain \( g_n \) from \( g_{n-1} \) one replaces each strictly increasing linear piece on the \( 2^{n-1} \) intervals of \( C_{n-1} \) by a piecewise linear function, which assumes the constant average value on the mid-third interval of \( C_n \) and is again linear on the remaining pieces (which are precisely the disjoint intervals of \( C_n \)).

a) Write down an expression for the general \( g_n(x) \). Show that the sequence of functions converges to a continuous limit and that this limit is precisely the Cantor-Lebesgue function \( f : [0, 1] \to [0, 1] \) constructed on Example Sheet 3.

b) Verify directly from the definition of absolute continuity that the Cantor-Lebesgue function is not absolutely continuous. (Recall that we have already shown this indirectly since the identity \( \int_a^b F'(x)dx = F(b) - F(a) \) fails for the Cantor-Lebesgue function.)
3 Absolute Continuity II (Problem 19 from [SS])

Show that if \( f : [a, b] \to \mathbb{R} \) is absolutely continuous, then

a) \( f \) maps sets of measure zero to sets of measure 0.

b) \( f \) maps measurable sets to measurable sets.

4 Absolute Continuity III

If \( F \) and \( G \) are absolutely continuous on \([a, b]\), then so is the product \( FG \) and the integration by parts formula

\[
\int_a^b (FG'' + GF') (x) \, dx = F(b)G(b) - F(a)G(a)
\]

holds.