Measure and Integration: Example Sheet 7

Fall 2016 [G. Holzegel]

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1 Hardy-Littlewood maximal function I (Exercise 4 in [SS])

a) Prove that if f is integrable on $\mathbb{R}^d$ and not identically zero, then

$$f^\star(x) \geq \frac{c}{|x|^d}$$

holds for some constant $c > 0$ and all $|x| \geq 1$. Conclude that $f^\star$ is not integrable on $\mathbb{R}^d$.

b) Consider an f supported on the unit ball with $\int |f| = 1$. Conclude that

$$m(\{x \mid f^\star(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

holds for some constant $c' > 0$ and all sufficiently small α. Remark: This shows that the weak-type inequality we proved in class, $m(\{x \mid f^\star(x) > \alpha\}) \leq \frac{C}{\alpha}$ for all α, cannot be improved.

2 Hardy-Littlewood maximal function II (Exercise 5 in [SS])

Consider the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2} & \text{if } |x| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

a) Verify that f is integrable.

b) Establish the inequality

$$f^\star(x) \geq \frac{c}{|x|(\log 1/|x|)}$$

for some constant $c > 0$ and all $|x| \leq \frac{1}{2}$. Conclude that the maximal function $f^\star$ is not locally integrable.

3 Points of Lebesgue density

Let $E \subset \mathbb{R}^d$ be a measurable set. We say that x is a point of Lebesgue density of E if

$$\lim_{\substack{m(B) \to 0 \\ x \in B}} \frac{m(B \cap E)}{m(B)} = 1\,.$$

In other words, sufficiently small balls around x are almost entirely covered by E.

a) Prove the following:

   i) Almost every $x \in E$ is a point of Lebesgue density of E

   ii) Almost every $x \notin E$ is not a point of Lebesgue density.

   Hint: This follows immediately from the Lebesgue differentiation theorem.

b) Prove that if a measurable subset $E \subset [0, 1]$ satisfies $m(E \cap I) \geq \alpha m(I)$ for some $\alpha > 0$ and all intervals $I \subset [0, 1]$ then E has measure 1.

4 Functions of bounded variation I

Prove the following: If $f : [a, b] \to \mathbb{R}$ is of bounded variation and continuous then so is the total variation $T_f(a, x)$.

5 Functions of bounded variation II (Exercise 11 of [SS])

Let $a, b > 0$ be positive constant and consider

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{for } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases}$$

a) Prove that f is of bounded variation if $[0, 1]$ if and only if $a > b$.

b) Set $a = b$. Construct for each $\alpha \in (0, 1)$ a function that satisfies the Lipschitz condition of exponent α

$$|f(x) - f(y)| \leq M_\alpha |x - y|^\alpha$$

but which is not of bounded variation. Here M_α is a constant.
Remark: Recall that in lectures we saw that if the above holds with $\alpha = 1$, then f is of bounded variation.
HINT: Estimate $|f(x + h) - f(x)|$ for $h > 0$ in two different ways (one being the mean value theorem, the other being by $C(x + h)^a$. Consider then the cases $x^{a+1} \geq h$ and $x^{a+1} < h$.).