Measure and Integration: Example Sheet 5

Fall 2016 [G. Holzegel]

December 3, 2016

1 A variant of the dominant convergence theorem

a) Left (f_n) and (g_n) be sequences of integrable functions with $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for almost every x and the limits f and g being also integrable. Prove that if $|f_n| \leq g_n$ and $\int g_n \to \int g$, then also $\int f_n \to \int f$. Hint: Mimick the proof of the dominant convergence theorem from lectures.

b) Let (f_n) be a sequence of integrable functions with $f_n(x) \to f(x)$ for almost every x and the limit f being integrable. Prove that

$$\int |f_n - f| \to 0 \quad \text{if and only if} \quad \int |f_n| \to \int |f|\,.$$

2 Implications and (non implications) of integrability

a) Show that if f in integrable on $\mathbb{R}$, then $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is uniformly continuous.

b) Show that there exists a non-negative continuous function f on $\mathbb{R}$ such that f is integrable on $\mathbb{R}$, yet $\limsup_{x\to\infty} f(x) = +\infty$. Hint: Construct a continuous version of $f_n = n \cdot \chi_{[n, n+\frac{1}{n^3}]}$.

3 Computation of Integrals

Show that

$$\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k = n!$$

Hint: You might want to look up the Gamma function.

4 Invariance properties of the Lebesgue integral

This question looks at the invariance properties of the Lebesgue integral (Stein-Shakarchi, page73-74).

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Define the translation

$$f_h(x) := f(x-h) \quad \text{for some } h \in \mathbb{R}^d\,.$$

a) Prove that if f is integrable, then so is f_h and

$$\int_{\mathbb{R}^d} f(x-h)\,dx = \int_{\mathbb{R}^d} f(x)\,dx\,.$$

Hint: First show this for the characteristic function $f = \chi_E$, then for simple functions. Finally use the approximation theorem from lectures and the monotone convergence theorem.

b) Prove that if f is integrable, then so are $f(\delta x)$ for $\delta > 0$ and $f(-x)$ with the relations

$$\delta^d \int_{\mathbb{R}^d} f(\delta x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \text{and} \quad \int_{\mathbb{R}^d} f(-x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx$$

Hint: Proceed exactly as in a).

c) Prove that if f is integrable, then

$$\|f_h - f\|_{L^1} \to 0 \quad \text{as } h \to 0.$$

Hint: Approximate f by a continuous function of compact support g and write $f_h - f = (f_h - g_h) + (g_h - g) + (g - f)$.

d) Prove that if f is integrable, then

$$\|f(\delta x) - f(x)\|_{L^1} \to 0 \quad \text{as } \delta \to 1.$$

5 The Riemann-Lebesgue Lemma

Let $f \in L^1\left(\mathbb{R}^d\right)$. We define the Fourier transform of f as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x \xi} dx\,.$$

a) Proof that $\hat{f}$ is continuous.

b) Prove that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.
Hint: Verify the formula

$$\hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}^d} \left[f(x) - f\left(x - \frac{1}{2} \frac{\xi}{|\xi|^2}\right)\right] e^{-2\pi i x \xi} dx$$

and use Problem 4c above.