Measure and Integration: Example Sheet 4

Fall 2016 [G. Holzegel]

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Problem 1 provides a proof omitted in lectures. Problem 2 is very easy. You may want to wait for Friday?s lecture with Problem 3.

1 Independence of the representation for Lebesgue integral on simple functions

Prove the first item in Proposition 3.1 of the notes. Here is a possible outline (from Stein-Shakarchi):

1. First a helpful Lemma: Prove that given a finite collection of sets $F_1, F_2, ..., F_N$ there exists another collection $F_1^\star, ..., F_M^\star$ with $M = 2^N - 1$ such that
   - $\bigcup_{n=1}^N F_n = \bigcup_{m=1}^M F_m^\star$.
   - The $F_m^\star$ are pairwise disjoint
   - $F_n = \bigcup_{F_m^\star \subset F_n} F_m^\star$ for every n

   Hint: Look at the sets $F_1' \cap F_2' \cap ... \cap F_N'$ with each F_n' being either F_n or $(F_n)^c$.

2. Let $\varphi = \sum_{k=1}^N a_k \chi_{E_k}$ and assume first that the E_k are disjoint (but the a_k not necessarily distinct). Prove that the integral is independent of the representation.

3. Assume now $\varphi = \sum_{k=1}^N a_k \chi_{E_k}$ arbitrary. Use the Lemma in Part 1 to write $\varphi = \sum_{j=1}^M a_j^\star \chi_{E_j^\star}$ for suitable $a_j^\star$ and note that this is a decomposition dealt with in Part 2.

2 Tchebychev Inequality

Suppose $f \geq 0$ and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x \mid f(x) > \alpha\}$ then

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f \,.$$

3 The Borel Cantelli Lemma revisited

a) Let $\sum_{k=1}^\infty a_k(x)$ be a series with each $a_k(x) \geq 0$ measurable. Then

$$\int \sum_{k=1}^\infty a_k(x)\, dx = \sum_{k=1}^\infty \int a_k(x)\, dx \,.$$

In particular, if the right hand side is finite, then $\sum_{k=1}^\infty a_k(x)$ converges for a.e. x.

b) Use Part a) to give an alternative proof of the Borel Cantelli Lemma from Exercise Sheet 2.
Hint: Choose $a_k = \chi_{E_k}$.