Measure and Integration: Example Sheet 4 (Solutions)

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1 Independence of the representation for Lebesgue integral on simple functions

1. Preliminary remark: It is useful to first draw a picture to see what's going on. We prove

Lemma 1.1. *Given a finite collection of sets $F_1, F_2, ..., F_N$ there exists another collection $F_1^\star, ..., F_M^\star$ with $M = 2^N - 1$ such that*

(a) $\bigcup_{n=1}^N F_n = \bigcup_{m=1}^M F_m^\star$.

(b) The $F_m^\star$ are pairwise disjoint

(c) For any fixed F_n ($n \in \{1, ..., N\}$) and $F_m^\star$ ($m \in \{1, ..., 2^N - 1\}$) we have either $F_m^\star \subset F_n$ or $F_m^\star \subset (F_n)^c$.

Note that the Lemma implies the statement on the problem sheet because fixing F_n the inclusion

$$\bigcup_{F_m^\star \subset F_n} F_m^\star \subset F_n$$

holds trivially while assuming $x \notin \bigcup_{F_m^\star \subset F_n} F_m^\star$ and $x \in F_n$ leads to a contradiction: We then would have $x \in F_k^\star \not\subset F_n$ for some k (as x has to be in *some* $F_k^\star$ by (a)) and hence by (c) $x \in F_k^\star \subset (F_n)^c$.

Proof. We use the hint and consider the $2^{N-1} - 1$ sets $F_m^\star$ given by $F_1' \cap \ldots \cap F_n'$ where F_i' is either F_i or $(F_i)^c$ and we are omitting the set where all F_i' are given by the complement (otherwise there would be 2^N sets, obviously). We refer to the expression for $F_m^\star$ as an *expansion* and to the F_i' appearing in the expansion as *the i^{th} entry* below.

The $F_m^\star$ are clearly pairwise disjoint as any pair $F_m^\star$, $F_{\tilde{m}}^\star$ differs in at least one "entry" (one of them being F_i and the other $(F_i)^c$) hence their intersection is empty. This shows (b). Item (c) is also immediate since a given $F_m^\star$ has i^{th}-entry either F_i or $(F_i)^c$ and is hence a subset of either F_i or its complement. Finally, we prove (a). We clearly have $\bigcup_{n=1}^N F_n \supset \bigcup_{m=1}^M F_m^\star$ since being in $F_m^\star$ implies being in at least one F_n (since not all entries in the definition of $F_m^\star$ can be complements). On the other hand, suppose $x \in F_n$. Then we have

$$F_n = \bigcup_{F_i', i \leq N, i \neq n} F_1' \cap \ldots \cap F_n \cap \ldots \cap F_N' \subset \bigcup_{m=1}^M F_m^\star$$

which proves the claim. [To see the last equality note again that the $\supset$-direction is trivial while assuming $x \in F_n$ we must have for any $i \neq n$ either $x \in F_i$ or $x \in (F_i)^c$ and this expansion necessarily appears in the union.] □

2. We consider a representation $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ with the E_k disjoint but the a_k not necessarily distinct. We find its canonical form: Let $a_1', ..., a_m'$ be the *distinct* values of a_k. For each a_m' we consider the sets
$$E_m' = \bigcup_{k \text{ with } a_k = a_m'} E_k \, .$$
The sets E_m' are still disjoint (a set E_k can only appear in the union of one E_m') and we have
$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} = \sum_m a_m' \chi_{E_m'} \, .$$
We now observe
$$\sum_m a_m' m\left(E_m'\right) = \sum_m a_m' \sum_{k \text{ with } a_k = a_m'} m\left(E_k\right) = \sum_m \sum_{k \text{ with } a_k = a_m'} a_k m\left(E_k\right) = \sum_{k=1}^{N} a_k m\left(E_k\right) \, .$$

3. We finally consider an arbitrary representation $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, i.e. the E_k not necessarily disjoint and the a_k not necessarily distinct. The proof consists in finding a representation for φ considered in 2., i.e. $\varphi = \sum_{j=1}^{M} a_j^\star \chi_{E_j^\star}$ with the E_j disjoint and showing that $\sum_k a_k m\left(E_k\right) = \sum_j a_j^\star m\left(E_j^\star\right)$. This indeed establishes independence of the representation because by 2. the second sum is equal to the integral of φ in its canonical representation.

   To find the representation we use the Lemma. Given the collection E_k, we find a pairwise disjoint collection $E_j^\star$ with the properties stated in the Lemma. Now for each a_k we define
$$a_j^\star = \sum_{k \mid E_j^\star \subset E_k} a_k \, ,$$
   that is we are summing over all k such that E_k contains $E_j^\star$ (again, draw a picture!). We have[1]
$$\sum_{j=1}^{M} a_j^\star \chi_{E_j^\star} = \sum_{j=1}^{M} \sum_{k \mid E_j^\star \subset E_k} a_k \chi_{E_j^\star} = \sum_{k=1}^{N} \sum_{j \mid E_j^\star \subset E_k} a_k \chi_{E_j^\star} = \sum_{k=1}^{N} a_k \sum_{j \mid E_j^\star \subset E_k} \chi_{E_j^\star} = \sum_{k=1}^{N} a_k \chi_{E_k} = \varphi$$
   and we observe (using the reasoning of the previous line)
$$\sum_{j=1}^{M} a_j^\star m\left(E_j^\star\right) = \sum_{k=1}^{N} \sum_{j \mid E_j^\star \subset E_k} a_k m\left(E_j^\star\right) = \sum_{k=1}^{N} a_k m\left(E_k\right) \, .$$

2 Tchebychev Inequality

We have
$$\int f \geq \int_{E_\alpha} f \geq \alpha \int_{E_\alpha} 1 = \alpha \cdot m\left(E_\alpha\right)$$
for any $\alpha > 0$.

3 The Borel Cantelli Lemma revisited

a) Define the sequence of measurable functions (f_N) by
$$f_N(x) = \sum_{k=1}^{N} a_k(x)$$

[1] For the second equality, think of summing over all pairs (j,k) with the property that $E_j^\star \subset E_k$.

Since $a_k(x) \geq 0$ the sequence f_N is increasing and non-negative, and also $f_N \to \sum_{k=1}^{\infty} a_k(x)$. The MCT implies that

$$\int \lim_{N\to\infty} f_N(x)dx = \lim_{N\to\infty} \int f_N(x)dx$$

holds in the extended sense. This can be written as

$$\int \sum_{k=1}^{\infty} a_k(x)\ dx = \lim_{N\to\infty} \sum_{k=1}^{N} \int a_k(x)\ dx = \sum_{k=1}^{\infty} \int a_k(x)\ dx\,, \tag{1}$$

which is what appears on the problem sheet. If the right hand side is finite, then so is the left hand side, which implies that $\sum_{k=1}^{\infty} a_k(x)$ is integrable, which in turn implies that $\sum_{k=1}^{\infty} a_k(x)$ is finite almost everywhere, which is equivalent to $\sum_{k=1}^{\infty} a_k(x)$ converging for a.e. x.

b) Recall that the Borel-Cantelli Lemma assumes a countable collection of sets (E_k) with $\sum_{k=1}^{\infty} m(E_k) < \infty$ and concludes that the set of points contained in infinitely many E_k has measure zero. To prove this statement we use the hint and apply the identity (1) with $a_k = \chi_{E_k}$. Then the right hand side is precisely $\sum_{k=1}^{\infty} m(E_k) < \infty$ and from a) we conclude that the sum

$$\sum_{k=1}^{\infty} \chi_{E_k}(x)$$

converges for a.e. $x \in \mathbb{R}^d$, i.e. for $x \in \mathbb{R}^d \setminus \mathcal{N}$ with $\mathcal{N}$ a measure zero set on which the sum diverges. Clearly the sum converges if and only if x is in the complement of the set $\{x \mid x \in E_k \text{ for infinitely many } k\}$. Hence $\mathcal{N} = \{x \mid x \in E_k \text{ for infinitely many } k\}$ and we are done.