1 Independence of the representation for Lebesgue integral on simple functions

1. Preliminary remark: It is useful to first draw a picture to see what’s going on. We prove

Lemma 1.1. Given a finite collection of sets \( F_1, F_2, \ldots, F_N \) there exists another collection \( F_1^*, \ldots, F_M^* \) with \( M = 2^N - 1 \) such that

(a) \( \bigcup_{n=1}^{N} F_n = \bigcup_{m=1}^{M} F_m^* \).
(b) The \( F_m^* \) are pairwise disjoint
(c) For any fixed \( F_n \) (\( n \in \{1, \ldots, N\} \)) and \( F_m^* \) (\( m \in \{1, \ldots, 2^N - 1\} \)) we have either \( F_m^* \subset F_n \) or \( F_m^* \subset (F_n)^c \).

Note that the Lemma implies the statement on the problem sheet because fixing \( F_n \) the inclusion

\[ \bigcup_{F_m^* \subset F_n} F_m^* \subset F_n \]

holds trivially while assuming \( x \notin \bigcup_{F_m^* \subset F_n} F_m^* \) and \( x \in F_n \) leads to a contradiction: We then would have \( x \in F_k^* \not\in F_n \) for some \( k \) (as \( x \) has to be in some \( F_k^* \) by (a)) and hence by (c) \( x \in F_k^* \subset (F_n)^c \).

Proof. We use the hint and consider the \( 2^N - 1 \) sets \( F_m^* \) given by \( F_k^* \cap \ldots \cap F_n^* \) where \( F_k^* \) is either \( F_k \) or \( (F_k)^c \) and we are omitting the set where all \( F_k^* \) are given by the complement (otherwise there would be \( 2^N \) sets, obviously). We refer to the expression for \( F_m^* \) as an expansion and to the \( F_k^* \) appearing in the expansion as the \( k^{th} \) entry below.

The \( F_m^* \) are clearly pairwise disjoint as any pair \( F_m^*, F_n^* \) differs in at least one “entry” (one of them being \( F_k \) and the other \((F_k)^c\)) hence their intersection is empty. This shows (b). Item (c) is also immediate since a given \( F_m^* \) has \( k^{th} \)-entry either \( F_k \) or \( (F_k)^c \) and is hence a subset of either \( F_k \) or its complement. Finally, we prove (a). We clearly have \( \bigcup_{n=1}^{N} F_n \supset \bigcup_{m=1}^{M} F_m^* \) since being in \( F_m^* \) implies being in at least one \( F_n \) (since not all entries in the definition of \( F_m^* \) can be complements). On the other hand, suppose \( x \in F_n \). Then we have

\[ F_n = \bigcup_{F_i \cap \ldots \cap F_n \cap \ldots \cap F_N' \subset \bigcup_{m=1}^{M} F_m^* \} \]

which proves the claim. [To see the last equality note again that the \( \supset \)-direction is trivial while assuming \( x \in F_n \) we must have for any \( i \neq n \) either \( x \in F_i \) or \( x \in (F_i)^c \) and this expansion necessarily appears in the union.]
2. We consider a representation \( \varphi = \sum_{k=1}^{N} a_k \chi_{E_k} \) with the \( E_k \) disjoint but the \( a_k \) not necessarily distinct. We find its canonical form: Let \( a'_1, \ldots, a'_m \) be the distinct values of \( a_k \). For each \( a'_m \), we consider the sets 
\[
E'_m = \bigcup_{k \text{ with } a_k = a'_m} E_k.
\]
The sets \( E'_m \) are still disjoint (a set \( E_k \) can only appear in the union of one \( E'_m \)) and we have
\[
\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} = \sum_{m} a'_m \chi_{E'_m}.
\]

We now observe 
\[
\sum_{m} a'_m m(\varphi) = \sum_{m} a'_m \sum_{k \text{ with } a_k = a'_m} m(E_k) = \sum_{m} \sum_{k \text{ with } a_k = a'_m} a_k m(E_k) = \sum_{k=1}^{N} a_k m(E_k).
\]

3. We finally consider an arbitrary representation \( \varphi = \sum_{k=1}^{N} a_k \chi_{E_k} \), i.e. the \( E_k \) not necessarily disjoint and the \( a_k \) not necessarily distinct. The proof consists in finding a representation for \( \varphi \) considered in 2, i.e. \( \varphi = \sum_{j=1}^{M} a'_j \chi_{E'_j} \) with the \( E_j \) disjoint and showing that \( \sum_{k=1}^{N} a_k m(E_k) = \sum_{j=1}^{M} a'_j m(E'_j) \). This indeed establishes independence of the representation because by 2, the second sum is equal to the integral of \( \varphi \) in its canonical representation.

To find the representation we use the Lemma. Given the collection \( E_k \), we find a pairwise disjoint collection \( E'_j \) with the properties stated in the Lemma. Now for each \( a_k \) we define 
\[
a'_j = \sum_{k \mid E'_j \subset E_k} a_k,
\]
that is we are summing over all \( k \) such that \( E_k \) contains \( E'_j \) (again, draw a picture!). We have\(^1\)
\[
\sum_{j=1}^{M} a'_j \chi_{E'_j} = \sum_{j=1}^{M} \sum_{k \mid E'_j \subset E_k} a_k \chi_{E'_j} = \sum_{k=1}^{N} \sum_{j \mid E'_j \subset E_k} a_k \chi_{E'_j} = \sum_{k=1}^{N} a_k \sum_{j \mid E'_j \subset E_k} \chi_{E'_j} = \sum_{k=1}^{N} a_k \chi_{E_k} = \varphi
\]
and we observe (using the reasoning of the previous line)
\[
\sum_{j=1}^{M} a'_j m(\varphi) = \sum_{j=1}^{M} \sum_{k \mid E'_j \subset E_k} a_k m(E'_j) = \sum_{k=1}^{N} a_k m(E_k).
\]

2 Tchebychev Inequality

We have 
\[
\int f \geq \int_{E'_{\alpha}} f \geq \alpha \int_{E'_{\alpha}} 1 = \alpha \cdot m(E_{\alpha})
\]
for any \( \alpha > 0 \).

3 The Borel Cantelli Lemma revisited

a) Define the sequence of measurable functions \( (f_N) \) by 
\[
f_N(x) = \sum_{k=1}^{N} a_k(x)
\]
\(^1\)For the second equality, think of summing over all pairs \((j,k)\) with the property that \( E'_j \subset E_k \).
Since \( a_k(x) \geq 0 \) the sequence \( f_N \) is increasing and non-negative, and also \( f_N \to \sum_{k=1}^{\infty} a_k(x) \). The MCT implies that
\[
\int \lim_{N \to \infty} f_N(x) \, dx = \lim_{N \to \infty} \int f_N(x) \, dx
\]
holds in the extended sense. This can be written as
\[
\int \sum_{k=1}^{\infty} a_k(x) \, dx = \lim_{N \to \infty} \sum_{k=1}^{N} \int a_k(x) \, dx = \sum_{k=1}^{\infty} \int a_k(x) \, dx,
\]
which is what appears on the problem sheet. If the right hand side is finite, then so is the left hand side, which implies that \( \sum_{k=1}^{\infty} a_k(x) \) is integrable, which in turn implies that \( \sum_{k=1}^{\infty} a_k(x) \) is finite almost everywhere, which is equivalent to \( \sum_{k=1}^{\infty} a_k(x) \) converging for a.e. \( x \).

b) Recall that the Borel-Cantelli Lemma assumes a countable collection of sets \( (E_k) \) with \( \sum_{k=1}^{\infty} m(E_k) < \infty \) and concludes that the set of points contained in infinitely many \( E_k \) has measure zero. To prove this statement we use the hint and apply the identity (1) with \( a_k = \chi_{E_k} \). Then the right hand side is precisely \( \sum_{k=1}^{\infty} m(E_k) < \infty \) and from a) we conclude that the sum
\[
\sum_{k=1}^{\infty} \chi_{E_k}(x)
\]
converges for a.e. \( x \in \mathbb{R}^d \), i.e. for \( x \in \mathbb{R}^d \setminus \mathcal{N} \) with \( \mathcal{N} \) a measure zero set on which the sum diverges. Clearly the sum converges if and only if \( x \) is in the complement of the set \( \{x \mid x \in E_k \text{ for infinitely many } k\} \). Hence \( \mathcal{N} = \{x \mid x \in E_k \text{ for infinitely many } k\} \) and we are done.