Measure and Integration: Example Sheet 5 (Solutions)

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1 A variant of the dominated convergence theorem

a) We start from
$$|f_n(x) - f(x)| \leq g_n(x) + g(x)$$
which holds for a.e. x by the triangle inequality (note the assumptions imply $|f(x)| \leq g(x)$ for a.e. x). After changing g on a set of measure zero, the sequence of functions given by $h_n(x) = g(x) + g_n(x) - |f_n(x) - f(x)|$ is non-negative and we can apply Fatou's Lemma to produce the inequality
$$\liminf_{n\to\infty} \int (g(x) + g_n(x) - |f_n(x) - f(x)|)\, dx \geq \int 2g(x)dx\,.$$
Now since we are assuming $\int g_n \to \int g$ and that $\int g < \infty$ we obtain, just as in the proof in class
$$\limsup_{n\to\infty} \int (|f_n(x) - f(x)|)\, dx \leq 0\,,$$
from which we deduce $\lim_{n\to\infty} \int |f_n(x) - f(x)|\, dx = 0$ and by the triangle inequality that $\int f_n \to \int f$. For the application in part b) we note that we actually proved the stronger statement that $\int |f_n - f| \to 0$.

b) Suppose $\int |f_n - f| \to 0$. By the triangle inequality for the integral and the pointwise reverse triangle inequality we have
$$\Big| \int \Big(|f_n(x)| - |f(x)|\Big) dx \Big| \leq \int \Big| |f_n(x)| - |f(x)| \Big|\, dx \leq \int \Big| f_n(x) - f(x) \Big|\, dx$$
and using the assumption the left hand side goes to zero. It follows that $\int |f_n| \to \int |f|$.

Suppose now $\int |f_n| \to \int |f|$. The idea is to use part a). We have $f_n(x) \to f(x)$ for almost every x and hence $|f_n(x)| \to |f(x)|$ for almost every x. If we set $g_n(x) = |f_n(x)|$ and $g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} |f_n(x)| = |f(x)|$ it is easily checked that all assumptions of part a) are satisfied, in particular $f_n(x) \leq |f_n(x)| = |g_n(x)|$ and $\int |f_n| \to \int |f|$. The stronger statement proven in part a) then implies $\int |f_n - f| \to 0$ as desired.

2 Implications and (non implications) of integrability

a) To show F is uniformly continuous we need to show given $\epsilon > 0$ we can find a δ such that $|F(y) - F(x)| < \epsilon$ holds whenever $|x - y| < \delta$. Given ϵ we find by Proposition 3.5 of the lecture notes (absolute continuity) a δ such that
$$\Big| \int_I f(t)dt \Big| < \epsilon \quad \text{whenever } |m\,(I)\,| < \delta.$$
Hence in particular
$$|F(y) - F(x)| = \Big| \int_x^y f(t)dt \Big| < \epsilon \quad \text{whenever } |y - x| < \delta.$$

b) We first define the following non-negative (piece-wise linear) continuous function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^5(x-n)+n & \text{if } x \in \left[n - \frac{1}{n^4}, n\right], \\ n & \text{if } x \in \left(n, n+\frac{1}{n^3}\right), \\ -n^5(x-(n+\frac{1}{n^3})) & \text{if } x \in \left[n+\frac{1}{n^3}, n+\frac{1}{n^3}+\frac{1}{n^4}\right], \\ 0 & \text{if otherwise.} \end{cases}$$

It's easiest to draw a picture! This function is clearly integrable with $\|f_n\|_{L^1} = \frac{1}{n^3} + \frac{1}{n^2}$. Moreover, for $n \geq 2$ f_m and f_n have disjoint support for $m \neq n$. We define for $n \geq 2$ the function

$$f_N = \sum_{n=2}^{N} f_n$$

which is continuous and integrable. Now observe that the function f defined by $f(x) := \lim_{N\to\infty} f_N(x)$ is

- continuous (indeed for fixed $x \in \mathbb{R}$ there is an N such that $f = f_N$ holds in a neighbourhood of x)
- non-negative (obvious)
- integrable (this follows from the f_N being monotone and the fact that $\int_{\mathbb{R}} f_N = \sum_{n=2}^{N} \frac{1}{n^2} + \frac{1}{n^3}$. Indeed, by MCT we have $\int f = \lim_{N\to\infty} \int f_N < \infty$)
- The sequence $x_m = m$ satisfies $f(x_m) = m$. Hence $\limsup_{x\to\infty} f(x) = +\infty$.

3 Computation of Integrals

We recall first that

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$$

Next we rewrite the desired integral as

$$\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx = \lim_{k\to\infty} \int_0^\infty \chi_{[0,k]} x^n \left(1 - \frac{x}{k}\right)^k dx = \lim_{k\to\infty} \int_0^\infty f_k(x) dx \tag{1}$$

for a sequence of measurable functions f_k defined by

$$f_k(x) := \chi_{[0,k]} x^n \left(1 - \frac{x}{k}\right)^k$$

It is clear that pointwise this function converges to $f(x) = x^n e^{-x}$, so the desired claim would follow if we can interchange the limit and the integral in the last expression of (1). This in turn follows from the MCT: Indeed, the f_k are non-negative (note $f_k(x) = 0$ for $x \geq k$) and also for any x we have $f_k(x) \leq f_{k+1}(x)$ for any k. To see this last claim, fix x and k. If $x \notin (0, k)$, then $f_k(x) = 0$ and the inequality is trivially true. For $0 < x < k$ fixed one shows using elementary calculus that the function $g_x(k) = (1 - x/k)^k$ increases in k.

4 Invariance properties of the Lebesgue integral

a) First, from the translation invariance of the Lebesgue measure (discussed in class) we recall that the set E and the translated set $E_h = \{x + h \mid x \in E\}$ have the same measure $m(E_h) = m(E)$. Since $f_h = \chi_{E_h}$ the identity for the integral follows. Second, by the linearity of the integral the identity then also holds for finite linear combinations of characteristic function, hence for simple functions. Third, an arbitrary non-negative function can be approximated by a strictly increasing sequence of simple functions (φ_n). But then both (φ_n) and $((\varphi_h)_n)$ are increasing sequences of simple functions converging to f and f_h respectively. The desired identity clearly holds for each φ_n and $(\varphi_h)_n$ and the MCT implies that it also holds for f and f_h. Fourth and finally, a general integrable f can be decomposed into its positive and negative part which are both integrable and for which the identity has already been established. Using linearity once more the result follows.

b) Indeed exactly the same as in a).

c) We follow the hint and for $\epsilon > 0$ prescribed we find a continuous function of compact support g with $\|f-g\|_{L^1} < \epsilon$. The assumptions on g imply that $g_h - g$ with $|h| < 1$ is (uniformly) compactly supported and continuous, hence $g_h - g$ is dominated by an integrable function and the DCT implies $\lim_{h\to 0}\int |g(x-h)-g(x)|dx = 0$. Therefore we can find a δ (depending on ϵ) with $\|g_h - g\| < \epsilon$ for $|h| < \delta$. Finally, by part a) we have $\|f_h - g_h\|_{L^1} = \|f-g\|_{L^1} < \epsilon$ and hence

$$\|f_h - f\|_{L^1} \leq 3\epsilon \quad \text{for all } |h| < \delta$$

which establishes that the desired limit exists and is equal to zero.

d) We adapt the proof in c). Since we are interested in the limit $\delta \to 1$ we a priori restrict to $\frac{1}{2} < \delta < 2$. Set $f_\delta(x) = f(\delta x)$, pick g continuous and compactly supported with $\|f-g\|_{L^1} < \epsilon$ and decompose $\|f_\delta - f\|_{L^1} = \|f_\delta - g_\delta\|_{L^1} + \|g_\delta - g\|_{L^1} + \|g-f\|_{L^1}$. We have that $g(\delta x) - g(x)$ is (uniformly) compactly supported and hence $\lim_{\delta\to 1}\int |g(\delta x) - g(x)|dx = 0$ by DCT. We can hence choose η such that $|\delta - 1| < \eta$ implies $\|g_\delta - g\|_{L^1} < \epsilon$. Finally, we have $\|f_\delta - g_\delta\|_{L^1} = \delta^{-d}\|f-g\|_{L^1}$ by part b) so that in total

$$\|f_\delta - f\|_{L^1} \leq \delta^{-d}\epsilon + 2\epsilon \leq 2^d\epsilon + 2\epsilon\,.$$

5 The Riemann-Lebesgue Lemma

[This makes reference to Section 3.5 of the notes (complex valued functions).]

a) To show that $\hat{f}$ is continuous at ξ we show that for any sequence (h_n) with $h_n \to 0$ we have $f(\xi + h_n) \to f(\xi)$. Indeed, given such a sequence (h_n) we have

$$\hat{f}(\xi + h_n) - \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}\left[e^{-2\pi i x h_n} - 1\right]dx\,. \tag{2}$$

Now the integrand on the right hand side defines a sequence of integrable functions dominated by $2|f(x)|$, which is integrable by assumption. Hence when taking the limit $n \to \infty$ of (2), the limit can be interchanged with the integral and we obtain $\lim_{n\to\infty} f(\xi + h_n) = f(\xi)$ as desired, as the integrand converges pointwise to zero a.e.

b) To verify the formula for $\xi \neq 0$, it suffices to observe that for $\xi \neq 0$ fixed, we have using the change of variables (translation) $y = x - \frac{1}{2}\frac{\xi}{|\xi|^2}$

$$\int f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)e^{-2\pi i x\xi}\,dx = \int f(y)\,e^{-2\pi i x y}e^{-\pi i}dy = -\int f(y)\,e^{-2\pi i \xi y}dy\,.$$

To prove the statement about the limit we compute

$$|\hat{f}(\xi)| \leq \frac{1}{2}\left\|f - f_{h=\frac{1}{2}\frac{\xi}{|\xi|^2}}\right\|_{L^1(\mathbb{R}^d)}\,.$$

By question 4c the right hand side goes to zero as $\xi \to \infty$ ($h \to 0$) and hence so does the left hand side.