Measure and Integration: Example Sheet 8 (Solutions)

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1 Absolute Continuity I

a) This is simple. If C is the Lipschitz constant for F we choose $\delta < C^{-1}\epsilon$ if $\epsilon > 0$ is prescribed. Then we have for any collection of disjoint intervals (a_k, b_k), $k = 1, ..., N$ of $[a, b]$ with $\sum_{k=1}^{N}(b_k - a_k) < \delta$ that

$$\sum_{n=1}^{N} |F(b_k) - F(a_k)| \leq \sum_{n=1}^{N} C|b_k - a_k| \leq C\delta < \epsilon \, .$$

b) Pick $f(x) = \sqrt{x}$ on the interval $[0, 1]$. This is clearly not Lipschitz ($\sqrt{y} - \sqrt{0} \leq C(y - 0)$ yields a contradiction for small enough y) but $\sqrt{x}$ is absolutely continuous because the function $\frac{1}{2\sqrt{x}}$ is integrable on $[0, 1]$ and hence by the Fundamental Theorem of Calculus (Theorem 4.7 in the notes)

$$\int_0^x \frac{1}{2\sqrt{y}} dy = \sqrt{x} \, ,$$

so the function $\sqrt{x}$ is absolutely continuous.

2 The Cantor-Lebesgue function revisited one more time

a) There are many ways to write the g_n more or less explicitly. Here is one suggestion. The complement of $\mathfrak{C}_n$ consists of the union of $2^n - 1$ intervals ("the open intervals inbetween the 2^n closed intervals of $\mathfrak{C}_n$") which we order from left to right and denote by I^n_j for $j = 1, ..., 2^n - 1$. On each of these intervals the function g_n is constant, its value being $f\left(I^n_j\right) = \{\frac{j}{2^n}\}$.[1] We can then define

$$g_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, \\ \frac{j}{2^n} & \text{if } x \in I^n_j, \\ \text{linear} & \text{on each interval of } \mathfrak{C}_n \text{ such that } g_n \text{ is continuous} \end{cases} \tag{1}$$

It is clear that each g_n is monotone increasing. Note that on each interval of $\mathfrak{C}_n$ the function g_n increases by $\frac{1}{2^n}$. When going from g_n to g_{n+1} we are replacing in each interval of $\mathfrak{C}_n$ a linear function with slope $\frac{3^n}{2^n}$ on an interval of length $\frac{1}{3^n}$ by a function linear with slope $\frac{3^{n+1}}{2^{n+1}}$ in the first and last third of the interval and constant in the middle. We easily see (draw a picture!)

$$|g_{n+1}(x) - g_n(x)| \leq \frac{1}{2^{n+1}} \tag{2}$$

in each interval of $\mathfrak{C}_n$ and since the $g_{n+1}(x) = g_n(x)$ in the complement of $\mathfrak{C}_n$ the bound (2) holds uniformly for all $x \in [0, 1]$. Therefore, in the telescopic sum

$$g_N(x) = g_1(x) + \sum_{n=1}^{N-1} (g_{n+1}(x) - g_n(x))$$

[1] Note that the I^n_j do not all have the same length!

the sum converges absolutely and uniformly in x we have that $g_N(x)$ converges uniformly to a $g(x)$. Since the limit of a uniformly converging sequence of continuous functions is again continuous, the limit $g(x)$ is continuous.

What is left is to show that g above is precisely the Cantor-Lebesgue function f from Example Sheets 1 and 3. Recall that on Example Sheet 1 we showed that the 2^n intervals constituting $\mathfrak{C}_n$ can be written as

$$C_{n,(a_k)} = \left[\sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{n} a_k 3^{-k} + \frac{1}{3^n}\right]$$

for $(a_k)_{k=1}^n$ a sequence with $a_k \in \{0,2\}$ (of which there are 2^n distinct ones). We first claim that the function g_n is a linear function on each $C_{n,(a_k)}$ with image the interval

$$g_n\left(C_{n,(a_k)}\right) = \left[\sum_{k=1}^{n} b_k 2^{-k}, \sum_{k=1}^{n} b_k 2^{-k} + \frac{1}{2^n}\right]$$

where $b_k = \frac{a_k}{2}$. This is certainly true for $n=1$. Assuming it now for n we observe that each interval of $C_{n+1,(\tilde{a}_k)}$ arises as the left or right third of an interval of $C_{n,(a_k)}$ as follows: The first n entries of $\tilde{a}_k$ agree with those of a_k, and moreover $\tilde{a}_{n+1} = 0$ if it is the left third and $\tilde{a}_{n+1} = 2$ if it is the right third of $C_{n,(a_k)}$. Recalling how g_{n+1} is obtained from g_n, we observe that g_{n+1} maps the left third, i.e. $\left[\sum_{k=1}^n a_k 3^{-k} + \frac{0}{3^{n+1}}, \sum_{k=1}^n a_k 3^{-k} + \frac{0}{3^{n+1}} + \frac{1}{3^{n+1}}\right]$ (linearly) to $\left[\sum_{k=1}^n b_k 2^{-k}, \sum_{k=1}^n b_k 2^{-k} + \frac{1}{2^{n+1}}\right]$, the middle third to the constant $\sum_{k=1}^n b_k 2^{-k} + \frac{1}{2^{n+1}}$ and the right third, which is the interval $\left[\sum_{k=1}^n a_k 3^{-k} + \frac{2}{3^{n+1}}, \sum_{k=1}^n a_k 3^{-k} + \frac{2}{3^{n+1}} + \frac{1}{3^{n+1}}\right]$, (linearly) to the interval

$$\left[\sum_{k=1}^{n} b_k 2^{-k} + \frac{1}{2^{n+1}}, \sum_{k=1}^{n} b_k 2^{-k} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\right] = \left[\sum_{k=1}^{n+1} \tilde{b}_k 2^{-k}, \sum_{k=1}^{n+1} \tilde{b}_k 2^{-k} + \frac{1}{2^{n+1}}\right]$$

This proves the claim. Next consider an arbitrary point $x \in \mathfrak{C}$, which by Sheet 1 has unique representation

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \tag{3}$$

for some sequence (a_k) with entries 0 or 2. Moreover, defining $b_k = \frac{a_k}{2}$ we have (cf. Sheet 1)

$$f(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}.$$

On the other hand, for any n, x lies in the interval $C_{n,(a_k)}$, where with a slight abuse of notation, $(a_k)_{k=1}^n$ denote the first n entires of the infinite sequence (a_k) in (3). By the claim above

$$|g_n(x) - \sum_{k=1}^{n} \frac{b_k}{2^k}| \leq \frac{1}{2^n}.$$

The triangle inequality and an elementary geometric sum allow us to conclude

$$|f(x) - g_n(x)| \leq \sum_{k=n+1}^{\infty} \frac{b_k}{2^k} + \frac{1}{2^n} \leq \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

As this works for any n and $g_n \to g$ we conclude that g and f agree for $x \in \mathfrak{C}$. Now for $x \in \mathfrak{C}^c$ (the complement being in $[0,1]$) we have

$$x \in \bigcup_{n=1}^{\infty} (\mathfrak{C}_n)^c$$

so x lies in one of the open intervals of some $\mathfrak{C}_n$. But both f and g are defined to be constant there with the constant depending on the endpoints of the (open) interval which belong to $\mathfrak{C}$ where f and g have already been shown to agree.

b) We claim that for $\epsilon = 1/2$ the following statement is true: For any $\delta > 0$ there exists a collection of disjoint intervals (a_k, b_k), $k = 1, ..., N$ such that $\sum_{k=1}^{N}(b_k - a_k) < \delta$ but $\sum_{k=1}^{N} |f(b_k) - f(a_k)| \geq 1/2$. To find such a collection of disjoint intervals given $\delta > 0$ we first pick n such that $(\frac{2}{3})^n < \delta$. We then look at the interior of the $N = 2^n$ closed disjoint intervals of $\mathfrak{C}_n \subset \mathfrak{C} \subset [0, 1]$ denoted (a_k, b_k), which, since each has length $\frac{1}{3^n}$ satisfy $\sum_{k=1}^{N}(b_k - a_k) = 2^n \frac{1}{3^n} < \delta$. On the other hand, we have that f changes by $\frac{1}{2^n}$ on each of these intervals, so $f(b_k) - f(a_k) = \frac{1}{2^n}$ and hence $\sum_{k=1}^{N} f(b_k) - f(a_k) = 1$. [Recall that the value of f at the endpoints of each interval of C_n is determined once and for all in the n^{th} step of the construction in a)!]

3 Absolute Continuity II

a) Let $\mathcal{N} \subset [a, b]$ be a set of measure zero. We can assume wlog that a and b are not in $\mathcal{N}$. (If we show $f(\mathcal{N})$ has measure zero for such $\mathcal{N}$ it also follows for $\mathcal{N} \cup \{a\} \cup \{b\}$.)

Let $\epsilon > 0$ be prescribed. Since f is absolutely continuous we can find a δ such that for any collection of disjoint intervals (a_k, b_k), $k = 1, ..., N$ of $[a, b]$ with $\sum_{k=1}^{N}(b_k - a_k) < \delta$ we have

$$\sum_{n=1}^{N} |f(b_k) - f(a_k)| < \epsilon \, .$$

For this $\delta > 0$ we find an open set $\mathcal{U}$ with $\mathcal{N} \subset \mathcal{U} \subset (a, b)$ and $m(\mathcal{U}) < \delta$. We can write $\mathcal{U}$ as a disjoint union of open intervals $(a_k, b_k) \subset (a, b)$, that is $\mathcal{U} = \bigcup_{k=1}^{\infty} (a_k, b_k)$. We have that

$$f(\mathcal{N}) \subset f\left(\bigcup_{k=1}^{\infty} (a_k, b_k)\right) = \bigcup_{k=1}^{\infty} f((a_k, b_k)) \, . \tag{4}$$

If f was monotone, we would have $\bigcup_{k=1}^{\infty} f((a_k, b_k)) = \bigcup_{k=1}^{\infty}(f(a_k), f(b_k))$ and by the absolute continuity

$$\sum_{k=1}^{n} |f(a_k) - f(b_k)| < \epsilon$$

for any n. Hence $\sum_{k=1}^{\infty} |f(a_k) - f(b_k)| < \epsilon$, which shows that $m\left(\bigcup_{k=1}^{\infty}(f(a_k), f(b_k))\right) = 0$ since the argument works for any $\epsilon > 0$. It follows that $f(\mathcal{N})$ is a subset of a measure zero set and hence measurable with measure zero.

If f is not monotone we refine the above argument. We first note that (4) holds replacing the open intervals by closed intervals $[a_k, b_k]$. We then further replace each interval $[a_k, b_k]$ by a smaller interval $\left[\tilde{a}_k, \tilde{b}_k\right] \subset [a_k, b_k]$ such that $f([a_k, b_k]) = \left[f(\tilde{a}_k), f(\tilde{b}_k)\right]$ as follows: If m denotes the minimum and M the maximum of f on $[a_k, b_k]$ (which exist by continuity and compactness) we define $\tilde{a}_k = \min_{x \in [a_k, b_k]}\{x \mid f(x) = m\}$ (the smallest x which assumes the minimum) and $\tilde{b}_k = \max_{x \in [a_k, b_k]}\{x \mid f(x) = M\}$ (the largest x which assumes the maximum). We now have

$$f(\mathcal{N}) \subset \bigcup_{k=1}^{\infty} \left[f(\tilde{a}_k), f(\tilde{b}_k)\right] \, . \tag{5}$$

The sets $\left(\tilde{a}_k, \tilde{b}_k\right)$ are disjoint and $\sum_{k=1}^{n} \left(\tilde{b}_k - \tilde{a}_k\right) < \delta$ holds for any n. Hence by absolute continuity

$$\sum_{k=1}^{\infty} \left(f(\tilde{b}_k) - f(\tilde{a}_k)\right) < \epsilon$$

for any n. Since the left hand side controls the measure of the set on the right hand side of (5) we conclude $m(f(\mathcal{N})) = 0$ as in the previous (monotone) case.

b) Let $E \subset [a,b]$ be a measurable set. By Proposition 2.4 of the notes we know that E is a union of an F_σ set with a set of measure zero,

$$E = \bigcup_{k=1}^{\infty} F_k \cup \mathcal{N}\,,$$

where the F_k are closed and $\mathcal{N}$ is a set of measure zero. Since the F_k are necessarily subsets of $[a,b]$ they are also compact. Since f is continuous the $f(F_k)$ are compact, hence measurable. Using that $f(\mathcal{N})$ is measurable with measure zero by part a) we obtain that $f(E)$ is a countable union of measurable set and is hence measurable.

4 Absolute Continuity III

We first show that FG is also absolutely continuous. Let $\epsilon > 0$ be given. Defining

$$C = \max\left(\max_{x\in[a,b]} F(x), \max_{x\in[a,b]} G(x) \right).$$

and δ to be be the minimum of the two δ's associated with $\tilde{\epsilon} = \frac{\epsilon}{2C}$ in the definition of absolute continuity for F and G respectively. We then have, for any collection of disjoint intervals (a_k, b_k), $k = 1, ..., N$ of $[a,b]$ with $\sum_{k=1}^{N}(b_k - a_k) < \delta$ the inequality

$$\begin{aligned}
\sum_{k=1}^{N} |F(b_k)G(b_k) - F(a_k)G(a_k)| &\leq \sum_{k=1}^{N} |F(b_k)(G(b_k) - G(a_k)) + G(a_k)(F(b_k) - F(a_k))| \\
&\leq C\sum_{k=1}^{N} |G(b_k) - G(a_k)| + C\sum_{k=1}^{N} |F(b_k) - F(a_k)| \\
&\leq \epsilon\,. \qquad (6)
\end{aligned}$$

This establishes absolute continuity of the function FG. Since FG is absolutely continuous the fundamental theorem of calculus applies in the form

$$\int_a^b (FG)'(x)dx = F(b)G(b) - F(a)G(a)\,.$$

All that is left is to show that

$$(FG)'(x) = F'(x)G(x) + F(x)G'(x)$$

holds for a.e. $x \in [a,b]$. We write out the difference quotient

$$\frac{(FG)(x+h) - (FG)(x)}{h} = F(x+h)\frac{G(x+h) - G(x)}{h} + G(x)\frac{F(x+h) - F(x)}{h} \qquad (7)$$

and take the limit $h \to 0$. For almost every x the limit on the left exists and is by definition equal to $(FG)'$ since FG is absolutely continuous. Similarly the limits on the right exists for almost every x and yield $FG' + GF'$ completing the proof.