Measure and Integration: Example Sheet 6 (Solutions)

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1 Modes of convergence

a) The function in i) converges to the zero function uniformly. It has L^1-norm equal to 1 for all n and hence does not converge to the zero function in L^1.

The function in *ii*) converges to zero pointwise but not uniformly. The L^1-norm is equal to 1 for all n and hence the function does not converge to the zero function in L^1.

The function in *iii*) converges to zero pointwise almost everywhere (everywhere except at $x = 0$). The L^1-norm is again equal to 1 for all n so f_n does not converge in L^1 to the zero function.

b) Define the following sequence of functions. We let

- $f_1 = \chi_{[0,1]}$.
- $f_2 = \chi_{[0,1/2]}$ and $f_3 = \chi_{[1/2,1]}$
- $f_4 = \chi_{[0,1/4]}$ and $f_5 = \chi_{[1/4,1/2]}$ and $f_6 = \chi_{[1/2,3/4]}$ and $f_7 = \chi_{[3/4,1]}$
- ...

I leave it to you to find an explicit expression for the general f_n.

It is clear from the definition that f_n converges to zero in L^1. However, f_n does not converge pointwise to 0 for any x. This is because if $f_n(x) \to 0$ for some $x \in [0,1]$ then one could find N such that $|f_n(x)| < \frac{1}{2}$ for any $n \geq N$. However, one can clearly find an n larger than N such that $|f_n(x)| = 1$ holds.

You should compare this with Corollary 3.1 of the notes (existence of a *sub*sequence which converges pointwise).

c) This is a consequence of the Chebychev inequality: For fixed $\epsilon > 0$ we have

$$m\left(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}\right) \leq \frac{1}{\epsilon} \int |f_n(x) - f(x)|$$

and the right hand side goes to zero as $n \to \infty$ proving that $f_n \to f$ in measure.

The example i) in a) converges to zero in measure but not in L^1.

2 Application of Fubini's Theorem

We follow the hint and let $A = (0, \infty) \times [0,1]$ and consider the two-dimensional integral

$$\int_A dxdy f(x,y) := \int_A dxdy \; e^{-sx} \sin(2xy)$$

We first verify that f is integrable over A by applying Tonelli's theorem. Indeed the iterated integral

$$\int_0^1 dy \int_0^\infty dx |f(x,y)| \leq \int_0^1 dy \int_0^\infty dx \; e^{-sx} < \infty$$

is finite for $s > 0$, hence by Tonelli f is integrable over I. By Fubini we can now compute $\int_A f(x,y)$ by either of the iterated integrals

$$I_1 = \int_0^1 dy \int_0^\infty dx \; e^{-sx} \sin(2xy) \quad \text{or} \quad I_2 = \int_0^\infty dx \int_0^1 dy \; e^{-sx} \sin(2xy)$$

and the results will agree, $I_1 = I_2$. We now compute for I_1 using integration by parts

$$\int_0^\infty dx \; e^{-sx} \sin(2xy) = \int_0^\infty \frac{1}{s} e^{-sx} \cos(2xy) 2y = \frac{2y}{s^2} - \int_0^\infty dx \frac{1}{s^2} e^{-sx} \sin(2xy) \, 4y^2$$

and hence

$$\int_0^\infty dx \; e^{-sx} \sin(2xy) = \frac{2y}{s^2} \frac{1}{1 + \frac{4y^2}{s^2}} = \frac{2y}{s^2 + 4y^2}$$

Integrating this in y (using substitution) yields

$$I_1 = \int_0^1 dy \int_0^\infty dx \; e^{-sx} \sin(2xy) = \frac{1}{4} \log\left(1 + \frac{4}{s^2}\right).$$

For I_2 we observe

$$I_2 = \int_0^\infty dx e^{-sx} \left(\frac{1 - \cos(2x)}{2x}\right)$$

and that $1 - \cos(2x) = 1 - \cos(x + x) = 1 - \cos^2 x + \sin^2 x = 2\sin^2 x$.

3 Application of the Dominated Convergence Theorem

We establish the identity at fixed $t \in I$. We first define the difference quotient

$$h_n(t,x) = \frac{f(t + t_n, x) - f(t,x)}{t_n}$$

where (t_n) is any sequence converging to zero with $t_n \neq 0$ for all n and such that $t + t_n \in I$. Clearly $h_n(t,x) \to \partial_t f(t,x)$ for all $x \in E$. Hence $\partial_t f(t,x)$ is a measurable function on E and in view of the assumption $|\partial_t f(t,x)| \leq g(x)$ also integrable. By the mean value theorem

$$h_n(t,x) = f'\left(\tilde{t}, x\right)$$

for a $\tilde{t} \in (t, t + t_n) \subset I$ and hence by assumption $|h_n(t,x)| \leq g(x)$ for sufficiently large n. It follows that the dominant convergence theorem applies to $h_n(t,x)$ producing

$$\lim_{n \to \infty} \int_E h_n(t,x) = \int_E \frac{\partial f}{\partial t}(t,x) < \infty.$$

Finally, observe that

$$\lim_{n \to \infty} \int_E h_n(t,x) = \lim_{n \to \infty} \frac{\int_E f(t + t_n, x) - \int_E f(t,x)}{t_n} = \frac{d}{dt} \int_E f(t,x) \, dx,$$

the last step following since we proved the result for any sequence (t_n) converging to zero.

4 The Gamma function

We fix $0 < t_0 < 1$ and apply the previous problem 3 with $I = (t_0, \infty)$ and $E = (0, \infty)$. Clearly f defined by $f(t,x) = e^{-tx}$ is in $L^1(E)$ for any $t \in (t_0, \infty)$ and the derivative satisfies

$$|\partial_t f(t,x)| = e^{-tx} x \leq g(x) := e^{-t_0 x} x \quad \text{for } x \in E.$$

and g is clearly integrable. Problem 3 now implies

$$\frac{d}{dt}\int_0^\infty e^{-tx}dx = \int_0^\infty e^{-tx}\,(-x)\,dx = -\frac{1}{t^2}$$

holds for $t \in (t_0, \infty)$. In particular

$$\int_0^\infty e^{-x}x^1 dx = 1\,,$$

which is the desired identity for $n = 1$. We can now use induction to establish the formula

$$\int_0^\infty e^{-tx}x^n\ dx = \frac{n!}{t^{n+1}} \tag{1}$$

which when evaluated at $t = 1$ produces the desired identity. Indeed, we established this formula for $n = 1$. Assuming it holds for $n-1$ with $n \geq 2$ we differentiate the formula

$$\int_0^\infty e^{-tx}x^{n-1}\ dx = \frac{(n-1)!}{t^n}$$

with respect to t by applying once more Problem 3 with the same I and E as above but with f replaced by $e^{-tx}x^{n-1}$ (whose partial derivative is dominated by the integrable function $e^{-t_0x}x^n$). This establishes (1).

Remark: Note that the result itself can also be proven using integration by parts and induction. The above solution avoids the integration by parts formula.

5 Another Application of Fubini

The fact that $|f(x) - f(y)|$ is integrable on the unit square means that we can apply Fubini's theorem and conclude that the iterated integral is finite:

$$\int_0^1 dx \int_0^1 dy|f(x) - f(y)| < \infty$$

from which we conclude that

$$\int_0^1 dy|f(x) - f(y)| < \infty$$

for almost every $x \in (0, 1)$. Let x_0 be such an x. Then

$$\int_0^1 dy|f(x_0) - f(y)| < \infty\,.$$

From this we conclude by the triangle inequality that

$$\int_0^1 dy|f(y)|dy \leq \int_0^1 dy|f(y) - f(x_0)|dy + 1 \cdot |f(x_0)| < \infty\,, \tag{2}$$

with the last inequality following from the fact that f is finite-valued (hence $|f(x_0)| < \infty$). Inequality (2) is the desired statement that f in integrable on $[0, 1]$.