Measure and Integration: Example Sheet 7 (Solutions)

Fall 2016 [G. Holzegel]

March 30, 2017

1 Hardy-Littlewood maximal function I

Recall the maximal function is defined by

$$f^\star(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)| dy$$

where we take the sup over all balls containing x. Note that one can take either open or closed balls here. [Indeed, the sup taken over closed balls containing x is clearly bigger or equal than the sup taken over open balls containing x as the closure of any open ball containing x also contains x. Conversely, if the sup taken over closed balls containing x is equal to k, then, given any $\delta > 0$ there is a closed ball containing x with $\frac{1}{m(\bar{B})} \int_{\bar{B}} |f(y)| dy \geq k - \delta$ and by the absolute continuity of the integral an open ball B' with $\bar{B} \subset B'$ with $\frac{1}{m(B')} \int_{B'} |f(y)| dy \geq k - 2\delta$. Since this works for any δ we must have that the sup over the open balls containing x is also greater or equal to k.] For the present question taking the sup over closed balls containing x is more convenient.

a) If f is not zero almost everywhere, we necessarily have

$$\int_{\mathbb{R}^d} |f(y)| dy \geq \delta \tag{1}$$

for some $\delta > 0$ (recall we proved that $\|f\|_{L^1} = 0$ implies f is zero almost everywhere). In particular, there exists some ball $B = \bar{B}(0, R)$ such that

$$\int_B |f(y)| dy \geq \frac{\delta}{2}.$$

[Indeed, assuming $\int_B |f(y)| dy < \frac{\delta}{2}$ for any $B = \bar{B}(0, R)$ we pick $\epsilon = \frac{\delta}{4}$ and a ball $B' = \bar{B}(0, R')$ with $\int_{\mathbb{R}^d \setminus B'} |f(y)| dy < \frac{\delta}{4}$ (cf. Proposition 3.5 in the notes) which is in contradiction with (1).] Now

- for $1 \leq |x| < R$ we choose the ball $B = B(0, R)$ which contains x to get a lower bound on $f^\star(x)$ through $f^\star(x) \geq \frac{\delta}{2k_d R^d} \geq \frac{\delta}{2k_d R^d} \frac{1}{|x|^d}$, where $m(B(0,R)) = k_d R^d$.[1]
- for $|x| \geq R$ we choose the ball $\bar{B}(0, |x|) \supset \bar{B}(0, R)$ which contains x to obtain a lower bound on $f^\star$, namely

$$f^\star(x) \geq \frac{\delta}{2 \cdot k_d |x|^d}.$$

Choosing $c = \min\left(\frac{\delta}{2k_d R^d}, \frac{\delta}{2 \cdot k_d}\right)$ the desired estimate on the maximal function follows. The non-integrability easily follows in polar coordinates (or using the method carried out in Question 2 below).

[1] We did not compute the constant $k_d = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ explicitly in class but $m(B(0,R)) \sim R^d$ follows directly from the scaling property of the Lebesgue measure (which we did prove) and the fact that the unit ball has finite measure.

b) By the estimate from part a) we have

$$\{x \mid f^\star(x) > \alpha\} \supset \{x \mid 1 \le |x| \le \frac{c^{\frac{1}{d}}}{\alpha^{\frac{1}{d}}}\}$$

for $c = \frac{1}{2k_d}$ since the assumption $\int_{B(0,1)} |f| = 1$ implies $\delta = 1$ and $R = 1$ in part a). Hence for $\alpha < 2^{-d}c$ we have therefore

$$m\left(\{x \mid f^\star(x) > \alpha\}\right) \ge k_d \left(\frac{c}{\alpha} - 1\right) \ge \frac{k_d c}{2} \cdot \frac{1}{\alpha}\,.$$

2 Hardy-Littlewood maximal function II

a) The most straightforward way would be to compute

$$\int_{\mathbb{R}} f(x)\ dx = 2\int_0^{\frac{1}{2}} f(x)dx = 2\int_0^{\frac{1}{2}} \frac{1}{r\,(\log r)^2} dr = 2\int_0^{\frac{1}{2}} dr \frac{d}{dr}\frac{-1}{\log r} = \frac{2}{\log 2}$$

but this requires careful justification of the fundamental theorem of calculus and the limit $r \to 0$ (which you should provide).

Here is a way to do it from first principles. We define the sets

$$E_{2^n} := \left\{x \mid \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}\right\}$$

which are disjoint and whose union is $(0, 1/2]$. We have (by MCT)

$$\int_{\mathbb{R}} f(x)\ dx = 2\int_{(0,1/2]} f(x)\ dx = 2\sum_{n=1}^{\infty}\int_{E_{2^n}} f(x)\ d^d x\,. \tag{2}$$

Now the integral can be estimated by

$$\int_{E_{2^n}} f(x) \le m\left(E_{2^n}\right) 2^{n+1}\frac{1}{n^2(\log 2)^2} = \left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right] 2^{n+1}\frac{1}{n^2(\log 2)^2} = \frac{1}{(\log 2)^2}\frac{1}{n^2}\,,$$

and this is summable in n and hence the right hand side of (2) is finite.

b) For $x = 0$ the inequality holds trivially. For $0 < x \le \frac{1}{2}$, the interval $I = \left[e^{-\frac{1}{x}}, x\right]$ contains x and satisfies $m\left(I\right) < x$. We have by the fundamental theorem of calculus

$$\int_{e^{-\frac{1}{x}}}^{x} f(x)dx = \int_{e^{-\frac{1}{x}}}^{x} \frac{1}{r\,(\log r)^2} dr = \int_{e^{-\frac{1}{x}}}^{x} dr\frac{d}{dr}\frac{-1}{\log r} = -x - \frac{1}{\log(x)} \ge \frac{1}{\log\frac{1}{x}}\left(1 - x\log\left(\frac{1}{x}\right)\right)$$

and since $x\log\left(\frac{1}{x}\right) \le \frac{1}{2}$ in $[0, 1/2]$ we conclude

$$f^\star(x) \ge \frac{1}{x}\int_{e^{-\frac{1}{x}}}^{x} f(x)dx \ge \frac{1}{2x\log(1/x)}\,.$$

Obviously, the same argument works for $-\frac{1}{2} \le x < 0$ thereby proving the desired bound for all $|x| < \frac{1}{2}$.

We finally show that $f^\star$ is not integrable locally around 0 adapting the idea in *a*). Using the notation E_{2^n} from a) and (2) with $f^\star$ replacing f we now establish a lower bound for the integral

$$\int_{E_{2^n}} f^\star(x) \ge m\left(E_{2^n}\right) 2^n\frac{1}{(n+1)(\log 2)} = \left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right] 2^n\frac{1}{(n+1)(\log 2)} = \frac{1}{2\log 2(n+1)}\,.$$

The sum over n now diverges and hence the function $f^\star$ is not integrable over $(0, 1/2)$.

3 Points of Lebesgue density

a) Choosing the characteristic function χ_E for f in the Lebesgue differentiation theorem we obtain the formula

$$\chi_E(x) = \lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B \chi_E = \lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{m(B\cap E)}{m(B)} .$$

for almost every x. Thus for almost every $x \in E$, x is a point of Lebesgue density as the left hand side is 0. For almost every $x \notin E$ we have

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{m(B\cap E)}{m(B)} = 0 \tag{3}$$

so in particular almost every $x \notin E$ is not a point of Lebesgue density.

Remark: Note that almost everywhere is important here. In general, there can be points in E for which $\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{m(B\cap E)}{m(B)}$ takes any value between 0 and 1. The closed unit square for instance has points where this limit is equal to 1/2 and 1/4 (where?) but of course these points form a measure 0 set.

b) By assumption we have for some $\alpha > 0$

$$\frac{m(I\cap E)}{m(I)} \geq \alpha > 0$$

for all intervals I. By part a), in particular (3), we must have

$$\lim_{\substack{m(I)\to 0 \\ x\in I}} \frac{m(I\cap E)}{m(I)} = 0 \tag{4}$$

for almost all $x \in E^c$. If $E \subset [0,1]$ has measure $1-\delta$ for $\delta > 0$, then the set $[0,1]\setminus E \subset E^c$ has measure $\delta > 0$ and hence in view of (4) there are points $x \in E^c$ and intervals I such that

$$\frac{m(I\cap E)}{m(I)} < \frac{\alpha}{2} .$$

Contradiction. [Note that if $[0,1]\setminus E$ had measure 0 we would not be able to conclude that (4) held for any point in $[0,1]\setminus E$ because it only holds up to a measure zero set!]

4 Functions of bounded variation I

To show continuity from the left at $t > a$ with $\epsilon > 0$ prescribed, we find a partition $t_0 = a, \ldots, t_N = t$ with

$$T_f(a,t) \geq \sum_{n=1}^{N} |f(t_i) - f(t_{i-1})| \geq T_f(a,t) - \frac{\epsilon}{2} . \tag{5}$$

Refining this partition if necessary we can assume – using the continuity of f – that t_{N-1} is sufficiently close to t such that $|f(s) - f(t)| < \frac{\epsilon}{2}$ holds for all $s \in (t_{N-1}, t)$. Now for any such s, the refined partition $t_0, \ldots, t_{N-1}, s, t_N = t$ still satisfies (5) and hence for such s

$$T_f(a,s) \geq \sum_{n=1}^{N-1} |f(t_i) - f(t_{i-1})| + |f(s) - f(t_{N-1})| \geq T_f(a,t) - \epsilon ,$$

where we add and subtract $|f(t) - f(s)|$ to verify the last inequality. This show continuity from the left as we have $0 \leq T_f(a,t) - T_f(a,s) \leq \epsilon$ for all $s \in (t_{N-1}, t)$ with $t_{N-1} < t$ depending on $\epsilon > 0$.

To show continuity at t from the right we choose a partition $t_0 = t, t_1 \dots, t_N = b$ such that

$$T_f(t,b) \geq \sum_{n=1}^{N} |f(t_i) - f(t_{i-1})| \geq T_f(t,b) - \frac{\epsilon}{2} \tag{6}$$

By refining the partition if necessary we can assume – using the continuity of f – that the first interval of the partition $(t_0 = t, t_1)$ is sufficiently small so that $|f(t) - f(s)| < \frac{\epsilon}{2}$ holds for all $s \in (t, t_1)$. This fixes t_1 (depending on ϵ). The refined partition $t_0 = t, s, t_1, ..., t_N$ still satisfies (6) and hence

$$T_f(t,b) - \frac{\epsilon}{2} \leq |f(t) - f(s)| + |f(s) - f(t_1)| + \sum_{n=2}^{N-1} |f(t_i) - f(t_{i-1})| \leq T_f(s,b) + \frac{\epsilon}{2}$$

holds for any $s \in (t, t_1)$. The triangle inequalities implies $T_f(t,s) \leq \epsilon$ for all such s and another application of the triangle inequality yields

$$0 \leq T_f(a,s) - T_f(a,t) \leq T(t,s) \leq \epsilon \quad \text{for all } s \in (t, t_1).$$

5 Functions of bounded variation II

a) We first establish that if $a > b$, then f is of BV. We have

$$f'(x) = ax^{a-1} \sin\left(x^{-b}\right) - bx^{a-b-1} \cos\left(x^{-b}\right)$$

for $x \in (0,1)$ and f' is also integrable in $[0,1]$ in view of

$$|f'(x)| \leq a \cdot x^{a-1} + b \cdot x^{a-b-1} \quad \text{and hence} \quad \int_0^1 dx |f'(x)| dx \leq 1 + \frac{b}{a-b}.$$

It follows that for any partition $0 = x_0 < x_1 < ... < x_N = 1$ we have by the FT for the Lebesgue integral

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{N} \left| \int_{x_{i-1}}^{x_i} dt f'(t)\, dt \right| \leq \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} |f'(x)| dx \leq \int_0^1 |f'(x)| dx \leq 1 + \frac{b}{a-b}.$$

We now establish that if $a \leq b$ then f is not of bounded variation. To do this, we construct a sequence of partitions whose variation is unbounded.

For $n \geq 1$ we define $x_n = \left(\frac{2}{\pi n}\right)^{1/b}$. Note that $|f(x_n)| = \left(\frac{2}{\pi n}\right)^{a/b}$ if n is odd, while $f(x_n) = 0$ if n is even. For any $N \geq 1$ we now consider a partition of $[0,1]$ by $N+1$ intervals of the form

$$0 < x_N < x_{N-1} < ... < x_2 < x_1 < 1.$$

The variation of this partition is (dropping the left and right outermost interval)

$$Var_f(\mathcal{P}) \geq \sum_{n=1}^{N-1} |f(x_{n+1}) - f(x_n)| \geq \sum_{\substack{n=1 \\ n \text{ odd}}}^{N-1} \left(\frac{2}{\pi n}\right)^{a/b} + \sum_{\substack{n=1 \\ n \text{ even}}}^{N-1} \left(\frac{2}{\pi(n+1)}\right)^{a/b}$$

Both sums on the right hand side diverge as $N \to \infty$ if $a \leq b$, the case $a = b$ being the borderline case (harmonic series). Hence the total variation of f on $[0,1]$ is not bounded.

b) Let us fix $\alpha \in (0,1)$ and set $b = a$, so $f(x) = x^a \sin x^{-a}$ on $(0,1]$ and $f(0) = 0$. We will choose $a(\alpha)$ such that the α-Lipschitz condition holds. Wlog we fix $y > x$ and set $y = x + h$ with $0 < h \leq 1$ and $x \in [0,1)$.

We estimate $|f(x+h) - f(x)| \leq (x+h)^a + x^a \leq 2(x+h)^a$ for $x \in [0,1)$ and also $|f(x+h) - f(x)| \leq |f'(\tilde{x})| h$ for some $\tilde{x} \in (x, x+h)$ by the mean value theorem. By part a) we estimate f' to obtain $|f(x+h) - f(x)| \leq \frac{2a}{x} h$ for $x \in (0,1]$.

Now if $h \leq x^{a+1}$ then $\frac{1}{x} \leq h^{-\frac{1}{a+1}}$ then $x \neq 0$ and the second estimate yields $|f(x+h) - f(x)| \leq \frac{2a}{x} h \leq 2ah^{1-\frac{1}{a+1}}$. Hence if we set $\alpha := 1 - \frac{1}{a+1} = \frac{a}{a+1}$ we satisfy the desired Lipschitz condition in this range of h. If $h > x^{a+1}$ the first estimate yields the desired Lipschitz condition via (using $h \leq 1$ and $a > 0$)

$$|f(x+h) - f(x)| \leq 2(h^{\frac{1}{a+1}} + h)^a \leq 2(h^{\frac{1}{a+1}} + h^{\frac{1}{a+1}})^a \leq 2 \cdot 2^a h^\alpha.$$