Measure and Integration: Example Sheet 6

Fall 2016 [G. Holzegel]

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1 Modes of convergence

There are several ways in which a sequence of real valued measurable functions (f_n) can converge to a limiting function f: For instance, pointwise, pointwise a.e. or uniformly. In lectures, we also introduced the L^1-norm and hence the notion of $f_n \to f$ in L^1.

a) Discuss the convergence of the following sequences of functions with regard to the aforementioned notions:

   (a) $f_n = \frac{1}{n}\chi_{(0,n)}$

   (b) $f_n = \chi_{(n,n+1)}$

   (c) $f_n = n \cdot \chi_{[0,1/n]}$

b) Give an example of a sequence (f_n) which converges to f in L^1 but not pointwise a.e.
HINT: Construct a sequence of indicator functions travelling over and over through the interval $[0,1]$ with decreasing length.

c) There is a further notion of convergence, which plays an important role in probability: We say that (f_n) is **Cauchy in measure** if for every $\epsilon > 0$ we have

$$m\left(\{x \mid |f_k(x) - f_n(x)| \geq \epsilon\}\right) \to 0 \quad \text{as } k, n \to \infty.$$

We say that (f_n) converges to f in measure if for every $\epsilon > 0$ we have

$$m\left(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}\right) \to 0 \quad \text{as } n \to \infty.$$

Prove that if $f_n \to f$ in L^1, then $f_n \to f$ in measure. Revisit a) to see that the converse does not hold.

2 Applications of Fubini's Theorem

Establish the formula

$$\int_0^\infty e^{-sx}\frac{1}{x}\sin^2 x \, dx = \frac{1}{4}\log\left(1 + \frac{4}{s^2}\right)$$

for $s > 0$ by integrating $e^{-sx}\sin(2xy)$ with respect to x and y.

3 Application of the Dominated Convergence Theorem

Suppose $I \subset \mathbb{R}$ is open and that $f : I \times E \to \mathbb{R}$, $(t,x) \mapsto f(t,x)$ for E a measurable subset of $\mathbb{R}^d$. Suppose f is in $L^1(E)$ for all t, differentiable in t for all x with the derivative satisfying $|\partial_t f(t,x)| \leq g(x)$ for some $g \in L^1(E)$. Then

$$\frac{d}{dt}\int_E f(t,x)\, dx = \int_E \frac{\partial f}{\partial t}(t,x)\, dx < \infty\,.$$

4 The Gamma function

Prove the formula for the Gamma function from the last sheet

$$\int_0^\infty x^n e^{-x}\ dx = n!$$

by differentiating the formula $\int_0^\infty e^{-tx}\ dx = t^{-1}$.

5 Another application of Fubini

Let f be a measurable, finite-valued function on $[0,1]$ and suppose that $|f(x) - f(y)|$ is integrable on the unit square $[0,1] \times [0,1]$. Show that f is integrable on $[0,1]$.