1 Modes of convergence

There are several ways in which a sequence of real valued measurable functions \((f_n)\) can converge to a limiting function \(f\): For instance, pointwise, pointwise a.e. or uniformly. In lectures, we also introduced the \(L^1\)-norm and hence the notion of \(f_n \to f\) in \(L^1\).

a) Discuss the convergence of the following sequences of functions with regard to the aforementioned notions:

(a) \(f_n = \frac{1}{n} \chi_{(0,n)}\)

(b) \(f_n = \chi_{(n,n+1)}\)

(c) \(f_n = n \cdot \chi_{[0,1/n]}\)

b) Give an example of a sequence \((f_n)\) which converges to \(f\) in \(L^1\) but not pointwise a.e.

HINT: Construct a sequence of indicator functions travelling over and over through the interval \([0,1]\) with decreasing length.

c) There is a further notion of convergence, which plays an important role in probability: We say that \((f_n)\) is **Cauchy in measure** if for every \(\epsilon > 0\) we have

\[
m(\{x \mid |f_k(x) - f_n(x)| \geq \epsilon\}) \to 0 \quad \text{as } k, n \to \infty.
\]

We say that \((f_n)\) converges to \(f\) in measure if for every \(\epsilon > 0\) we have

\[
m(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \quad \text{as } n \to \infty.
\]

Prove that if \(f_n \to f\) in \(L^1\), then \(f_n \to f\) in measure. Revisit a) to see that the converse does not hold.

2 Applications of Fubini’s Theorem

Establish the formula

\[
\int_0^\infty e^{-sx} \frac{1}{x} \sin^2 x \, dx = \frac{1}{4} \log \left(1 + \frac{4}{s^2}\right)
\]

for \(s > 0\) by integrating \(e^{-sx} \sin(2xy)\) with respect to \(x\) and \(y\).

3 Application of the Dominated Convergence Theorem

Suppose \(I \subset \mathbb{R}\) is open and that \(f: I \times E \to \mathbb{R}\), \((t,x) \mapsto f(t,x)\) for \(E\) a measurable subset of \(\mathbb{R}^d\). Suppose \(f\) is in \(L^1(E)\) for all \(t\), differentiable in \(t\) for all \(x\) with the derivative satisfying \(|\partial_t f(t,x)| \leq g(x)\) for some \(g \in L^1(E)\). Then

\[
\frac{d}{dt} \int_E f(t,x) \, dx = \int_E \frac{\partial f}{\partial t}(t,x) \, dx < \infty.
\]
4 The Gamma function

Prove the formula for the Gamma function from the last sheet
\[ \int_0^\infty x^n e^{-x} \, dx = n! \]
by differentiating the formula \( \int_0^\infty e^{-tx} \, dx = t^{-1} \).

5 Another application of Fubini

Let \( f \) be a measurable, finite-valued function on \([0, 1]\) and suppose that \(|f(x) - f(y)|\) is integrable on the unit square \([0, 1] \times [0, 1]\). Show that \( f \) is integrable on \([0, 1]\).