VON NEUMANN LECTURES: MOTIVIC INTEGRATION, TROPICAL GEOMETRY, AND RATIONALITY OF ALGEBRAIC VARIETIES

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Disclaimer: these notes are still under construction, please send me any typos or mistakes you find (j.nicaise@imperial.ac.uk).

1. Lecture 1 (17 October)

In the first part of this course, we will study semialgebraic geometry over the field of Puiseux series. Algebraic geometry over a field F is the study of sets defined by polynomial equations over F. In semialgebraic geometry, we extend our language by adding an extra structure on F that we can use to define a larger class of sets. The most classical example is the case where $F = \mathbb{R}$ and we consider the ordering $\leq$ on $\mathbb{R}$. Semialgebraic sets over $\mathbb{R}$ are then subsets of $\mathbb{R}^n$ that can be defined by polynomial equations and inequalities. In the case where F is the field of Puiseux series, the extra structure we consider is given by the t-adic valuation.

1.1. **The field of Puiseux series.** Let k be an algebraically closed field of characteristic zero. The ring $k[\![t]\!]$ of formal power series over k is defined as

$$k[\![t]\!] = \{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in k \text{ for all } i\}$$

with the obvious operations of addition and multiplication. This is a local ring with maximal ideal (t), since an element $\sum_{i=0}^{\infty} a_i t^i$ in $k[\![t]\!]$ is invertible if and only if the constant term a_0 is different from zero. The residue field of this local ring is equal to k. The ring $k[\![t]\!]$ is domain, and its quotient field is the field of formal Laurent series

$$k(\!(t)\!) = \{\sum_{i=N}^{\infty} a_i t^i \mid N \in \mathbb{Z},\, a_i \in k \text{ for all } i\}.$$

This field is not algebraically closed: the element t does not have a d-th root for any integer $d \geq 2$. It turns out that this is the only obstruction: we obtain an algebraic closure of $k(\!(t)\!)$ by adding a d-th root of t for all $d \geq 2$. Since $k(\!(\sqrt[d]{t})\!)$ is a subfield of $k(\!(\sqrt[e]{t})\!)$ whenever d divides e, we can write this algebraic closure as an increasing union

$$K = \bigcup_{d>0} k(\!(t^{1/d})\!).$$

The field K is called the field of *Puiseux series* over k. The fact that it is algebraically closed was essentially proved by Newton in his study of parameterizations of branches of plane curves; it was rediscovered almost 200 years later by Puiseux. The proof provides an algorithmic method to find roots of polynomials over K by an approximation argument that uses the co-called Newton polygon. An elementary introduction, which also explains the relation with the geometry of plane curves, can be found in Chapter 2 of "Singular points of plane curves" by C.T.C. Wall (2004). Beware that the assumption that k has characteristic zero is essential.

Exercise 1.1.1. Show that, if k has characteristic $p > 0$, then the equation $x^p - tx = 1$ has no solutions in K.

On top of the algebraic structure, the field K carries the *t-adic valuation*

$$v\colon K^* \to \mathbb{Q},\ \sum_{i=N}^{\infty} a_i t^{i/d} \mapsto N/d$$

where N is an integer, d is a positive integer, and the coefficients a_i are elements of k such that $a_N \neq 0$. In other words, the t-adic valuation of an element of K^* is the smallest exponent of t that appears in the t-adic expansion with non-zero coefficient. We extend v to the whole of K by setting $v(0) = \infty$. The following exercise shows that the t-adic valuation v satisfies the usual axioms of a valuation.

Exercise 1.1.2. Show that for all non-zero elements a and b of K, we have

(1) $v(ab) = v(a) + v(b)$;
(2) $v(a + b) \geq \min\{v(a), v(b)\}$.

These axioms imply that

$$R = \{a \in K \mid v(a) \geq 0\}$$

is a subring of K, called the *valuation ring* of v, and that

$$\mathfrak{m} = \{a \in K \mid v(a) > 0\}$$

is an ideal in R. Moreover, K is the quotient field of R (since we can push any element of K into R by multiplying with a large power of an element with positive valuation) and the ideal $\mathfrak{m}$ is the unique maximal ideal of R (since every element a in $R \setminus \mathfrak{m}$ has valuation zero, so that its inverse in K has valuation zero and therefore still lies in R). It follows that R is a local integral domain.

Explicitly, we have

$$R = \bigcup_{d>0} k[\![t^{1/d}]\!]$$

(Puiseux series without negative exponents). The maximal ideal $\mathfrak{m}$ is generated by the elements $t^{1/d}$ with $d > 0$, but not by any finite number of them, so that the ring R is not noetherian. The residue field of R is equal to k: the ring morphism

$$R \to k,\ \sum_{i=0}^{\infty} a_i t^{i/d} \mapsto a_0$$

that maps an element of R to its constant term is surjective, and its kernel is the maximal ideal $\mathfrak{m}$.

An equivalent way to think about (real-valued) valuations is in terms of absolute values: we can transform v into an absolute value $|\cdot|$ on K by setting $|a| = \exp(-v(a))$ for every a in K (with the convention that $\exp(-\infty) = 0$). We can think of R and $\mathfrak{m}$ as the closed, resp. open, unit disk in K: we have $R = \{a \in K \mid |a| \leq 1\}$ and $\mathfrak{m} = \{a \in K \mid |a| < 1\}$.

It follows from Axiom 2 in Exercise 1.1.2 that this absolute value satisfies a stronger version of the triangle inequality, the so-called *ultrametric* or *non-archimedean* triangle inequality:

$$|a+b| \leq \max\{|a|,\, |b|\}.$$

This drastically affects the geometry in K; for instance, it implies that all triangles are isosceles, and every point of a closed disk is a center of the disk. This is the starting point for the theory of non-archimedean geometry, which lie beyond the scope of this course.

1.2. **Semialgebraic sets.** Using the t-adic valuation v on K, we can extend the class of geometric objects from algebraic varieties to so-called semialgebraic sets over K.

Definition 1.2.1. Let n be a positive integer. A subset S of K^n is called *semialgebraic* if we can write it as a finite Boolean combination of sets of the form

$$\{a \in K^n \mid v(f(a)) \geq v(g(a))\}$$

with $f, g \in K[x_1, \ldots, x_n]$.

Finite Boolean combinations are formed by a finite succession of the following operations: finite intersections, finite unions and complements.

Exercise 1.2.2. Let f and g be polynomials in $K[x_1, \ldots, x_n]$. Show that

$$\{a \in K^n \mid v(f(a)) \Box v(g(a))\}$$

is semialgebraic when $\Box$ is any of the following symbols: $\geq, \leq, >, <, =$.

Example 1.2.3. A subset of K^n is called *constructible* if it can be written as a finite Boolean combination of sets of the form

$$\{a \in K^n \mid f(a) = 0\}$$

with $f \in K[x_1, \ldots, x_n]$. Every constructible set is semialgebraic, because $f(a) = 0$ if and only if $v(f(a)) \geq v(0)$.

Thus, we recover all the sets from algebraic geometry as a special subclass of semialgebraic sets. However, the class of semialgebraic sets is much larger, as illustrated by the following exercise.

Exercise 1.2.4. Show that every constructible set in K is finite or has finite complement. Deduce that R and $\mathfrak{m}$ are semialgebraic sets in K that are not constructible.

We will now discuss two more elaborate constructions of semialgebraic sets that show how they arise naturally in algebraic geometry.

1.3. **The Milnor fibration: topology.** Let f be a non-constant polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ and assume that $f(0) = 0$. Then f defines a polynomial map $f\colon \mathbb{C}^n \to \mathbb{C}$ and a hypersurface

$$V(f) = \{a \in \mathbb{C}^n \mid f(a) = 0\}.$$

For now, we can think of $V(f)$ simply as a topological subspace of $\mathbb{C}^n$. We would like to understand the topology of $V(f)$ around the point 0 in $\mathbb{C}^n$. If 0 is not a critical point of f, then this is easy: the implicit function theorem implies that 0 has an open neighbourhood in $V(f)$ that is homeomorphic to $\mathbb{C}^{n-1}$. The topology becomes much more intricate when 0 is a critical point of f. An important tool in this context is the *Milnor fibration*. It was constructed in a slightly different form by John Milnor in the beautiful book "Singular points of complex hypersurfaces" (1968), under the assumption that 0 is an isolated critical point of f. The construction was then generalized by Lê Dũng Tráng in the form that we will present.

The basic idea behind the Milnor fibration is that we try to understand the topology of $V(f)$ around 0 by looking at the smooth fibers $f^{-1}(b)$ for values of b close to 0 in $\mathbb{C}$, and how these fibers are transformed when we let b turn around the origin. Let D be a small open disk of radius $\eta > 0$ around 0 in $\mathbb{C}$, and let B be a small open ball of radius $\varepsilon > 0$ around 0 in $\mathbb{C}^n$. We write D^* for the punctured disk $D = D \setminus \{0\}$. Then $f^{-1}(D)$ is a small tube around $V(f)$ in $\mathbb{C}^n$, and $f^{-1}(D^*)$ is the punctured tube obtained by removing $V(f)$. All the fibers of f in $f^{-1}(D^*)$ are smooth when η is sufficiently small. We then zoom in around the origin in $\mathbb{C}^n$ by intersecting with the open ball B (we made pictures in class).

Theorem 1.3.1 (Milnor, Lê)**.** *When ε is sufficiently small, and η is sufficiently small with respect to ε, then the map*

$$f\colon B \cap f^{-1}(D^*) \to D^*$$

is a locally trivial fibration (topologically, or even in the $\mathcal{C}^\infty$-category if you know what that means). It is called the Milnor fibration *of f at* 0*.*

Recall that, when X, S and F are topological spaces, a continuous map $h\colon X \to S$ is called a locally trivial fibration with fiber F if we can cover S by opens U such that there exists a homeomorphism $h^{-1}(U) \to U \times F$ that makes the triangle

$$\begin{tikzcd} h^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "h"'] & & U \times F \arrow[dl, "\text{projection}"] \\ & U & \end{tikzcd}$$

commute. This implies in particular that every fiber of h is homeomorphic with F (in a non-canonical way).

In class we did the example of the projection of a Möbius strip onto a circle. Over any small arc of the circle, the Möbius strip just looks like the product of the arc with a closed interval (the fiber). Something interesting happens when we move around the circle: the fiber gets tilted upside-down (due to the twist in the construction of the Möbius strip). A similar phenomenon occurs for every locally trivial fibration $X \to S$: for any open U as in the definition, we can use the chosen homeomorphism with $U \times F$ to move from fiber to fiber by moving the U-coordinate and fixing the F-coordinate. For any loop in S based at some point s_0, we can cover the loop by finitely many U and compose the parallel translations of the fibers to get a homeomorphism from $h^{-1}(s_0)$ to itself. It depends on the choices of the various U and the homeomorphisms with $U \times F$, but one can show that it is independent of these choices up to isotopy (continuous deformations of homeomorphisms), and that it also only depends on the homotopy class of the loop. Thus, we get an action of the fundamental group $\pi_1(S, s_0)$ by homeomorphisms on $h^{-1}(s_0)$, up to isotopy. This is called the *monodromy action*. The important ingredients of the fibration $X \to S$ are the geometry of the fiber F and the monodromy action.

These constructions apply in particular to the Milnor fibration. If 0 is not a critical point of f, then we can reduce to the case where $f = x_1$ by means of a holomorphic coordinate change on $\mathbb{C}^n$ around 0. Now it is straightforward to check that the Milnor fibration extends to a *trivial* fibration $B \cap f^{-1}(D) \to D$: we can find a homeomorphism $B \cap f^{-1}(D) \to D \times B'$ where B' is an open ball in $\mathbb{C}^{n-1}$ and such that the homeomorphism commutes with f on the source and with the projection to D on the target. It follows that the fibers of the Milnor fibration are contractible, and that the monodromy acts as the identity. This means that, in the general case, any non-trivial topology in the fibers and any non-trivial monodromy are symptoms of a singularity of f at 0. One can then use the exact form of these symptoms to classify the singularities of polynomial maps f.

Example 1.3.2. If f has an isolated singularity at 0, then Milnor proved that the fibers of the Milnor fibration are homotopy equivalent to a wedge sum of μ spheres of dimension $n-1$ (that is, μ spheres of dimension $n-1$ joined together in a point). The number μ is called the Milnor number of the singularity and can be computed purely in algebraic terms: it is the dimension of the complex vector space

$$\mathbb{C}[[x_1, \ldots, x_n]]/(\partial_{x_1} f, \ldots, \partial_{x_n} f).$$

This is one of the most important invariants of isolated hypersurface singularities.

A more detailed discussion of the Milnor fibration (especially in the case $n = 2$) can be found in chapter 6 of "Singular points of plane curves" by C.T.C. Wall. The book by Milnor also remains an excellent reference.

In order to explain the link with semialgebraic geometry, we need one final topological construction. We can build a model of the fiber of the Milnor fibration

$$f \colon B \cap f^{-1}(D^*) \to D^*$$

that does not require us to choose a specific point in D^*, resulting in a more canonical construction. This is done by performing a base change to the universal covering space $\widetilde{D}^* \to D^*$, where

$$\widetilde{D}^* = \{w \in \mathbb{C} \mid \Re(w) < \log \eta\}$$

and the map $\widetilde{D}^* \to D^*$ is given by exponentiation (we made pictures in class). We can read off the fundamental group $\pi_1(D^*) = \mathbb{Z}$ of D^* from the universal cover by considering the group of *covering transformations*, homeomorphisms from $\widetilde{D}^*$ to itself that commute with the map to D^*. These are precisely the maps of the form $w \mapsto w + 2m\pi i$ with $m \in \mathbb{Z}$. Considering a straight path from w to $w + 2\pi i$ and applying the exponential function, we obtain a loop in D^* starting from $\exp(w)$ that circles around the origin once, counterclockwise.

Base changing the Milnor fibration to $\widetilde{D}^*$, we get a locally trivial fibration

$$(B \cap f^{-1}(D^*) \times_{D^*} \widetilde{D}^* \to \widetilde{D}^*$$

with the same fibers as the Milnor fibration. The upshot is that the new base $\widetilde{D}^*$ is contractible, which implies that the fibration is globally trivial (an actual product) and the total space $(B \cap f^{-1}(D^*) \times_{D^*} \widetilde{D}^*$ is homotopy equivalent to each of the fibers. We call this space the (universal) *Milnor fiber* of f at 0. The fundamental group $\pi_1(D^*)$ acts on the Milnor fiber *via* the covering transformations on $\widetilde{D}^*$, and this recovers the monodromy action.

In class, we did the example of the Möbius strip projecting to a circle. The universal covering space of the circle is the real line; the covering map is

$$\mathbb{R} \to S^1,\ x \mapsto \exp(2\pi i x).$$

The covering transformations on $\mathbb{R}$ are the translations by integers. Base changing the Möbius strip to this covering space, we get an infinitely long rectangular strip. The translation by 1 on the line acts on this strip by translating it by 1 and switching it upside-down.

2. Lecture 2 (20 October)

2.1. **The Milnor fibration: semialgebraic geometry.**

The construction of the topological Milnor fibration cannot be translated into algebraic geometry in a direct way[1], because we used the metric on $\mathbb{C}$ to construct the disk D and the ball B. However, we can carry out an analogous construction in semialgebraic geometry, which also provides some geometric intuition about the Puiseux series field K.

A fundamental dogma of algebraic geometry is that the geometry of a space is determined by the algebraic properties of the ring of functions on the space; this is the bridge between geometry and algebra. We can think of $k[\![t]\!]$ as the ring of power series that converge on an infinitesimally small disk D around 0 in $\mathbb{C}$; the incarnation of that disk in algebraic geometry is then the spectrum $\operatorname{Spec} k[\![t]\!]$. Of course, the underlying topological space of $\operatorname{Spec} k[\![t]\!]$ is very different from a disk: it consists of a unique closed point (t) and a unique open point (0). Still, it is justified to think of this space equipped with the structure sheaf $\mathcal{O}_{\operatorname{Spec} k[\![t]\!]}$ as the correct analog of D in algebraic geometry. Removing the origin $t = 0$ of the disk corresponds to inverting the element t in the ring of functions; thus the punctured disk D^* corresponds to $\operatorname{Spec} k(\!(t)\!)$.

One of the most profound, and most beautiful, insights of algebraic geometry is the surprising connection between number theory and topology, as manifested in the theory of algebraic fundamental groups: these simultaneously generalize Galois groups of fields in number theory and fundamental groups of topological spaces. An even deeper development of these ideas is the theory of étale cohomology, which was one of Grothendieck's main motivations to develop scheme theory and led to a proof of the famous Weil conjectures on algebraic varieties over finite fields. We will now see these ideas at work in a very special case.

The finite topological covers of a punctured disk D^* are all of the form

$$D^* \to D^*,\ z \mapsto z^d$$

(to be precise, we should replace the radius of the disk by its d-th root in the source to get a well-defined map, but we ignore this issue because we think of the radius as being infinitesimally small). In our algebraic analog, this becomes the morphism

$$\operatorname{Spec} k(\!(\sqrt[d]{t})\!) \to \operatorname{Spec} k(\!(t)\!).$$

Letting d grow and passing to the limit, we get the morphism

$$\operatorname{Spec} K \to \operatorname{Spec} k(\!(t)\!)$$

as an algebraic approximation of the universal covering space $\widetilde{D}^* \to D^*$. On the algebraic side, the role of the group of covering transformations of $\widetilde{D}^*$ (i.e., the fundamental group of D^*) is played by the Galois group $\operatorname{Gal}(K/k(\!(t)\!))$. If $k = \mathbb{C}$,

[1]One can perform a similar construction using the étale topology, but the result is less geometric than the semialgebraic approach.

then a loop in D^* going around the origin once counterclockwise corresponds to the Galois automorphism of K given by

$$\sum_{j=N}^{\infty} a_j t^{j/d} \mapsto \sum_{j=N}^{\infty} a_j \xi_d^j t^{j/d}$$

with $\xi_d = \exp(2\pi i/d)$. This automorphism is a topological generator for the profinite group $\mathrm{Gal}(K/k(\!(t)\!))$. If you have not seen Galois groups of infinite algebraic extensions before: they carry a natural topology, and "topological generator" means that the set of integer powers of the element is dense in the whole Galois group.

With this dictionary in place, we can build the semialgebraic version of the Milnor fibration. Let f be a non-constant polynomial in $k[x_1, \ldots, x_n]$ with $f(0) = 0$. Then f defines a morphism

$$f\colon \mathbb{A}^n_k \to \mathbb{A}^1_k = \operatorname{Spec} k[t].$$

The analog of the Milnor tube $f^{-1}(D)$ is the base change

$$\mathscr{Y} = \mathbb{A}^n_k \times_{k[t]} k[\![t]\!]$$

which is a separated $k[\![t]\!]$-scheme of finite type. More explicitly,

$$\mathscr{Y} = \operatorname{Spec} k[\![t]\!][x_1, \ldots, x_n]/(f - t)$$

(prove this as an exercise on fibered products if it is not immediately obvious).

We find the hypersurface $V(f) = f^{-1}(0)$ in $\mathscr{Y}$ by setting $t = 0$:

$$V(f) = \mathscr{Y} \times_{k[\![t]\!]} k = \operatorname{Spec} k[x_1, \ldots, x_n]/(f).$$

Removing $V(f)$ amounts to inverting t: the punctured tube $f^{-1}(D^*)$ becomes the separated $k(\!(t)\!)$-scheme of finite type

$$\mathscr{Y} \times_{k[\![t]\!]} k(\!(t)\!) = \operatorname{Spec} k(\!(t)\!)[x_1, \ldots, x_n]/(f - t).$$

This is the fiber of $\mathscr{Y}$ over the generic point of $\operatorname{Spec} k[\![t]\!]$. Passing to the universal covering space $\widetilde{D}^*$ amounts to a further base change to K: the topological space $f^{-1}(D^*) \times_{D^*} \widetilde{D}^*$ corresponds to

$$\mathscr{Y} \times_{k[\![t]\!]} K$$

and the monodromy action becomes the Galois action of $\mathrm{Gal}(K/k(\!(t)\!))$.

So far, we have not had any need for semialgebraic geometry: the space

$$Y = \mathscr{Y} \times_{k[\![t]\!]} K$$

is a separated K-scheme of finite type, which we can write more explicitly as

$$Y = \operatorname{Spec} K[x_1, \ldots, x_n]/(f - t).$$

But we still need to find the analog of the Milnor ball B to zoom in around $0 \in V(f)$. So let us think a bit further about how we can visualize closed points on Y. Since K is an increasing union of field extensions $k(\!(t^{1/d})\!)$, each closed point a in Y has affine coordinates in $k(\!(t^{1/d})\!)$ for some $d > 0$. As we have seen, we should think of the spectrum of this field as a punctured disk D^* that lies as a degree d topological cover over $D^* \approx \operatorname{Spec} k(\!(t)\!)$ inside the Milnor tube $\mathscr{Y} \approx f^{-1}(D)$. Now we want to express that this punctured disk lies "close" to the point 0 in $V(f)$. Topologically, we can impose this by requiring that the punctured disk can be extended to a disk D whose origin lies at $0 \in V(f)$. Algebraically, this means that, if we send the parameter $\sqrt[d]{t}$

to 0 in the coordinates of our point a, then we find the origin $0 \in V(f)$. This means precisely that only positive powers of t occur in the coordinates of a; equivalently, that each coordinate has positive valuation. In this way, we finally arrive at the definition of the semialgebraic Milnor fiber.

Definition 2.1.1. The semialgebraic Milnor fiber of f at 0 is the semialgebraic set

$$\mathrm{MF}^{\mathrm{sa}}_{f,0} = \{a \in K^n \mid f(a) = t,\, v(a_i) > 0 \text{ for all } i\}.$$

The monodromy action on the semialgebraic Milnor fiber is the action of the Galois group $\mathrm{Gal}(K/k(\!(t)\!))$ on $\mathrm{MF}^{\mathrm{sa}}_{f,0}$.

Note that the Galois group indeed acts on this set, because the equation $f(x) = t$ has coefficients in $k(\!(t)\!)$ (so that any Galois conjugate of a solution a is again a solution) and the Galois action on K preserves the valuation v.

At this point, our definition of the semialgebraic Milnor fiber merely arose by analogy. In future lectures we will define various invariants of semialgebraic sets. One can then show that the values of these invariants on the semialgebraic Milnor fiber agree with the those of the topological Milnor fiber. A basic example of such an invariant is the Euler characteristic.

2.2. **The tropicalization map.** Let us now explore another source of semialgebraic sets. For every positive integer n, the n-dimensional *tropicalization map* takes coordinatewise valuations of elements in $(K^*)^n$:

$$\mathrm{trop}\colon (K^*)^n \to \mathbb{Q}^n,\ a \mapsto (v(a_1), \ldots, v(a_n)).$$

The aim of tropical geometry[2] is to study the geometry of algebraic varieties X in $(K^*)^n$ by analyzing the shape of $\mathrm{trop}(X)$. The Bieri-Groves theorem states that $\mathrm{trop}(X)$ is a finite union of polytopes in $\mathbb{Q}^n$.

Definition 2.2.1. A *polytope* in $\mathbb{Q}^n$ is a finite intersection of closed half-spaces of the form

$$H = \{w \in \mathbb{Q}^n \mid \ell(w) \geq c\}$$

where $\ell\colon \mathbb{Q}^n \to \mathbb{Q}$ is a $\mathbb{Q}$-linear map and c is a rational number.

Note that we do not require polytopes to be bounded: for instance, $\mathbb{Q}^n$ is itself a polytope. The bounded polytopes in $\mathbb{Q}^n$ are precisely the convex hulls of finite sets of points in $\mathbb{Q}^n$.

Proposition 2.2.2. *For every polytope P in $\mathbb{Q}^n$, the inverse image* $\mathrm{trop}^{-1}(P)$ *is a semialgebraic subset of K^n.*

Proof. Since intersections commute with taking inverse images, it suffices to consider the case where P is a half-space H as in Definition 2.2.1. Rescaling the inequality $\ell(w) \geq c$, we may assume that the linear form ℓ has integer coefficients:

$$\ell(w) = b_1 w_1 + \ldots + b_n w_n$$

with $b_1, \ldots, b_n \in \mathbb{Z}$. Permuting the coordinates on K^n and $\mathbb{Q}^n$, we may further assume that there exists an element m in $\{1, \ldots, n\}$ such that $b_1, \ldots, b_m$ are non-negative and $b_{m+1}, \ldots, b_n$ are negative. Then we can rewrite the equation for H as

$$b_1 w_1 + \ldots + b_m w_m \geq c - b_{m+1} w_{m+1} - \ldots - b_n w_n.$$

[2]The name "tropical geometry" was given to honour one of the pioneers of this field, the Brazilian computer scientist Imre Simon.

Since the valuation v takes products to sums, we now have

$$\mathrm{trop}^{-1}(H) = \{a \in K^n \mid a_i \neq 0 \text{ for all } i, \text{ and } v(a_1^{b_1} \cdots a_m^{b_m}) \geq v(t^c a_{m+1}^{-b_{m+1}} \cdots a_n^{-b_n})\}.$$

The two arguments of v that appear in this description are polynomials in the coordinates of a with coefficients in K, so that $\mathrm{trop}^{-1}(H)$ is semialgebraic. $\square$

Example 2.2.3. The following sets are semialgebraic in K:

$$\begin{aligned} \mathrm{trop}^{-1}(0) &= \{a \in K^* \mid v(a) = 0\} = R^*, \\ \mathrm{trop}^{-1}([0,1]) &= \{a \in K^* \mid 0 \leq v(a) \leq 1\}, \\ \mathrm{trop}^{-1}(\mathbb{Q}_{\geq 0}) &= \{a \in K^* \mid v(a) \geq 0\} = R. \end{aligned}$$

Remarks 2.2.4.

(1) The proposition immediately implies that, if Γ is a finite Boolean combination of polytopes in $\mathbb{Q}^n$, then $\mathrm{trop}^{-1}(\Gamma)$ is a semialgebraic subset of K^n (Boolean combinations commute with taking inverse images).

(2) It is not hard to show that, for every constructible subset X of K^n that has infinite intersection with $(K^*)^n$, the image $\mathrm{trop}(X \cap (K^*)^n)$ is unbounded. This is similar to the property that algebraic subvarieties of $\mathbb{C}^n$ of positive dimension are never compact. It follows that, for every non-empty bounded polytope P, the inverse image $\mathrm{trop}^{-1}(P)$ is semialgebraic but not constructible in K^n.

2.3. **Semialgebraic sets in K-varieties.** In algebraic geometry, we can pass from affine varieties to more general varieties and schemes by a gluing construction. Similarly, we can globalize the definition of semialgebraic sets in K^n to subsets in algebraic K-varieties. Recall that, for every K-scheme X of finite type, the set $X(K)$ is defined as the set of morphism of K-schemes $\mathrm{Spec}\, K \to X$, and we can identify $X(K)$ with the set of closed points of X.

Definition 2.3.1. Let X be a K-scheme of finite type. A subset S of $X(K)$ is called *semialgebraic* if we can cover X by affine open subschemes U such that $S \cap U(K)$ is a finite Boolean combination of sets of the form

$$\{a \in U(K) \mid v(f(a)) \geq v(g(a))\}$$

with $f, g \in \mathcal{O}_X(U)$. Equivalently, for every closed embedding $U \to \mathbb{A}^n_K$, the image of $S \cap U(K)$ in $\mathbb{A}^n_K(K) = K^n$ is semialgebraic in the sense of Definition 1.2.1.

Note that any finite Boolean combination of semialgebraic subsets in $X(K)$ is again semialgebraic. The following exercise shows that the definition does not depend on the choice of the opens U.

Exercise 2.3.2. Let S be a semialgebraic set in $X(K)$. Show that, for *every* open U in X and *every* locally closed embedding $U \to \mathbb{A}^n_K$, the image of $S \cap U(K)$ in K^n is semialgebraic.

Example 2.3.3. Let X be a K-scheme of finite type. A *constructible* subset of $X(K)$ is a finite Boolean combination of Zariski-closed subsets of $X(K)$. Every constructible subset is semialgebraic; this follows at once from the affine case in Example 1.2.3.

Exercise 2.3.4. Let X be a K-scheme of finite type, and let U be a (locally closed) subscheme of X. Show that every semialgebraic subset of $U(K)$ is also semialgebraic in $X(K)$.

A powerful method to construct semialgebraic sets (most of which will not be constructible) is the *specialization map* that will be introduced in the following lecture.

3. Lecture 3 (24 October)

3.1. **Schemes over the valuation ring.** This section collects some reminders about schemes over the valuation ring R in the field of Puiseux series. The underlying topological space of the spectrum $\operatorname{Spec} R$ consists of the closed point $\mathfrak{m}$ and the generic point (0). The residue fields at $\mathfrak{m}$ and (0) are k and K, respectively, so that we can think of $\operatorname{Spec} R$ as the points $\operatorname{Spec} k$ and $\operatorname{Spec} K$ glued together.

More generally, every R-scheme $\mathscr{X}$ consists of a *generic fiber* $\mathscr{X} \times_R K$, which is a scheme over K, and a *special fiber* $\mathscr{X} \times_R k$, which is a scheme over k. These are glued together inside $\mathscr{X}$: the generic fiber is open in $\mathscr{X}$, whereas the special fiber is closed in $\mathscr{X}$. We made a schematic picture in class.

Example 3.1.1. If

$$\mathscr{X} = \operatorname{Spec} R[x_1, \ldots, x_n]/(f_1, \ldots, f_\ell),$$

then the generic fiber is the affine K-scheme

$$\mathscr{X} \times_R K = \operatorname{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_\ell)$$

and the special fiber is the affine k-scheme

$$\mathscr{X} \times_R k = \operatorname{Spec} k[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_\ell)$$

where we write $\overline{f}_j$ for the polynomial obtained from f_j by reducing the coefficients modulo $\mathfrak{m}$.

For every R-algebra A, the set $\mathscr{X}(A)$ is by definition the set of morphisms of R-schemes $\operatorname{Spec} A \to \mathscr{X}$. By the universal property of the fibered product, we have $\mathscr{X}(A) = (\mathscr{X} \times_R A)(A)$. This applies in particular to $A = K$ and $A = k$, so that we can identify $\mathscr{X}(K)$ with the set of closed points on the generic fiber, and $\mathscr{X}(k)$ with the set of closed points on the special fiber.

The inclusion $R \to K$ induces a morphism $\operatorname{Spec} K \to \operatorname{Spec} R$ and therefore a map $\mathscr{X}(R) \to \mathscr{X}(K)$ by composition. Similarly, the projection morphism $R \to k = R/\mathfrak{m}$ gives rise to a map

$$\operatorname{sp}_{\mathscr{X}} \colon \mathscr{X}(R) \to \mathscr{X}(k)$$

called the *specialization map.*

Example 3.1.2. If

$$\mathscr{X} = \operatorname{Spec} R[x_1, \ldots, x_n]/(f_1, \ldots, f_\ell),$$

then

$$\begin{aligned}
\mathscr{X}(K) &= \{a \in K^n \mid f_j(a) = 0 \text{ for all } j\} \\
\mathscr{X}(R) &= \{a \in R^n \mid f_j(a) = 0 \text{ for all } j\} \\
\mathscr{X}(k) &= \{a \in k^n \mid \overline{f}_j(a) = 0 \text{ for all } j\}
\end{aligned}$$

where $\overline{f}_j$ again denotes the reduction of f_j modulo $\mathfrak{m}$. The map $\mathscr{X}(R) \to \mathscr{X}(K)$ is the obvious inclusion, and the map $\operatorname{sp}_{\mathscr{X}} \colon \mathscr{X}(R) \to \mathscr{X}(k)$ reduces the coordinates of a modulo $\mathfrak{m}$.

It follows directly from the definition that the specialization map is functorial in $\mathscr{X}$: every morphism of R-schemes $\mathscr{X} \to \mathscr{Y}$ induces maps $\mathscr{X}(R) \to \mathscr{Y}(R)$ and $\mathscr{X}(k) \to \mathscr{Y}(k)$, and these commute with specialization.

Exercise 3.1.3. Let $\mathscr{X}$ be a R-scheme of finite type. Show that a point $a \in \mathscr{X}(K)$ extends to a point in $\mathscr{X}(R)$ (in other words, lies in the image of $\mathscr{X}(R) \to \mathscr{X}(K)$) if and only if there exists an affine open neighbourhood

$$\mathscr{U} = \operatorname{Spec} R[x_1, \ldots, x_n]/I$$

of a in $\mathscr{X}$ such that $x_i(a) \in R$ for $i = 1, \ldots, n$.

Exercise 3.1.4. Prove a topological version of the criterion in Exercise 3.1.3: a point a in $\mathscr{X}(K)$ extends to a point in $\mathscr{X}(R)$ if and only if the Zariski closure of a in $\mathscr{X}$ intersects the special fiber $\mathscr{X} \times_R k$. Moreover, the set of intersection points corresponds bijectively to the set of extensions of a to $\mathscr{X}(R)$.

If $\mathscr{X}$ is separated, then each point in $\mathscr{X}(K)$ has at most one extension to $\mathscr{X}(R)$, so that the map $\mathscr{X}(R) \to \mathscr{X}(K)$ is injective. This is a special case of the valuative criterion of separatedness. In this case, we can view $\mathscr{X}(R)$ as a subset of $\mathscr{X}(K)$. For an example where $\mathscr{X}(R) \to \mathscr{X}(K)$ is not injective, take two copies of $\operatorname{Spec} R$ and glue them along the generic points to get "$\operatorname{Spec} R$ with two origins". This scheme has only one K-point but two R-points.

The map $\mathscr{X}(R) \to \mathscr{X}(K)$ is bijective if and only if $\mathscr{X}$ is proper over R. The "if" part follows from the valuative criterion for properness. The "only if" part is more difficult to prove; let us just look at an elementary example.

Example 3.1.5. Let $\mathscr{X} = \operatorname{Proj} R[x_0, x_1]$, the projective line over R. This is a proper R-scheme with generic fiber $\mathbb{P}^1_K$ and special fiber $\mathbb{P}^1_k$. A K-point on $\mathscr{X}$ is the same thing as a couple of homogeneous coordinates in K (up to scaling by a factor in K^*) and a R-point on $\mathscr{X}$ is the same thing as a couple of homogeneous coordinates in R which do not both lie in $\mathfrak{m}$ (up to scaling by a factor in R^*)[3]. This immediately implies that $\mathscr{X}(R) \to \mathscr{X}(K)$ is bijective: multiplying a couple of homogeneous coordinates in K by a suitable fractional power of t, we can rewrite it as $[a_0, a_1]$ with a_0 and a_1 in R and at least one of them a unit; this expression is unique up to a scalar in R^*. The image of $[a_0, a_1]$ under the specialization map $\operatorname{sp}_{\mathscr{X}}$ is the point on $\mathbb{P}^1_k$ with homogeneous coordinates $[\overline{a}_0, \overline{a}_1]$, where $\overline{(\cdot)}$ denotes reduction modulo $\mathfrak{m}$.

Let $\mathscr{Y}$ be the R-scheme that we obtain from $\mathscr{X}$ by removing the origin $[0, 1]$ from the special fiber $\mathbb{P}^1_k$. The scheme $\mathscr{Y}$ is no longer proper over R and $\mathscr{Y}(R) \to \mathscr{Y}(K)$ is no longer bijective: we have $\mathscr{Y}(K) = \mathscr{X}(K)$ (since we have not altered the generic fiber) but in $\mathscr{Y}(R)$ we are missing all the points a from $\mathscr{X}(R)$ with $\operatorname{sp}_{\mathscr{X}}(a) = [0, 1]$ (equivalently, all the points $[a_0, a_1]$ with $v(a_0) > v(a_1)$).

For every proper K-scheme X, there exists a proper R-scheme $\mathscr{X}$ whose generic fiber $\mathscr{X} \times_R K$ is isomorphic to X. This is easy to show when X is projective: fix a closed embedding $X \to \mathbb{P}^n_K$ and take for $\mathscr{X}$ the schematic closure of X in $\mathbb{P}^n_R$. The proof of the general case is much more subtle; it is a special case of Nagata's compactification theorem.

[3]The case of R-points is not entirely trivial and relies on the fact that R is a local ring.

3.2. **The specialization map.** Now, we will see how we can use the specialization map on R-schemes to produce further examples of semialgebraic sets.

Proposition 3.2.1. *Let $\mathscr{X}$ be a R-scheme of finite type, and let C be a constructible subset of $\mathscr{X}(k)$. Then the image of $\mathrm{sp}_{\mathscr{X}}^{-1}(C)$ under the map $\mathscr{X}(R) \to \mathscr{X}(K)$ is a semialgebraic subset of $\mathscr{X}(K)$.*

Proof. If $\mathscr{U}_1, \ldots, \mathscr{U}_r$ are open subschemes that cover $\mathscr{X}$, then it is easy to see that

$$\mathrm{sp}_{\mathscr{X}}^{-1}(C) = \mathrm{sp}_{\mathscr{U}_1}^{-1}(C \cap \mathscr{U}_1(k)) \cup \ldots \cup \mathrm{sp}_{\mathscr{U}_r}^{-1}(C \cap \mathscr{U}_r(k)).$$

Prove this as an exercise: the key point is showing that every R-point $\operatorname{Spec} R \to \mathscr{X}$ factors through at least one of the opens $\mathscr{U}_i$. It follows from Exercise 2.3.4 that every semialgebraic subset in one of the opens $\mathscr{U}_i(K)$ is also semialgebraic in $\mathscr{X}(K)$. Since a finite union of semialgebraic sets is again semialgebraic, it suffices to prove the result for the opens $\mathscr{U}_i$, which reduces the problem to the case where $\mathscr{X}$ is affine. We write

$$\mathscr{X} = \operatorname{Spec} R[x_1, \ldots, x_n]/I$$

for some ideal I. Taking inverse images commutes with Boolean operations, so that we can further reduce to the case where C is of the form

$$C = \{b \in \mathscr{X}(k) \mid g(b) = 0\}$$

for some polynomial g in $k[x_1, \ldots, x_n]$.

Let $\widetilde{g}$ be any polynomial in $R[x_1, \ldots, x_n]$ whose reduction modulo $\mathfrak{m}$ is equal to g. Then for every element a in R^n, we have that $g(\overline{a}) = 0$ if and only if $\widetilde{g}(a) \in \mathfrak{m}$, where $\overline{(\cdot)}$ denotes reduction of coordinates modulo $\mathfrak{m}$. This condition is also equivalent to $v(\widetilde{g}(a)) > 0$. Therefore, we can write

$$\mathrm{sp}_{\mathscr{X}}^{-1}(C) = \{a \in \mathscr{X}(K) \mid v(x_i(a)) \geq 0 \text{ for } i = 1, \ldots, n, \text{ and } v(\widetilde{g}(a)) > 0\}$$

which is a semialgebraic set in $\mathscr{X}(K)$. $\square$

Example 3.2.2. Let X be a separated k-scheme of finite type, equipped with a morphism

$$f\colon X \to \mathbb{A}^1_k = \operatorname{Spec} k[t].$$

Set $\mathscr{X} = X \times_{k[t]} R$; this can also be viewed as the closed subscheme of $X \times_k R$ defined by the equation $f = t$. Then $\mathscr{X}(K) = \{a \in X(K) \mid f(a) = t\}$ and $\mathscr{X}(k) = \{a \in X(k) \mid f(a) = 0\}$. Thus $\mathscr{X}(k)$ captures the geometry of the zero locus of f in X, and $\mathscr{X}(K)$ captures the geometry of the fibers of f that lie very close to the fiber over 0. Inside $\mathscr{X}(K)$ we can consider the semialgebraic subset

$$\mathscr{X}(R) = \{a \in X(R) \mid f(a) = t\}$$

and, for every point x in $\mathscr{X}(k)$, we can consider the smaller semialgebraic subset $\mathrm{sp}_{\mathscr{X}}^{-1}(x)$ consisting of the points a in $\mathscr{X}(R)$ that map to x under the reduction of the coordinates modulo $\mathfrak{m}$. We call $\mathrm{sp}_{\mathscr{X}}^{-1}(x)$ the semialgebraic Milnor fiber of f at x; in the case where $X = \mathbb{A}^n_k$ and x is the origin, this is the same construction as in Definition 2.1.1.

3.3. **Semialgebraic maps and quantifier elimination.** In order to define a category of geometric objects, we need to specify not only the objects themselves but also the maps between them. This will tell us, in particular, which objects we consider to be isomorphic. Geometry then becomes the study of properties that are invariant under isomorphism, and the main goal is classifying objects up to isomorphism.

Definition 3.3.1. Let X and Y be K-schemes of finite type. Let S and T be semialgebraic subsets of $X(K)$ and $Y(K)$, respectively. A map $f\colon S \to T$ is called *semialgebraic* if its graph is a semialgebraic subset of $X(K)\times Y(K) = (X\times_K Y)(K)$.

Exercise 3.3.2. Let X be a K-scheme of finite type and let S be a semialgebraic subset of $X(K)$. Show that the inclusion map $S \to X(K)$ is semialgebraic.

Exercise 3.3.3. Let X and Y be K-schemes of finite type. Let S and T be semialgebraic subsets of $X(K)$ and $Y(K)$, respectively. Show that, for every morphism of K-schemes $f\colon X \to Y$ such that $f(S) \subset T$, the map $f|_S\colon S \to T$ is semialgebraic.

It is straightforward to check that a composition of semialgebraic maps is again semialgebraic. When $f\colon S \to T$ is a semialgebraic bijection, then its inverse $f^{-1}\colon T \to S$ is again semialgebraic, because the graph of f^{-1} is simply the transpose of the graph of f. Thus the isomorphisms in the category of semialgebraic sets are the semialgebraic bijections.

The following theorem is one of the cornerstones of semialgebraic geometry over K.

Theorem 3.3.4. *The image of a semialgebraic set under a semialgebraic map is again semialgebraic.*

This is a special case of the theorem of *quantifier elimination for algebraically closed valued fields* proved by A. Robinson (1956). To see what this statement has to do with quantifiers, consider the case of a semialgebraic map $f\colon K^m \to K^n$. For every point a in K^m, the image $f(a)$ is the unique point b in K^n satisfying some semialgebraic expression

$$\varphi(a_1,\ldots,a_m,b_1,\ldots,b_n)$$

in the coordinates of a and b (namely, the expression defining the graph of f). The image of f is then the set of points b in K^n satisfying the formula

$$(\exists a_1)\ldots(\exists a_m)(\varphi(a_1,\ldots,a_m,b_1,\ldots,b_n)$$

involving existential quantifiers. Quantifiers were not allowed in our definition of a semialgebraic set, so we must show that there exists an equivalent description of the image in terms of a semialgebraic formula without quantifiers. This is the process of eliminating quantifiers.

Example 3.3.5. Chevalley's theorem in algebraic geometry states that the image of a constructible set under a morphism of algebraic varieties over an algebraically closed field is again constructible[4]; this is equivalent to quantifier elimination for algebraically closed fields, which was proved by Tarski (1948).

[4]There is a much more general versions of this theorem for morphisms of finite presentation between schemes.

Another famous example is Tarski's theorem on quantifier elimination for the ordered field $(\mathbb{R}, \leq)$. The field $\mathbb{R}$ itself does not have quantifier elimination, because the image of the polynomial map

$$f\colon \mathbb{R} \to \mathbb{R},\, x \mapsto x^2$$

is not constructible. Tarski proved that this problem disappears by adding the ordering $\leq$ to the structure of $\mathbb{R}$: we can define semialgebraic sets in $\mathbb{R}^n$ as finite Boolean combinations of sets of the form

$$\{a \in \mathbb{R}^n \mid g(a) \geq 0\}$$

with g a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$. Semialgebraic maps are then defined accordingly. Tarski's theorem states that the image of a semialgebraic set under a semialgebraic map is again semialgebraic. The proof gives an algorithm to eliminate quantifiers from semialgebraic formulas. In the example of our map f above, the expression $(\exists a)(a^2 = b)$ describing the image of f gets replaced by the quantifier-free expression $b \geq 0$.

Exercise 3.3.6. Let X and Y be K-schemes of finite type.

(1) Let $f\colon X \to Y$ be a morphism of K-schemes. Show that, for every semialgebraic subset T of $Y(K)$, the inverse image $f^{-1}(T)$ is semialgebraic in $X(K)$, without using Theorem 3.3.4.
(2) Let S be a semialgebraic subset of $X(K)$, and let $f\colon S \to Y(K)$ be a semialgebraic map. Show that, for every semialgebraic subset T of $Y(K)$, the inverse image $f^{-1}(T)$ is semialgebraic in $X(K)$. Here you will need Theorem 3.3.4.

4. Lecture 4 (27 October)

4.1. **There are many (but not too many) semialgebraic isomorphisms.** Let X and Y be K-schemes of finite type. We have seen in Exercise 3.3.3 that, for every morphism $X \to Y$ of K-schemes of finite type, the induced map $X(K) \to Y(K)$ is semialgebraic. In particular, if X and Y are isomorphic as K-schemes, then $X(K)$ and $Y(K)$ are isomorphic as semialgebraic sets. However, there are many examples where X and Y are not isomorphic as K-schemes but $X(K)$ and $Y(K)$ are still isomorphic as semialgebraic sets: working in the semialgebraic category gives us much more leeway to construct isomorphisms.

Example 4.1.1. Let X be a K-scheme of finite type, let S be a semialgebraic set in $X(K)$, and let $\{S_1, \ldots, S_r\}$ be a partition of S into semialgebraic subsets. Then S is isomorphic to the disjoint union $S_1 \sqcup \ldots \sqcup S_r$ (viewed as a semialgebraic set in the disjoint union of r copies of $X(K)$). Indeed, the projection to $X(K)$ defines a semialgebraic bijection from $S_1 \sqcup \ldots \sqcup S_r$ to S.

In particular, the semialgebraic set $K = \mathbb{A}^1_K(K)$ is isomorphic to $K^* \sqcup \{0\}$, even though the K-scheme $\mathbb{A}^1_K$ is not isomorphic to the disjoint union of $\mathbb{A}^1_K \setminus \{0\}$ and $\{0\}$.

This illustrates that semialgebraic geometry is *piecewise* geometry, since we can break any semialgebraic set into finitely many semialgebraic pieces without changing its geometry. There are more sophisticated approaches to semialgebraic geometry (based on non-archimedean geometry) that prevent this issue. For us, it is a feature rather than a bug, because the piecewise nature will make it possible to arrive at a complete classification of semialgebraic sets, while it preserves enough interesting structure for applications.

The following proposition produces a more profound class of examples of non-isomorphic K-varieties that become isomorphic as semialgebraic sets.

Proposition 4.1.2. *Let $\mathscr{X}$ and $\mathscr{Y}$ be smooth and proper R-schemes, and assume that the special fibers $\mathscr{X} \times_R k$ and $\mathscr{Y} \times_R k$ are isomorphic (as k-schemes). Then $\mathscr{X}(K)$ and $\mathscr{Y}(K)$ are isomorphic as semialgebraic sets.*

We will prove this proposition in Section 5.1; this requires a preliminary discussion on étale morphisms and henselian local rings. We can already look at an interesting example.

Example 4.1.3. To any elliptic curve E over a field F, one can attach its *j-invariant* $j(E) \in F$. If F is algebraically closed, then two elliptic curves are isomorphic if and only if they have the same j-invariant, and every value in F is the j-invariant of a unique isomorphism class of elliptic curves. See for instance Silverman's book "The arithmetic of elliptic curves" (1986).

Let E and E' be elliptic curves over K such that $j(E)$ and $j(E')$ lie in R and are congruent modulo $\mathfrak{m}$. Then one can show that E extends to a smooth and proper R-scheme $\mathscr{E}$ such that $\mathscr{E} \times_R K \cong E$ and such that $\mathscr{E} \times_R k$ is an elliptic curve over k with j-invariant $\overline{j(E)}$, the reduction of $j(E)$ modulo $\mathfrak{m}$. Similarly, E' extends to a smooth and proper R-scheme $\mathscr{E}'$ with the analogous properties. Since $j(E)$ and $j(E')$ are congruent modulo $\mathfrak{m}$, they have the same reduction in k, so that $\mathscr{E} \times_R k$ and $\mathscr{E}' \times_R k$ are isomorphic. Proposition 4.1.2 now implies that $E(K)$ and $E'(K)$ are isomorphic as semialgebraic sets. However, E and E' are not isomorphic as K-schemes unless $j(E) = j(E')$.

4.2. **Étale morphisms.** Consider the map

$$f\colon \mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto z^2.$$

This map is locally biholomorphic: we can cover the source by opens U (in the Euclidean topology) such that $f(U)$ is open in $\mathbb{C}^*$ and $f\colon U \to f(U)$ is a holomorphic bijection with holomorphic inverse. However, this is not a local isomorphism of complex algebraic varieties (with respect to the Zariski topology), because it is not birational. The issue is that the Zariski topology is much coarser than the Euclidean topology. On the other hand, we can still express the property of being locally biholomorphic in a purely algebraic way: by the inverse function theorem, this is equivalent to saying that the derivative is everywhere invertible. Pushing this idea further leads to the definition of an étale morphism.

Definition 4.2.1. A morphism of schemes $f\colon X \to Y$ is called *étale* if it is flat and unramified.

Unramified means that f is locally of finite presentation and that, for every point y on Y, the fiber $X \times_Y y$ is a disjoint union of spectra of finite separable extensions of the residue field $\kappa(y)$. The *flatness* assumption then expresses that these fibers vary in a continuous way as y moves over Y; for instance, it rules out immersions that are not open. For the purpose of this course, the main things you need to know about étale morphisms are summarized in the following list.

(1) Let X and Y be smooth schemes over an algebraically closed field F. Then a morphism of F-schemes $f\colon X \to Y$ is étale if and only if, for every point

x in $X(F)$, the induced map on Zariski tangents spaces $T_x X \to T_{f(x)} Y$ is an isomorphism. If $F = \mathbb{C}$, this is also equivalent to the property that $f\colon X(\mathbb{C}) \to Y(\mathbb{C})$ is locally biholomorphic.

(2) If $F \subset F'$ is an extension of fields, then $\operatorname{Spec} F' \to \operatorname{Spec} F$ is étale if and only if F' is a finite separable extension of F.

(3) Every open immersion of schemes is étale. If $f\colon X \to Y$ is an étale morphism of schemes, then $f(X)$ is open in Y.

(4) A composition of étale morphisms is étale, and any base change of an étale morphism is étale.

(5) A *standard étale morphism* is a morphism of the form

$$\operatorname{Spec} \frac{A[T][1/g(T)]}{(h(T))} \to \operatorname{Spec} A$$

where $g(T)$ and $h(T)$ are polynomials in $A[T]$, the polynomial $h(T)$ is monic, and the derivative $h'(T)$ is invertible in the ring

$$\frac{A[T][1/g(T)]}{(h(T))}.$$

As the name suggests, standard étale morphisms are étale (because we forced the derivative $h'(T)$ to be invertible). Moreover, every étale morphism is locally of this form: a morphism of schemes $f\colon X \to Y$ is étale if and only if, for every point x in X, we can find an affine open neighbourhood U of x in X and an affine open neighbourhood V of $f(x)$ in Y such that $f(U) \subset V$ and such that the morphism $f\colon U \to V$ is a standard étale morphism.

4.3. **Henselian local rings.** Let X and Y be smooth $\mathbb{C}$-schemes of finite type and let $f\colon X \to Y$ be an étale morphism. Then, as we have seen, the map $X(\mathbb{C}) \to Y(\mathbb{C})$ is locally biholomorphic. In particular, it is a local homeomorphism: we can cover $X(\mathbb{C})$ by opens U (in the Euclidean topology) such that $f(U)$ is open in $Y(\mathbb{C})$ and such that $f\colon U \to f(U)$ is a homeomorphism. This implies that, for every point x of $X(\mathbb{C})$ and every sufficiently small open neighbourhood V of $f(x)$ in $Y(\mathbb{C})$, there exists a unique open neighbourhood U of x in $X(\mathbb{C})$ such that f maps U homeomorphically onto V.

In algebraic geometry, the role of these "sufficiently small opens" is played by spectra of henselian local rings. We will give two equivalent definitions of a henselian local ring: the first is the direct translation of the above topological property, but the second is easier to check in practice.

Proposition 4.3.1. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Then the following properties are equivalent.*

(1) *For every étale morphism of schemes $X \to Y$, every morphism $\operatorname{Spec} A \to Y$ and every morphism $\operatorname{Spec} \kappa \to X$ such that the square*

$$\begin{array}{ccc} \operatorname{Spec} \kappa & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

commutes, there exists a unique morphism $\operatorname{Spec} A \to X$ such that both triangles in the diagram

$$\begin{tikzcd} \operatorname{Spec}\kappa \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

commute. Informally: we can uniquely lift a morphism $\operatorname{Spec} A \to Y$ *to* X *as soon as we have chosen a lifting of* $\operatorname{Spec}\kappa \to Y$ *to* X.

(2) *For every monic polynomial* $f(T)$ *in* $A[T]$ *and every simple root* a *of* $\overline{f}(T)$ *in* κ, *there exists a unique root* b *of* $f(T)$ *in* A *such that* $\overline{b} = a$. *Here* $\overline{(\cdot)}$ *denotes reduction modulo* $\mathfrak{m}$.

Proof. We first prove that (1) implies (2). Set $Y = \operatorname{Spec} A$ and

$$X = \operatorname{Spec} \frac{A[T][1/f'(T)]}{f(T)}.$$

Then giving a simple root a of $\overline{f}(T)$ in κ is the same thing as giving a morphism of A-schemes $\operatorname{Spec}\kappa \to X$. The lifting property in (1) implies that this morphism extends uniquely to a morphism of A-schemes $\operatorname{Spec} A \to X$, that is, a root b of $f(T)$ in A such that $\overline{b} = a$:

$$\begin{tikzcd} \operatorname{Spec}\kappa \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r, "\mathrm{Id}"'] \arrow[ur, dashed, "\exists! b"] & Y \end{tikzcd}$$

Observe that any root c of $f(T)$ in A with $\overline{c} = a$ satisfies $\overline{f'(c)} = \overline{f'}(a) \neq 0$ so that $f'(c)$ is invertible in A; therefore, c corresponds to a morphism of A-schemes $\operatorname{Spec} A \to X$, and the uniqueness statement in (1) implies uniqueness in (2).

Now, we prove that (2) implies (1). Let $X \to Y$ be an étale morphism of schemes that fits into a commutative square

$$\begin{tikzcd} \operatorname{Spec}\kappa \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

In order to lift the morphism $\operatorname{Spec} A \to Y$ to X, we can reduce to the case where $\operatorname{Spec} A \to Y$ is the identity on $\operatorname{Spec} A$ by performing a base change, replacing X by $X \times_Y \operatorname{Spec} A$. Since every étale morphism of schemes is locally of standard form, we may further assume that the A-scheme X is given by

$$X = \operatorname{Spec} \frac{A[T][1/g(T)]}{(h(T))}$$

where $g(T)$ and $h(T)$ are polynomials in $A[T]$, the polynomial $h(T)$ is monic, and the derivative $h'(T)$ is invertible in the ring

$$\frac{A[T][1/g(T)]}{(h(T))}.$$

The given morphism $\operatorname{Spec}\kappa \to X$ then corresponds to a simple root a of $\overline{h}(T)$ in κ. Applying (2) to the polynomial $h(T)$, we find that there exists a unique root b of $h(T)$ in A such that $\overline{b} = a$. This root corresponds to a morphism

$$\operatorname{Spec} A \to \operatorname{Spec} A[T]/(h(T))$$

that factors through the open

$$X = \operatorname{Spec} \frac{A[T][1/g(T)]}{(h(T))}$$

because $\overline{g(b)} = \overline{g}(a) \neq 0$ so that $g(b)$ is invertible in A. $\square$

Definition 4.3.2. We say that a local ring A is *henselian* if it satisfies the equivalent properties in Proposition 4.3.1.

Example 4.3.3.

(1) For every prime p, the ring $\mathbb{Z}_p$ of p-adic integers is henselian. For every field κ, the formal power series ring $\kappa[\![t]\!]$ is henselian. One can prove that these rings satisfy property (2) in Proposition 4.3.1 by means of the Newton approximation method. Contrary that what happens over the real numbers, here the sequence of approximate roots will always converge to a root. More generally, one can use this argument to show that every complete noetherian local ring is henselian (and even every complete and separated local ring). This result is often called *Hensel's lemma*, after Kurt Hensel, who discovered the p-adic numbers.

(2) The ring $R = \cup_{d>0} k[\![t^{1/d}]\!]$ is henselian: for each polynomial $f(T)$ in $R[T]$, all the coefficients of f will lie in $k[\![t^{1/d}]\!]$ for some sufficiently divisible value of d, so that Hensel's lemma for the power series ring $k[\![t^{1/d}]\!]$ also implies that R satifies property (2) in Proposition 4.3.1.

(3) Consider the ring $\mathcal{O}_{\mathbb{C},0}$ of germs of holomorphic functions around the origin of $\mathbb{C}$ (that is, the ring of formal power series in $\mathbb{C}[\![t]\!]$ with positive radius of convergence). It follows directly from the inverse function theorem that this ring is henselian.

(4) The local ring $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}},0} = \mathbb{C}[t]_{(t)}$ is *not* henselian: let $f(T)$ be the polynomial $T^2 - (t+1)$ over $\mathbb{C}[t]_{(t)}$. The reduction of f modulo (t) has a simple root at $T = 1$, but f has no roots in $\mathbb{C}[t]_{(t)}$, because $t+1$ has no square root in the field of rational functions $\mathbb{C}(t)$. Observe that $t+1$ does have a square root in $\mathcal{O}_{\mathbb{C},0}$ and $\mathbb{C}[\![t]\!]$ (given by the Taylor expansion of $\sqrt{t+1}$ around $t = 0$).

5. Lecture 5 (31 October)

5.1. **Proof of Proposition 4.1.2.** The henselian property of R has the following useful consequence.

Lemma 5.1.1. *Let $h\colon \mathscr{X} \to \mathscr{Y}$ be an étale morphism of R-schemes. Then the square*

$$\begin{tikzcd} \mathscr{X}(R) \arrow[r, "\mathrm{sp}_{\mathscr{X}}"] \arrow[d, "h"'] & \mathscr{X}(k) \arrow[d, "h"] \\ \mathscr{Y}(R) \arrow[r, "\mathrm{sp}_{\mathscr{Y}}"'] & \mathscr{Y}(k) \end{tikzcd}$$

is Cartesian. In other words, for every point b in $\mathscr{Y}(R)$ and every point $\overline{a}$ in $\mathscr{X}(k)$ such that $\mathrm{sp}_{\mathscr{Y}}(b) = h(\overline{a})$, there exists a unique element a in $\mathscr{X}(R)$ satisfying $h(a) = b$ and $\mathrm{sp}_{\mathscr{X}}(a) = \overline{a}$.

Proof. This is a special case of property (1) in Proposition 4.3.1, applied to the henselian local ring $A = R$. $\square$

We are finally in position to prove Proposition 4.1.2; let us first recall the statement.

Proposition (4.1.2)**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be smooth and proper R-schemes, and assume that the special fibers $\mathscr{X}\times_R k$ and $\mathscr{Y}\times_R k$ are isomorphic (as k-schemes). Then $\mathscr{X}(K)$ and $\mathscr{Y}(K)$ are isomorphic as semialgebraic sets.*

Proof. We fix an isomorphism between $\mathscr{X}\times_R k$ and $\mathscr{Y}\times_R k$, which we use to identify these k-schemes in what follows. Let $\{U_1,\ldots,U_r\}$ be a cover of $\mathscr{X}\times_R k$ by open subschemes. For every non-empty subset J of $\{1,\ldots,r\}$, we write

$$C_J=\bigcap_{j\in J}U_j(k),\qquad C_J^o=C_J\setminus\bigcup_{j\notin J}U_j(k).$$

The sets C_J^o are constructible and form a partition of $\mathscr{X}(k)=\mathscr{Y}(k)$. Consequently, the sets $\mathrm{sp}_{\mathscr{X}}^{-1}(C_J^o)$ and $\mathrm{sp}_{\mathscr{Y}}^{-1}(C_J^o)$ form a partition of $\mathscr{X}(R)$, resp. $\mathscr{Y}(R)$, into semialgebraic pieces. Since $\mathscr{X}$ and $\mathscr{Y}$ are proper over R, we have $\mathscr{X}(R)=\mathscr{X}(K)$ and $\mathscr{Y}(R)=\mathscr{Y}(K)$.

Suppose that, for each j in $\{1,\ldots,r\}$, there exists an isomorphism of semialgebraic sets

$$\mathrm{sp}_{\mathscr{X}}^{-1}(U_j(k))\to\mathrm{sp}_{\mathscr{Y}}^{-1}(U_j(k))$$

that commutes with the specialization maps $\mathrm{sp}_{\mathscr{X}}$ and $\mathrm{sp}_{\mathscr{Y}}$. Then this isomorphism restricts to an isomorphism

$$\mathrm{sp}_{\mathscr{X}}^{-1}(C_J^o)\to\mathrm{sp}_{\mathscr{Y}}^{-1}(C_J^o)$$

for every subset J of $\{1,\ldots,r\}$ that contains j. It follows that the semialgebraic sets $\mathrm{sp}_{\mathscr{X}}^{-1}(C_J^o)$ and $\mathrm{sp}_{\mathscr{Y}}^{-1}(C_J^o)$ are isomorphic for each non-empty subset J of $\{1,\ldots,r\}$. Since these sets form a partition of $\mathscr{X}(K)$ and $\mathscr{Y}(K)$, respectively, we know by Example 4.1.1 that $\mathscr{X}(K)$ is isomorphic to the disjoint union of the sets $\mathrm{sp}_{\mathscr{X}}^{-1}(C_J^o)$, and similarly for $\mathscr{Y}$. Therefore, $\mathscr{X}(K)$ and $\mathscr{Y}(K)$ are isomorphic as semialgebraic sets.

This means that it suffices to prove that around every point x of $\mathscr{X}\times_R k=\mathscr{Y}\times_R k$, we can find an open subcheme U such that there exists an isomorphism of semialgebraic sets

$$\mathrm{sp}_{\mathscr{X}}^{-1}(U(k))\to\mathrm{sp}_{\mathscr{Y}}^{-1}(U(k))$$

that commutes with the specialization maps $\mathrm{sp}_{\mathscr{X}}$ and $\mathrm{sp}_{\mathscr{Y}}$. Since $\mathscr{X}$ is smooth over R, the special fiber $\mathscr{X}\times_R k$ is smooth over k. The theory of smooth morphisms tells us that a k-scheme Z is smooth if and only if it can be covered by opens U that admit an étale morphism to an affine space $\mathbb{A}^n_k$. Let U be such an open in $\mathscr{X}\times_R k$ that contains x, with an étale morphism

$$U\to\mathbb{A}^n_k=\operatorname{Spec}k[T_1,\ldots,T_n].$$

Pulling back the coordinate functions $T_1,\ldots,T_n$, we get regular functions $f_1,\ldots,f_n$ on U. Shrinking U around x, we may assume that there exist an open subscheme $\mathscr{U}$ of $\mathscr{X}$ and an open subscheme $\mathscr{V}$ of $\mathscr{Y}$ such that U is the intersection of $\mathscr{U}$ and $\mathscr{V}$ with he special fiber $\mathscr{X}\times_R k=\mathscr{Y}\times_R k$, and such that the functions f_j extend to regular functions g_j and h_j on $\mathscr{U}$ and $\mathscr{V}$, respectively. These extensions define morphisms $\mathscr{U}\to\mathbb{A}^n_R$ and $\mathscr{V}\to\mathbb{A}^n_R$ that restrict to $U\to\mathbb{A}^n_k$ on the special fibers.

The R-schemes $\mathscr{U}$ and $\mathscr{V}$ are flat over R because they are smooth. Now the fact that the morphisms $\mathscr{U}\to A^n_R$ and $\mathscr{V}\to\mathbb{A}^n_R$ are étale on the special fibers implies that they are also étale on some open neighbourhoods of the special fibers in $\mathscr{U}$ and $\mathscr{V}$ (take this as a black box if you have not seen flatness before). Replacing $\mathscr{U}$ and $\mathscr{V}$ by these open neighbourhoods, we have constructed étale morphisms $\mathscr{U} \to \mathbb{A}^n_R$ and $\mathscr{V} \to \mathbb{A}^n_R$ with the same restriction to the special fiber U. Note that $\mathscr{U}(R) = \mathrm{sp}_{\mathscr{X}}^{-1}(U(k))$ and $\mathscr{V}(R) = \mathrm{sp}_{\mathscr{Y}}^{-1}(U(k))$, so that it suffices to construct a semialgebraic isomorphism $\mathscr{U}(R) \to \mathscr{V}(R)$ that commutes with the specialization maps.

Consider the fibered product

$$\mathscr{W} = \mathscr{U} \times_{\mathbb{A}^n_R} \mathscr{V}.$$

All the morphisms in the diagram

$$\begin{tikzcd} & \mathscr{W} \arrow[dl] \arrow[dr] & \\ \mathscr{U} \arrow[dr] & & \mathscr{V} \arrow[dl] \\ & \mathbb{A}^n_R & \end{tikzcd}$$

are étale, because being étale is preserved by base change. The special fiber of $\mathscr{W}$ is

$$\mathscr{W} \times_R k = U \times_k U$$

so that we can consider U as a subscheme of $\mathscr{W} \times_R k$ *via* the diagonal embedding

$$\Delta \colon U \to U \times_k U.$$

It follows from Lemma 5.1.1 that the semialgebraic set

$$\mathrm{sp}_{\mathscr{W}}^{-1}(U(k)) \subset \mathscr{U}(R) \times \mathscr{V}(R)$$

is the graph of a bijection $\mathscr{U}(R) \to \mathscr{V}(R)$: every point b in $\mathscr{U}(R)$ lifts uniquely to a point a in $\mathscr{W}(R)$ such that $\mathrm{sp}_{\mathscr{U}}(b) = \mathrm{sp}_{\mathscr{W}}(a)$ in $U(k)$, and the analogous property holds for $\mathscr{V}$. This semialgebraic bijection commutes with the specialization maps on $\mathscr{U}$ and $\mathscr{V}$. □

Exercise 5.1.2. Adapt the proof of Proposition 4.1.2 to prove the following more general result: let $\mathscr{X}$ and $\mathscr{Y}$ be smooth separated R-schemes of finite type of the same dimension. Let U be a subscheme of $\mathscr{X} \times_R k$ and let V be a subscheme of $\mathscr{Y} \times_R k$ such that U and V are isomorphic as k-schemes. Then $\mathrm{sp}_{\mathscr{X}}^{-1}(U(k))$ is isomorphic to $\mathrm{sp}_{\mathscr{Y}}^{-1}(V(k))$ as semialgebraic sets.

6. Lecture 6 (2 November)

In the previous lectures, we have seen how semialgebraic geometry of K is connected to convex geometry over $\mathbb{Q}$ by means of the tropicalization map (t-adic valuation), and to algebraic geometry over k by means of the specialization map (reduction modulo $\mathfrak{m}$). We summarize these connections in the following diagram:

$$\begin{tikzcd}
& \text{Semialgebraic geometry}/K \arrow[dl, "\mathrm{trop}"'] \arrow[dr, "\mathrm{sp}"] & \\
\text{Convex geometry}/\mathbb{Q} \arrow[ur, dashed, bend left=60, "\mathrm{trop}^{-1}"] & & \text{Algebraic geometry}/k \arrow[ul, dashed, bend right=60, "\mathrm{sp}^{-1}"']
\end{tikzcd}$$

Remarkably, semialgebraic geometry over K is controlled entirely by convex geometry over $\mathbb{Q}$ and algebraic geometry over k, in the following sense:

(1) Any semialgebraic set over K can be built from pieces of the form $\mathrm{trop}^{-1}(\Gamma)$ and $\mathrm{sp}_{\mathscr{X}}^{-1}(C)$, where Γ is a definable set in $\mathbb{Q}^n$ for some $n \geq 0$, $\mathscr{X}$ is a smooth separated R-scheme of finite type, and C is a constructible subset of $\mathscr{X}(k)$.

(2) The description of a semialgebraic set in terms of such pieces is not unique, but we can control all the different descriptions of this type.

These statements lie at the core of Hrushovski and Kazhdan's theory of *motivic integration*, which aims to classify semialgebraic sets over K and construct geometric invariants of such sets. To make them precise, we need to introduce various *Grothendieck rings*.

6.1. **The Grothendieck ring of semialgebraic sets.** We denote by $\mathbf{K}_0(\mathrm{VF}_K)$ the abelian group defined by the following presentation:

- generators: isomorphism classes $[S]$ of semialgebraic sets S over K;
- relations: for every semialgebraic set S and every semialgebraic subset T of S, we have
$$[S] = [T] + [S \setminus T].$$

These relations are called *scissor relations*, because they allow us to cut a semialgebraic set into semialgebraic pieces. Note that S is isomorphic to the disjoint union of T and $S \setminus T$, by Example 4.1.1, so that we would obtain the same group by replacing the scissor relations by the relations $[S \sqcup S'] = [S] + [S']$ for all semialgebraic sets S and S'. The scissor relations imply that $[\emptyset] = [\emptyset] + [\emptyset]$ so that $[\emptyset] = 0$ in $\mathbf{K}_0(\mathrm{VF}_K)$.

If X and X' are K-schemes of finite type, and S and S' are semialgebraic subsets of $X(K)$ and $X'(K)$, respectively, then it is easy to show that $S \times S'$ is a semialgebraic subset of $X(K) \times X'(K) = (X \times_K X')(K)$. There exists a unique ring structure on $\mathbf{K}_0(\mathrm{VF}_K)$ with the property that $[S] \cdot [S'] = [S \times S']$ for all semialgebraic sets S and S'. The identity element for the multiplication is the isomorphism class of a point. We call $\mathbf{K}_0(\mathrm{VF}_K)$ the *Grothendieck ring of semialgebraic sets over K*.

6.2. **The Grothendieck ring of definable sets over $\mathbb{Q}$.**

Definition 6.2.1. A subset of $\mathbb{Q}^n$ is called *definable* if we can write it as a finite Boolean combination of polytopes in $\mathbb{Q}^n$ (in the sense of Definition 2.2.1). Two definable subsets Γ and Γ' of $\mathbb{Q}^n$ are called *isomorphic* if there exists a unimodular affine transformation
$$\phi\colon \mathbb{Q}^n \to \mathbb{Q}^n,\ w \mapsto Aw + b$$
with $A \in \mathrm{GL}_n(\mathbb{Z})$ and $b \in \mathbb{Q}^n$ such that $\Gamma' = \phi(\Gamma)$.

Here *unimodular* refers to the fact that we force the matrix A and its inverse to have coefficients in $\mathbb{Z}$. For instance, a homothety with scaling factor different from ± 1 is not unimodular. Note that we do not consider isomorphisms between definable sets in $\mathbb{Q}^m$ and $\mathbb{Q}^n$ for $m \neq n$; in particular, a point in $\mathbb{Q}^m$ is not isomorphic to a point in $\mathbb{Q}^n$ unless $m = n$.

For every $n \geq 0$, we denote by $\mathbf{K}_0(\mathbb{Q}[n])$ the abelian group defined by the following presentation:

- generators: isomorphism classes $[\Gamma]$ of definable subsets Γ in $\mathbb{Q}^n$;
- relations: if $\Gamma' \subset \Gamma$ are definable subsets in $\mathbb{Q}^n$, then

$$[\Gamma] = [\Gamma'] + [\Gamma \setminus \Gamma'].$$

The scissor relations again imply that $[\emptyset] = 0$ in $\mathbf{K}_0(\mathbb{Q}[n])$. We also consider the graded abelian group

$$\mathbf{K}_0^{\dim}(\mathbb{Q}) = \bigoplus_{n \geq 0} \mathbf{K}_0(\mathbb{Q}[n]).$$

If Γ is a definable subset of $\mathbb{Q}^n$ and we want to make the grading explicit, we write $[\Gamma]_n$ for the class of Γ in $\mathbf{K}_0^{\dim}(\mathbb{Q})$.

If Γ is a definable subset of $\mathbb{Q}^m$ and Γ' is a definable subset of $\mathbb{Q}^n$, then it is easy to see that $\Gamma \times \Gamma'$ is a definable subset of $\mathbb{Q}^{m+n}$. There exists a unique graded ring structure on $\mathbf{K}_0^{\dim}(\mathbb{Q})$ such that

$$[\Gamma]_m \cdot [\Gamma']_n = [\Gamma \times \Gamma']_{m+n}$$

for all such Γ and Γ'. The identity element for the ring multiplication is $[\mathrm{pt}]_0$, the class of the point in $\mathbb{Q}^0$.

The fact that we can not only add in $\mathbf{K}_0^{\dim}(\mathbb{Q})$, but also subtract, leads to some unexpected identities between classes of non-isomorphic definable sets.

Example 6.2.2. For all $a, b \in \mathbb{Q}$, the definable subsets $\mathbb{Q}_{\geq a} = [a, +\infty)$ and $\mathbb{Q}_{\geq b} = [b, +\infty)$ are isomorphic *via* translation. It follows from the scissor relations that, when $a \leq b$, we have

$$[\mathbb{Q}_{\geq a}] = [\,[a, b)\,] + [\mathbb{Q}_{\geq b}]$$

in $\mathbf{K}_0(\mathbb{Q}[1])$, so that the class of $[a, b)$ is equal to zero.

This also implies that

$$[\,[a, b]\,] = [\,[a, b)\,] + [\,\{b\}\,] = [\,\{0\}\,]$$

in $\mathbf{K}_0(\mathbb{Q}[1])$: every bounded closed interval has the same class as a point in the Grothendieck group of definable subsets of $\mathbb{Q}$. By induction on the dimension, one can show more generally that $[P]_n = [\,\{0\}\,]_n$ for every $n \geq 0$ and every bounded polytope P in $\mathbb{Q}^n$.

Example 6.2.3. Let Γ be the definable subset $\mathbb{Q}_{>0} \times \mathbb{Q}_{\geq 0}$ in $\mathbb{Q}^2$. Then we can partition Γ into the definable subsets

$$\begin{aligned} \Gamma_1 &= \{(u, v) \in \mathbb{Q}^2 \mid u > v \geq 0\} \\ \Gamma_2 &= \{(u, v) \in \mathbb{Q}^2 \mid v \geq u > 0\} \end{aligned}$$

Each of these subsets is isomorphic to Γ, so that

$$[\Gamma] = [\Gamma_1] + [\Gamma_2] = [\Gamma] + [\Gamma]$$

in $\mathbf{K}_0(\mathbb{Q}[2])$, which implies $[\Gamma] = 0$. We will later show that $[\mathbb{Q}_{>0}]$ and $[\mathbb{Q}_{\geq 0}]$ are different from zero in $\mathbf{K}_0(\mathbb{Q}[1])$; thus, these elements are non-trivial zero divisors in $\mathbf{K}_0^{\dim}(\mathbb{Q})$.

Seeing these examples, you may wonder if there is any interesting geometric information left in $\mathbf{K}^{\dim}(\mathbb{Q})$. Such information can be extracted by constructing suitable *additive invariants* of definable sets in $\mathbb{Q}^n$; this means that we attach a value in an abelian group G to any definable set in $\mathbb{Q}^n$ such that this value is stable under isomorphism and additive with respect to partitions. Such invariants induce a group morphism $\mathbf{K}^{\dim}(\mathbb{Q}) \to G$.

Proposition 6.2.4. *There is a unique way to attach a value $\chi(\Gamma) \in \mathbb{Z}$ to every definable subset Γ in $\mathbb{Q}^n$, for every $n \geq 0$, such that the following properties are satisfied:*

(1) *for every* bounded *non-empty polytope P, we have $\chi(P) = 1$;*
(2) *we have $\chi(\mathbb{Q}_{\geq 0}) = 0$;*
(3) *(additivity) if $\Gamma' \subset \Gamma$ are definable subsets of $\mathbb{Q}^n$, then*
$$\chi(\Gamma) = \chi(\Gamma') + \chi(\Gamma \setminus \Gamma');$$
(4) *(multiplicativity) if Γ is a definable subset of $\mathbb{Q}^m$ and Γ' is a definable subset of $\mathbb{Q}^n$, then*
$$\chi(\Gamma \times \Gamma') = \chi(\Gamma) \cdot \chi(\Gamma');$$
(5) *(invariance) for every definable subset Γ of $\mathbb{Q}^n$ and every affine transformation*
$$\phi\colon \mathbb{Q}^n \to \mathbb{Q}^n,\ w \mapsto Aw + b$$
with $A \in \mathrm{GL}_n(\mathbb{Q})$ and $b \in \mathbb{Q}^n$, we have
$$\chi(\Gamma) = \chi(\phi(\Gamma)).$$
In particular, χ takes the same value on isomorphic definable sets.

Note that, in the invariance axiom, we do not require ϕ to be unimodular. The invariant χ is called the *Euler characteristic*. The cleanest way to prove Proposition 6.2.4 is to use the theory of *o*-minimality in mathematical logic; see the book “Tame topology and *o*-minimal structures” by L. van den Dries. The strategy is to decompose any definable set into elementary pieces, for which the value of χ is determined by the axioms in Proposition 6.2.4, and to show that the sum of the Euler characteristics of the pieces is independent of the choice of the decomposition. The Euler characteristic also has a topological interpretation: to any definable subset in $\mathbb{Q}^n$ one can attach a corresponding set in $\mathbb{R}^n$ in a canonical way, defined by the same linear inequalities and Boolean operations; for such sets we can use the Euler characteristic with compact supports to define the invariant χ.

Since χ is additive, multiplicative and constant on isomorphism classes, it defines a ring morphism
$$\chi\colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \to \mathbb{Z},\ [\Gamma]_n \mapsto \chi(\Gamma),$$
which we can use to distinguish certain classes in the Grothendieck ring of definable sets.

Example 6.2.5. Additivity implies that
$$\chi(\mathbb{Q}_{>0}) = \chi(\mathbb{Q}_{\geq 0}) - \chi(\{0\}) = 0 - 1 = -1.$$
This shows in particular that $[\mathbb{Q}_{>0}] \neq 0$ in $\mathbf{K}_0(\mathbb{Q}[1])$.

We can tweak the construction of the Euler characteristic to obtain a second invariant.

Proposition 6.2.6. *There is a unique way to attach a value $\chi'(\Gamma) \in \mathbb{Z}$ to every definable subset Γ in $\mathbb{Q}^n$, for every $n \geq 0$, such that the following properties are satisfied:*

(1) *for every non-empty polytope P, we have $\chi'(P) = 1$;*
(2) *(additivity) if $\Gamma' \subset \Gamma$ are definable subsets of $\mathbb{Q}^n$, then*
$$\chi'(\Gamma) = \chi'(\Gamma') + \chi'(\Gamma \setminus \Gamma');$$
(3) *(multiplicativity) if Γ is a definable subset of $\mathbb{Q}^m$ and Γ' is a definable subset of $\mathbb{Q}^n$, then*
$$\chi'(\Gamma \times \Gamma') = \chi'(\Gamma) \cdot \chi'(\Gamma');$$
(4) *(invariance) for every definable subset Γ of $\mathbb{Q}^n$ and every affine transformation*
$$\phi\colon \mathbb{Q}^n \to \mathbb{Q}^n,\ w \mapsto Aw + b$$
with $A \in \mathrm{GL}_n(\mathbb{Q})$ and $b \in \mathbb{Q}^n$, we have
$$\chi(\Gamma) = \chi(\phi(\Gamma)).$$
In particular, χ takes the same value on isomorphic definable sets.

The key difference with the Euler characteristic χ is the behaviour on unbounded polytopes. The invariant χ' is called the *bounded Euler characteristic*, for the following reason: if Γ is a definable subset of $\mathbb{Q}^n$, then one can show that for all sufficiently large elements a of $\mathbb{Q}$, the value
$$\chi(\Gamma \cap [-a, a]^n)$$
is independent of a. This limit value is precisely the bounded Euler characteristic $\chi'(\Gamma)$. We again obtain a ring morphism
$$\chi'\colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \to \mathbb{Z},\ [\Gamma]_n \mapsto \chi'(\Gamma).$$

Example 6.2.7. Since $\mathbb{Q}_{\geq 0}$ is a polytope in $\mathbb{Q}$, we have $\chi'(\mathbb{Q}_{\geq 0}) = 1$, so that $[\mathbb{Q}_{\geq 0}] \neq 0$ in $\mathbf{K}_0(\mathbb{Q}[1])$.

It turns out that the invariants χ and χ' completely characterize homogeneous classes of fixed degree in $\mathbf{K}_0^{\dim}(\mathbb{Q})$. More precisely, we have the following theorem.

Theorem 6.2.8. *The ring morphism*
$$\mathbf{K}_0^{\dim}(\mathbb{Q}) \to \mathbb{Z}[x, y]/(xy),\ [\Gamma]_n \mapsto (-1)^n \chi(\Gamma) x^n + \chi'(\Gamma) y^n$$
is an isomorphism of graded rings. Its inverse maps x to $[\mathbb{Q}_{>0}]_1$ and y to $[\mathbb{Q}_{\geq 0}]_1$.

Proof. We only give a sketch of the argument. We have seen in Example 6.2.3 that $[\mathbb{Q}_{>0}]_1 \cdot [\mathbb{Q}_{\geq 0}]_1 = 0$ in $\mathbf{K}_0^{\dim}(\mathbb{Q})$. Consequently, the unique ring morphism $\mathbb{Z}[x, y] \to \mathbf{K}_0^{\dim}(\mathbb{Q})$ that maps x to $[\mathbb{Q}_{>0}]_1$ and y to $[\mathbb{Q}_{\geq 0}]_1$ factors through a graded ring morphism
$$\mathbb{Z}[x, y]/(xy) \to \mathbf{K}_0^{\dim}(\mathbb{Q}).$$
We have $\chi(\mathbb{Q}_{\geq 0}) = 0$ and $\chi'(\mathbb{Q}_{\geq 0}) = 1$. Writing $\mathbb{Q}_{\geq 0}$ as the disjoint union of $\{0\}$ and $\mathbb{Q}_{>0}$, the additivity of χ and χ' implies that $\chi(\mathbb{Q}_{>0}) = -1$ and $\chi'(\mathbb{Q}_{>0}) = 0$. It follows that the morphism
$$\mathbb{Z}[x, y]/(xy) \to \mathbf{K}_0^{\dim}(\mathbb{Q})$$

is left inverse to the morphism in the statement. Therefore, we only need to show that every element in $\mathbf{K}_0^{\dim}(\mathbb{Q})$ can be written as a polynomial combination of $[\mathbb{Q}_{>0}]_1$ and $[\mathbb{Q}_{\geq 0}]_1$ with coefficients in $\mathbb{Z}$. This follows from a (somewhat involved) decomposition argument. $\square$

A surprising consequence of Theorem 6.2.8 is that the class of a definable set is preserved by *all* affine transformations, not only the unimodular ones.

Corollary 6.2.9. *Let Γ be a definable subset in $\mathbb{Q}^n$, and let*

$$\phi\colon \mathbb{Q}^n \to \mathbb{Q}^n,\, w \mapsto Aw + b$$

be an affine transformation, with $A \in \mathrm{GL}_n(\mathbb{Q})$ and $b \in \mathbb{Q}^n$. Then $[\Gamma] = [\phi(\Gamma)]$ in $\mathbf{K}_0^{\dim}(\mathbb{Q})$.

Proof. This follows at once from the invariance of χ and χ' under affine transformations. $\square$

Corollary 6.2.9 implies that we would have gotten exactly the same Grothendieck ring if we had allowed arbitrary affine transformations in our definition of isomorphisms of definable sets, which begs the question why we used the more restrictive definition in the first place. The reason is provided by the following exercise, which is false with respect to arbitrary affine transformations.

Exercise 6.2.10. Let Γ and Γ' be isomorphic definable sets in $\mathbb{Q}^n$. Show that $\mathrm{trop}^{-1}(\Gamma)$ and $\mathrm{trop}^{-1}(\Gamma')$ are isomorphic semialgebraic subsets of $(K^*)^n$.

It follows that there exists a unique ring morphism

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \to \mathbf{K}_0(\mathrm{VF}_K)$$

that maps $[\Gamma]_n$ to $[\mathrm{trop}^{-1}(\Gamma)]$ for every $n \geq 0$ and every definable subset Γ in $\mathbb{Q}^n$, where

$$\mathrm{trop}\colon (K^*)^n \to \mathbb{Q}^n$$

is the tropicalization map in dimension n. Indeed, applying trop^{-1} commutes with taking partitions and products, so that Θ respects the scissor relations and the ring structure.

7. Lecture 7 (7 November)

In the previous lecture, we have seen how one can use the tropicalization map to connect the Grothendieck ring of definable sets in $\mathbb{Q}^n$ to the Grothendieck ring of semialgebraic sets over K. We will now build a similar bridge between k-varieties and semialgebraic sets using the specialization map.

7.1. **The Grothendieck rings of varieties.** Let F be a field. For every $n \geq 0$, we denote by $\mathbf{K}_0(\mathrm{Var}_F^{\leq n})$ the abelian group defined by the following presentation:

- generators: isomorphism classes $[X]_n$ of F-schemes X of finite type and of dimension at most n;
- relations: whenever Y is a closed subscheme of X, we have

$$[X]_n = [Y]_n + [X \setminus Y]_n.$$

As usual, the scissor relations imply that $[\emptyset]_n = 0$. They also imply that $\mathbf{K}_0(\mathrm{Var}_F^{\leq n})$ does not detect any non-reduced structure: if X is a F-scheme of finite type of dimension at most n, and X_{red} is its maximal reduced closed subscheme (defined by the nilradical in $\mathcal{O}_X$), then $X_{\mathrm{red}} \to X$ is a closed embedding with empty complement, so that

$$[X]_n = [X_{\mathrm{red}}]_n + [\emptyset]_n = [X_{\mathrm{red}}]_n.$$

We combine the groups $\mathbf{K}_0(\mathrm{Var}_F^{\leq n})$ into a graded abelian group

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_F) = \bigoplus_{n\geq 0} \mathbf{K}_0(\mathrm{Var}_F^{\leq n}).$$

Note that a F-scheme of finite type X defines a distinct element $[X]_n$ in $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)$ for each $n \geq \dim(X)$. The graded abelian group $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)$ has a unique graded ring structure such that, for all F-schemes X and X' of finite type and of dimensions at most m, resp. n, we have

$$[X]_m \cdot [X']_n = [X \times_F X']_{m+n}.$$

The identity element for the multiplication is $[\operatorname{Spec} F]_0$, the class of a point placed in degree 0. We call $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)$ the *Grothendieck ring of F-varieties graded by dimension.*

We can also define a different ring by removing the bounds on dimension. We denote by $\mathbf{K}_0(\mathrm{Var}_F)$ the abelian group defined by the following presentation:

- generators: isomorphism classes $[X]$ of F-schemes X of finite type;
- relations: whenever Y is a closed subscheme of X, we have

$$[X] = [Y] + [X \setminus Y].$$

This abelian group has a unique ring structure satisfying $[X] \cdot [X'] = [X \times_F X']$ for all F-schemes X and X' of finite type, with identity element $[\operatorname{Spec} F]$. The resulting ring is called the *Grothendieck ring of F-varieties.* We will later see that it contains strictly less information than the graded version. The next proposition explains how the two rings are related.

Proposition 7.1.1. *The forgetful map*

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_F) \to \mathbf{K}_0(\mathrm{Var}_F),\ [X]_n \mapsto [X]$$

is a surjective ring morphism, and its kernel is the ideal I generated by $[\operatorname{Spec} F]_1 - [\operatorname{Spec} F]_0$.

Proof. It is obvious that this map is a ring morphism, and that it is surjective. It is also clear that it sends $[\operatorname{Spec} F]_1 - [\operatorname{Spec} F]_0$ to 0, so that it factors through a ring morphism

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_F)/I \to \mathbf{K}_0(\mathrm{Var}_F).$$

This is an isomorphism: its inverse sends $[X]$ to $[X]_n$, for every F-scheme X of finite type and any $n \geq \dim(X)$. Note that the residue class of $[X]_n$ modulo I does not depend on n because

$$[X]_{n+1} = [X]_n \cdot [\operatorname{Spec} F]_1 = [X]_n \cdot [\operatorname{Spec} F]_0 = [X]_n$$

in $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)/I$. $\square$

Proposition 7.1.2. *There exists a unique ring morphism*

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{VF}_K)$$

such that, for every smooth separated R-scheme $\mathscr{X}$ of finite type and every subscheme Z of $\mathscr{X} \times_R k$, we have

$$\Theta([Z]_n) = [\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))]$$

where n denotes the dimension of $\mathscr{X} \times_R K$.

Proof. Uniqueness. Assume that a morphism Θ as in the statement exists. Let Z be a k-scheme of finite type and let n be an integer such that $n \geq \dim(Z)$. We will prove that there is only one possible value for $\Theta([Z]_n)$. Since k has characteristic zero, we can partition Z into smooth separated subschemes (use generic smoothness of the underlying reduced scheme Z_{red} and noetherian induction). By the scissor relations in $\mathbf{K}_0^{\dim}(\mathrm{Var}_k)$, we may assume that Z is itself smooth and separated. The ring R is a k-algebra *via* the morphism $k \to R$ that maps an element a in k to the constant Puiseux series a. Set

$$\mathscr{X} = (Z \times_k R) \times_R \mathbb{A}_R^{n-\dim(Z)}.$$

This is a smooth separated R-scheme with $\dim(\mathscr{X} \times_R K) = n$, and we can view Z as a subscheme of $\mathscr{X} \times_R k$ *via* the embedding

$$\mathrm{Id} \times \{0\}\colon Z \to \mathscr{X} \times_R k = Z \times \mathbb{A}_k^{n-\dim(Z)}$$

By the property in the statement, we must have

$$\Theta([Z]_n) = [\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))] = [Z(R) \times \mathfrak{m}^{n-\dim(Z)}].$$

Existence. Let Z be a k-scheme of finite type and let n be an integer such that $n \geq \dim(Z)$. Then our uniqueness proof shows that we can partition Z into subschemes $Z_1, \ldots, Z_r$ such that for every i in $\{1, \ldots, r\}$, there exists an embedding of Z_i into a smooth separated R-scheme $\mathscr{X}_i$ with $\dim(\mathscr{X}_i \times_R K) = n$. Now we define

$$\Theta([Z]_n) = \sum_{i=1}^{r} [\mathrm{sp}_{\mathscr{X}_i}^{-1}(Z_i(k))].$$

It follows from Exercise 5.1.2 that this expression does not depend on the choice of the embeddings $Z_i \to \mathscr{X}_i$. It is also independent of the choice of the partition for Z, because any two partitions have a common refinement, and the scissor relations imply that our expression is invariant under refinements of the partition. It follows directly from the construction that Θ respects the scissor relations in $\mathbf{K}_0^{\dim}(\mathrm{Var}_k)$ and that it is multiplicative, so that we obtain a ring morphism

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{VF}_K)$$

with the desired properties. □

The ring morphisms

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \to \mathbf{K}_0(\mathrm{VF}_K)$$
$$\Theta\colon \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{VF}_K)$$

together induce a ring morphism

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{VF}_K),\ a \otimes b \mapsto \Theta(a) \cdot \Theta(b).$$

The core of the theory of motivic integration of Hrushovski and Kazhdan consists of the following two miraculous properties.

Miracle A. The morphism

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{VF}_K)$$

is surjective. In this sense, every semialgebraic set over K can be built[5] from pieces of the form $\mathrm{trop}^{-1}(\Gamma)$ and $\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))$, with Γ a definable subset of $\mathbb{Q}^n$, and with $\mathscr{X}$ a smooth separated R-scheme and Z a subscheme of $\mathscr{X} \times_R k$.

Miracle B. The morphism Θ is not injective, but we can completely describe its kernel.

Example 7.1.3. The semialgebraic set R^* can be written as $\mathrm{trop}^{-1}(0)$, where $\mathrm{trop}\colon K^* \to \mathbb{Q}$ is the tropicalization map in dimension 1, but also as

$$R^* = \mathbb{G}_{m,R}(R) = \mathrm{sp}^{-1}_{\mathbb{G}_{m,R}}(\mathbb{G}_{m,k}(k)).$$

Consequently,

$$[R^*] = \Theta([\,\{0\}\,]_1 \otimes 1) = \Theta(1 \otimes [\mathbb{G}_{m,k}]_1)$$

and the element

$$[\,\{0\}\,]_1 \otimes 1 - 1 \otimes [\mathbb{G}_{m,k}]_1$$

lies in the kernel of Θ.

Example 7.1.4. The semialgebraic set $\mathfrak{m}$ can be written as $\mathrm{sp}^{-1}_{\mathbb{A}^1_R}(0)$, so that

$$[\mathfrak{m}] = \Theta(1 \otimes [\mathrm{Spec}\, k]_1).$$

But we can also write

$$\begin{aligned} [\mathfrak{m}] &= [\mathfrak{m} \setminus \{0\}] + [\,\{0\}\,] \\ &= [\mathrm{trop}^{-1}(\mathbb{Q}_{>0})] + [\mathrm{sp}^{-1}_{\mathrm{Spec}\, R}(\mathrm{Spec}\, k)] \\ &= \Theta([\mathbb{Q}_{>0}]_1 \otimes 1) + \Theta(1 \otimes [\mathrm{Spec}\, k]_0). \end{aligned}$$

It follows that the element

$$[\mathbb{Q}_{>0}]_1 \otimes 1 + 1 \otimes ([\mathrm{Spec}\, k]_0 - [\mathrm{Spec}\, k]_1)$$

also lies in the kernel of Θ.

Hrushovski and Kazhdan proved that the elements in Examples 7.1.3 and 7.1.4 generate the kernel of Θ. It is quite remarkable that these simple examples in dimension one suffice to explain all possible relations in the kernel of Θ. Our two miracles combine into the following theorem, which is the main result of this course.

Theorem 7.1.5 (Hrushovski-Kazhdan)**.** *The morphism Θ induces an isomorphism of rings*

$$\frac{\mathbf{K}_0^{\dim}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbf{K}_0^{\dim}(\mathrm{Var}_k)}{([\{0\}]_1 \otimes 1 - 1 \otimes [\mathbb{G}_{m,k}]_1,\ [\mathbb{Q}_{>0}]_1 \otimes 1 + 1 \otimes ([\mathrm{Spec}\, k]_0 - [\mathrm{Spec}\, k]_1))} \to \mathbf{K}_0(\mathrm{VF}_K).$$

[5]Beware that we lose information by taking classes of semialgebraic sets in the Grothendieck ring; for instance, $[\mathrm{trop}^{-1}(\,[0,1)\,)] = 0$ because the class of this interval is 0 in the Grothendieck ring of definable sets. Hrushovski and Kazhdan also prove finer results at the level of Grothendieck *semirings* where classes can be added but not subtracted; the class of a semialgebraic set in the Grothendieck semiring remembers the isomorphism type.

This theorem expresses in a precise way how semialgebraic geometry over K is controlled by convex geometry over $\mathbb{Q}$ and algebraic geometry over k. The proof of this theorem is quite involved and uses mathematical logic (model theory of algebraically closed valued fields); we currently do not have a purely geometric proof of this statement.

We can formulate Theorem 7.1.5 in a different way by using the explicit description of $\mathbf{K}_0^{\dim}(\mathbb{Q})$ in Theorem 6.2.8. There we proved that $\mathbf{K}_0^{\dim}(\mathbb{Q})$ is the polynomial ring in the classes $[\mathbb{Q}_{>0}]_1$ and $[\mathbb{Q}_{\geq 0}]_1$ modulo the relation

$$[\mathbb{Q}_{>0}]_1 \cdot [\mathbb{Q}_{\geq 0}]_1 = 0.$$

The element $[\mathbb{Q}_{>0}]_1 \otimes 1$ is equal to $1 \otimes ([\operatorname{Spec} k]_1 - [\operatorname{Spec} k]_0)$ modulo the kernel of Θ. Moreover,

$$[\mathbb{Q}_{\geq 0}]_1 \otimes 1 = ([\mathbb{Q}_{>0}]_1 + [\,\{0\}\,]_1) \otimes 1$$

is equal to

$$1 \otimes ([\operatorname{Spec} k]_1 - [\operatorname{Spec} k]_0 + [\mathbb{G}_{m,k}]_1) = 1 \otimes ([\mathbb{A}^1_k]_1 - [\operatorname{Spec} k]_0)$$

modulo the kernel of Θ. Thus, we can rewrite Theorem 7.1.5 in the following form.

Theorem 7.1.6. *The morphism*

$$\Theta\colon \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{VF}_K)$$

is surjective and factors through an isomorphism of rings

$$\frac{\mathbf{K}_0^{\dim}(\mathrm{Var}_k)}{([\operatorname{Spec} k]_1 - [\operatorname{Spec} k]_0)([\mathbb{A}^1_k]_1 - [\operatorname{Spec} k]_0)} \to \mathbf{K}_0(\mathrm{VF}_K).$$

In particular, pieces of the form $\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))$, with $\mathscr{X}$ a smooth separated R-scheme of finite type and Z a subscheme of $\mathscr{X} \times_R k$, already suffice to describe all semialgebraic sets (at least at the level of Grothendieck rings). This description looks simpler than the one in Theorem 7.1.6, but in concrete calculations it is usually easier to find expressions also involving sets of the form $\mathrm{trop}^{-1}(\Gamma)$. Moreover, Theorem 7.1.5 can be refined to a statement about Grothendieck semirings (retaining more information) while this is not possible for Theorem 7.1.6. For these reasons, Theorem 7.1.5 is more natural than Theorem 7.1.6.

8. Lecture 8 (10 November)

8.1. **The motivic volume.** Using Theorem 7.1.5 (or its variant Theorem 7.1.6) we can define various invariants of semialgebraic sets by constructing ring morphisms from $\mathbf{K}_0(\mathrm{VF}_K)$ to rings that we understand better. An important example is the *motivic volume*, which was originally defined by Hrushovski and Kazhdan.

Corollary 8.1.1. *There exists a unique ring morphism*

$$\mathrm{Vol}\colon \mathbf{K}_0(\mathrm{VF}_K) \to \mathbf{K}_0(\mathrm{Var}_k)$$

that satisfies the following properties.

(1) *For every separated smooth R-scheme $\mathscr{X}$ of finite type and every subscheme Z of $\mathscr{X} \times_R k$, we have*

$$\mathrm{Vol}([\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))]) = [Z].$$

(2) *For every definable subset Γ of $\mathbb{Q}^n$, we have*

$$\mathrm{Vol}([\mathrm{trop}^{-1}(\Gamma)]) = \chi'(\Gamma)[\mathbb{G}^n_{m,k}].$$

Proof. Uniqueness is clear, because the classes of semialgebraic sets of the form $\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))$ and $[\mathrm{trop}^{-1}(\Gamma)]$ generate the ring $\mathbf{K}_0(\mathrm{VF}_K)$. So let us prove existence. There exists a unique ring morphism

$$\mathbf{K}_0^{\dim}(\mathbb{Q}) \to \mathbf{K}_0(\mathrm{Var}_k)$$

that maps $[\Gamma]_n$ to $\chi'(\Gamma)[\mathbb{G}_{m,k}^n]$ for every $n \geq 0$ and every definable subset Γ in $\mathbb{Q}^n$. Together with the forgetful morphism

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{Var}_k),\ [Z]_n \mapsto [Z],$$

it induces a ring morphism

$$\Psi \colon \mathbf{K}_0^{\dim}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{Var}_k).$$

We will show that Ψ is zero on the kernel of the morphism Θ in Theorem 7.1.5. We have

$$\Psi([\,\{0\}\,]_1 \otimes 1 - 1 \otimes [\mathbb{G}_{m,k}]_1) = [\mathbb{G}_{m,k}] - [\mathbb{G}_{m,k}] = 0;$$

this explains why we have added the factor $[\mathbb{G}_{m,k}^n]$ in our expression for $\mathrm{Vol}([\mathrm{trop}^{-1}(\Gamma)]_n)$ in the statement. We also have

$$\Psi([\mathbb{Q}_{>0}]_1 \otimes 1 - 1 \otimes ([\mathrm{Spec}\, k]_1 - [\mathrm{Spec}\, k]_0) = \chi'(\mathbb{Q}_{>0})[\mathbb{G}_{m,k}] + [\mathrm{Spec}\, k] - [\mathrm{Spec}\, k] = 0$$

because $\chi'(\mathbb{Q}_{>0}) = 0$. This explains why we used the bounded Euler characteristic χ', rather than the Euler characteristic χ, in the construction of Vol (note that $\chi(\mathbb{Q}_{>0}) = -1$). It follows that Ψ factors through a ring morphism

$$\mathbf{K}_0(\mathrm{VF}_K) \to \mathbf{K}_0(\mathrm{Var}_k)$$

with the desired properties. □

Remark 8.1.2. We could also have used Theorem 7.1.6 to construct the morphism Vol: the forgetful morphism

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_k) \to \mathbf{K}_0(\mathrm{Var}_k) = \mathbf{K}_0^{\dim}(\mathrm{Var}_k)/([\mathrm{Spec}\, k]_1 - [\mathrm{Spec}\, k]_0)$$

from Proposition 7.1.1 factors through a ring morphism

$$\frac{\mathbf{K}_0^{\dim}(\mathrm{Var}_k)}{([\mathrm{Spec}\, k]_1 - [\mathrm{Spec}\, k]_0)([\mathbb{A}_k^1]_1 - [\mathrm{Spec}\, k]_0)} \to \mathbf{K}_0(\mathrm{Var}_k)$$

whose source is isomorphic to $\mathbf{K}_0(\mathrm{VF}_K)$. This provides the finer statement that Vol is already uniquely determined by the first property in Corollary 8.1.1, and that Vol is a surjective ring morphism with kernel generated by

$$\Theta([\mathrm{Spec}\, k]_1 - [\mathrm{Spec}\, k]_0) = [\mathfrak{m} \setminus \{0\}].$$

For concrete calculations, it is often convenient to use the second property in Corollary 8.1.1, as well, which is why we included it in the statement.

When S is a semialgebraic set over K, we will usually write $\mathrm{Vol}(S)$ instead of $\mathrm{Vol}([S])$ to unburden the notation. We call this invariant the motivic volume of S. The name *motivic volume* refers to the fact that we can view it as a measure of the size of semialgebraic sets, where the measure takes values in the Grothendieck ring $\mathbf{K}_0(\mathrm{Var}_k)$. This idea goes back to the very origins of motivic integration: Batyrev used p-adic integration (a classical integration theory with measures taking values in $\mathbb{R}$) to prove that birational Calabi-Yau varieties over $\mathbb{C}$ have the same betti numbers. Kontsevich realized that one could prove a stronger result (equality of Hodge numbers) by upgrading p-adic integration to a geometric integration theory, taking values in the Grothendieck ring of varieties. This idea kickstarted the development of motivic integration, with the theory of Hrushovski and Kazhdan as an important branch.

Example 8.1.3. Let us show how to compute the motivic volume on a simple example. Let $\mathscr{X}$ be a separated R-scheme of finite type such that $\mathscr{X} \times_R K$ has dimension one. We make the following assumptions about the geometry of $\mathscr{X}$.

(1) The special fiber $\mathscr{X} \times_R k$ consists of two irreducible components, E_1 and E_2, that intersect at a unique point O.
(2) The open subscheme $\mathscr{X} \setminus \{O\}$ is smooth over $\operatorname{Spec} R$.
(3) We can find an open neighbourhood $\mathscr{U}$ of O in $\mathscr{X}$ that admits an étale morphism
$$h\colon \mathscr{U} \to \mathscr{Y} = \operatorname{Spec} R[x,y]/(xy - t^q)$$
with $q \in \mathbb{Q}_{>0}$. Such a morphism automatically maps O to the origin $\mathbf{0} = (0,0)$ of
$$\mathscr{Y} \times_R k = \operatorname{Spec} k[x,y]/(xy).$$

We set $E_1^o = E_1 \setminus \{O\}$ and $E_2^o = E_2 \setminus \{O\}$. Then E_1^o, E_2^o and $\{O\}$ form a partition of $\mathscr{X} \times_R k$ into subschemes, which induces a partition
$$\mathscr{X}(R) = \operatorname{sp}_{\mathscr{X}}^{-1}(E_1^o(k)) \sqcup \operatorname{sp}_{\mathscr{X}}^{-1}(E_2^o(k)) \sqcup \operatorname{sp}_{\mathscr{X}}^{-1}(O)$$
of $\mathscr{X}(R)$ into semialgebraic subsets.

Every R-point in $\operatorname{sp}_{\mathscr{X}}^{-1}(E_1^o)$ factors through the open subscheme $\mathscr{X} \setminus \{O\}$, which is smooth over $\operatorname{Spec} R$ by assumption. Therefore, the class
$$[\operatorname{sp}_{\mathscr{X}}^{-1}(E_1^o)] = [\operatorname{sp}_{\mathscr{X}\setminus O}^{-1}(E_1^o)] \in \mathbf{K}_0(\mathrm{VF}_K)$$
corresponds to $1 \otimes [E_1^o]_1$ under the isomorphism in Theorem 7.1.5. Similarly, we can write $[\operatorname{sp}_{\mathscr{X}}^{-1}(E_2^o)]$ as $1 \otimes [E_2^o]_k$.

Since
$$h\colon \mathscr{U} \to \mathscr{Y} = \operatorname{Spec} R[x,y]/(xy - t^q)$$
is étale and R is henselian, the map
$$\operatorname{sp}_{\mathscr{U}}^{-1}(O) \to \operatorname{sp}_{\mathscr{Y}}^{-1}(\mathbf{0})$$
induced by h is bijective, and therefore an isomorphism of semialgebraic sets. We have
$$\operatorname{sp}_{\mathscr{Y}}^{-1}(\mathbf{0}) = \{(a,b) \in \mathfrak{m}^2 \mid ab = t^q\}$$
and *via* the projection onto the first coordinate, this semialgebraic set is isomorphic to
$$\{a \in K^* \mid 0 < v(a) < q\} = \operatorname{trop}^{-1}(\,(0,q)\,).$$
It follows that, under the isomorphism in Theorem 7.1.5, the class $[\operatorname{sp}_{\mathscr{X}}^{-1}(O)]$ corresponds to
$$[\,(0,q)\,]_1 \otimes 1 = [\,[0,q)\,]_1 \otimes 1 - [\,\{0\}\,]_1 \otimes 1 = -[\,\{0\}\,]_1 \otimes 1$$
where the last equality follows from Example 6.2.2.

Putting everything together, we see that we can express the class of $\mathscr{X}(R)$ as
$$1 \otimes ([E_1^o]_1 + [E_2^o]_1) - [\,\{0\}\,]_1 \otimes 1$$
in the ring
$$\frac{\mathbf{K}_0^{\dim}(\mathbb{Q}) \otimes \mathbf{K}_0^{\dim}(\mathrm{Var}_k)}{\ker(\Theta)} \cong \mathbf{K}_0(\mathrm{VF}_K).$$

Applying the morphism Vol yields

$$\mathrm{Vol}(\mathscr{X}(R)) = [E_1^o] + [E_2^o] - [\mathbb{G}_{m,k}]$$

in $\mathbf{K}_0(\mathrm{Var}_k)$. This is the motivic volume of the semialgebraic set $\mathscr{X}(R)$.

A special case of this calculation is

$$\mathscr{X} = \mathrm{Spec}\,[x,y]/(xy - t^q)$$

with $q \in \mathbb{Q}_{>0}$. Then $E_1 \cong E_2 \cong \mathbb{A}^1_k$ so that E_1^o and E_2^o are both isomorphic to $\mathbb{A}^1_k \setminus \{0\} = \mathbb{G}_{m,k}$. Our formula for the motivic volume becomes

$$\mathrm{Vol}(\mathscr{X}(R)) = [\mathbb{G}_{m,k}]$$

which is consistent with the fact that $\mathscr{X}(R)$ is isomorphic with $\mathrm{trop}^{-1}(\,[0,q]\,)$ *via* the projection onto one of the coordinates.

8.2. **Strictly semistable models.** The geometric set-up in Example 8.1.3 can be generalized in the following way.

Definition 8.2.1. Let $\mathscr{X}$ be a separated R-scheme of finite type. We say that $\mathscr{X}$ is *strictly semistable* if we can cover $\mathscr{X}$ by open subschemes that admit an étale morphism to a scheme of the form

$$\mathrm{Spec}\,R[x_1,\ldots,x_n]/(x_1\cdots x_m - t^q)$$

with $0 < m \leq n$ and $q \in \mathbb{Q}_{>0}$.

Note that the special fiber of $\mathrm{Spec}\,R[x_1,\ldots,x_n]/(x_1\cdots x_m - t^q)$ is a union of coordinate hyperplanes in $\mathbb{A}^n_k$. From this, one can deduce that the irreducible components of $\mathscr{X} \times_R k$ are smooth and intersect each other transversely; this is the key feature of strictly semistable schemes. The definition also implies that the generic fiber of a strictly semistable R-scheme is smooth, because it is locally étale over a smooth K-scheme. Conversely, the following deep result expresses that every separated R-scheme of finite type with smooth generic fiber can be modified into a strictly semistable scheme.

Theorem 8.2.2. *Let $\mathscr{X}$ be a separated R-scheme of finite type such that $\mathscr{X} \times_R K$ is smooth over* $\mathrm{Spec}\,K$*. Then there exists a proper morphism of R-schemes $h\colon \mathscr{X}' \to \mathscr{X}$ such that the induced morphism $\mathscr{X}' \times_R K \to \mathscr{X} \times_R K$ is an isomorphism, and such that $\mathscr{X}'$ is strictly semistable.*

This result follows from Hironaka's resolution of singularities in characteristic zero, one of the cornerstones of algebraic geometry (and one of the biggest open problems in positive characteristic). The fact that h restricts to an isomorphism between the generic fibers implies that it induces an isomorphism of semialgebraic sets $\mathscr{X}(K') \to \mathscr{X}(K)$. The fact that h is proper implies that this further restricts to an isomorphism of semialgebraic sets $\mathscr{X}'(R) \to \mathscr{X}(R)$, by the valuative criterion for properness. Thus, we can use the strictly semistable R-scheme $\mathscr{X}'$ to describe the semialgebraic set $\mathscr{X}(R)$.

Corollary 8.2.3. *For every smooth and proper K-scheme X, there exists a strictly semistable proper R-scheme $\mathscr{X}$ whose generic fiber $\mathscr{X} \times_R K$ is isomorphic to X.*

Proof. By Nagata's compactification theorem, there exists a proper R-scheme whose generic fiber is isomorphic with X. By Theorem 8.2.2, we can modify it into a strictly semistable proper R-scheme without altering the generic fiber. $\square$

A scheme $\mathscr{X}$ as in the statement of the corollary is called a *strictly semistable R-model* for X. The existence of such models is a key tool in the study of schemes and semialgebraic sets over K.

We are now ready to state a generalization of Example 8.1.3 to arbitrary dimensions. Let $\mathscr{X}$ be a strictly semistable R-scheme, and assume that $\mathscr{X} \times_R K$ has pure dimension n (that is, each irreducible component of $\mathscr{X} \times_R K$ has dimension n). We denote by E_i, $i \in I$, the irreducible components of the special fiber $\mathscr{X} \times_R k$. For every nonempty subset J of I, we set

$$E_J = \bigcap_{j \in J} E_j, \qquad E_J^o = E_J \setminus \left(\bigcup_{i \notin J} E_i \right).$$

The set E_J is closed in $\mathscr{X} \times_R k$, and E_J^o is locally closed. We endow them with their induced reduced subscheme structures. The schemes E_J and E_J^o are either empty or of pure dimension $n - |J| - 1$. As J varies over the nonempty subsets of I, the schemes E_J^o form a partition of $\mathscr{X} \times_R k$ into subschemes. In the set-up of Example 8.1.3, we had $E_{\{1,2\}} = E_{\{1,2\}}^o = \{O\}$.

Theorem 8.2.4. *For every subscheme Z of $\mathscr{X} \times_R k$, we have*

$$[\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))] = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} [\,\{0\}\,]_{|J|-1} \otimes [E_J^o \cap Z]_{n-|J|-1}$$

in $\mathbf{K}_0^{\dim}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbf{K}_0^{\dim}(\mathrm{Var}_k)/\ker(\Theta)$, *and*

$$\mathrm{Vol}(\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} [\mathbb{G}_{m,k}^{|J|-1}][E_J^o \cap Z]$$

in $\mathbf{K}_0(\mathrm{Var}_k)$.

In particular,

$$[\mathscr{X}(R)] = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} [\,\{0\}\,]_{|J|-1} \otimes [E_J^o]_{n-|J|-1}$$

and

$$\mathrm{Vol}(\mathscr{X}(R)) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} [\mathbb{G}_{m,k}^{|J|-1}][E_J^o].$$

The second part of the statement follows from the first by setting $Z = \mathscr{X} \times_R k$. The formula for the motivic volume is an immediate consequence of the expression for the class of $\mathrm{sp}_{\mathscr{X}}^{-1}(Z(k))$. This expression can be proved in a similar way as in Example 8.1.3, but the argument becomes more involved because for $1 < |J| < n+1$ the class of $\mathrm{sp}_{\mathscr{X}}^{-1}((E_J^o \cap Z)(k))$ will involve contributions from both $\mathbf{K}_0^{\dim}(\mathbb{Q})$ and $\mathbf{K}^{\dim}(\mathrm{Var}_k)$.

Theorem 8.2.4 enables us, in particular, to write down explicit expressions for the motivic volumes of semialgebraic sets of the following forms:

- $X(K)$, with X a smooth and proper K-scheme;
- the semialgebraic Milnor fiber $\mathrm{MF}_{f,x}^{\mathrm{sa}}$ associated with a regular function f on a smooth k-scheme of finite type, and a k-point x in the zero locus of f (see Definition 2.1.1).

Indeed, Theorem 8.2.2 guarantees that we can reduce the problem to a set-up covered by Theorem 8.2.4, at least in theory: finding explicit strictly semistable models can be complicated in practice. In sufficiently generic situations, such models can be constructed using tropical geometry.

9. Lecture 9 (14 November)

In the previous lecture, we have defined the motivic volume that attaches an element of $\mathbf{K}_0(\mathrm{Var}_k)$ to each semialgebraic set over K, and we have explained how to compute it explicitly on a strictly semistable model. Of course, this construction is only useful if we know how to extract information from classes in $\mathbf{K}_0(\mathrm{Var}_k)$. In the following lectures, we will try to answer the following closely related questions.

(1) Which properties of a k-scheme of finite type can we read off from its class in the Grothendieck ring of varieties?
(2) What does it mean when two k-schemes of finite type have the same class in the Grothendieck ring of varieties?

We will start, more generally, over an arbitrary field F, although we will soon be obliged to restrict to characteristic zero to obtain the most powerful results.

9.1. **Additive invariants.** An *additive invariant* of F-varieties is a map α from the set of isomorphism classes of F-schemes of finite type to some abelian group A, such that[6]

$$\alpha(X) = \alpha(X \setminus Y) + \alpha(Y)$$

for every F-scheme X of finite type and every closed subscheme Y of X. This means that α respects the scissor relations in the definition of $\mathbf{K}_0(\mathrm{Var}_F)$, and giving such an additive invariant is equivalent to giving a group morphism

$$\alpha\colon \mathbf{K}_0(\mathrm{Var}_F) \to A,\ [X] \mapsto \alpha(X).$$

In other words, the map that attaches to every F-scheme of finite type its class in the Grothendieck ring of F-varieties is the *universal* additive invariant of F-varieties, in the sense that every additive invariant factors through it.

We say that an additive invariant α is *multiplicative* if the target A is a ring and

$$\alpha(X \times_F X') = \alpha(X) \cdot \alpha(X')$$

for all F-schemes X and X' of finite type. This is equivalent to saying that the group morphism

$$\alpha\colon \mathbf{K}_0(\mathrm{Var}_F) \to A$$

is also a morphism of rings.

These definitions give a tautological answer to our first question: the properties we can read off from the Grothendieck ring of varieties are those that can be expressed in terms of additive invariants. Therefore, the information content of $\mathbf{K}_0(\mathrm{Var}_F)$ depends on how many additive invariants we can construct. Let us consider some important examples.

9.1.1. *Point counting.* An elementary (but interesting!) example of an additive invariant is the number of rational points over a finite field. If F is finite and X is a F-scheme of finite type, then $X(F)$ is a finite set, because $\mathbb{A}^n_F(F) = F^n$ is finite for every $n \geq 0$, and X can be covered by finitely many affine opens. Counting F-points is obviously additive and multiplicative, so that we get a ring morphism

$$\#\colon \mathbf{K}_0(\mathrm{Var}_F) \to \mathbb{Z},\ [X] \mapsto \#X(F).$$

[6]We slightly abuse notation by writing $\alpha(X)$ for the value of α on the isomorphism class of X.

9.1.2. *The Euler characteristic.* If $F = \mathbb{C}$ then, for every $\mathbb{C}$-scheme X of finite type, we can consider the set $X(\mathbb{C})$ of closed points with its Euclidean topology; if X is a subscheme of $\mathbb{A}^n_{\mathbb{C}}$ this is just the induced topology from $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^n$. Using the machinery of algebraic topology, one can attach to $X(\mathbb{C})$ its *singular cohomology spaces with compact supports* which measure the shape of $X(\mathbb{C})$: for each j in $\{0, \ldots, 2\dim(X)\}$ we get a finite dimensional $\mathbb{Q}$-vector space $H^j_c(X(\mathbb{C}), \mathbb{Q})$. The *Euler characteristic* with compact supports of X is then defined as

$$\chi(X) = \sum_{j \geq 0} (-1)^j \dim H^j_c(X(\mathbb{C}), \mathbb{Q}).$$

The fundamental properties of cohomology with compact supports imply that χ is an additive and multiplicative invariant; it defines a ring morphism

$$\chi \colon \mathbf{K}_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}.$$

Over arbitrary fields F, we can still define a similar invariant by replacing singular cohomology by étale cohomology.

9.1.3. *The Betti polynomial.* For any field F, the *Betti polynomial* of a F-scheme X of finite type is an element $P(X; t) \in \mathbb{Z}[t]$ that satisfies the following properties.

(1) It is additive and multiplicative with respect to X, so that it defines a ring morphism
$$P \colon \mathbf{K}_0(\mathrm{Var}_F) \to \mathbb{Z}[t],\ [X] \mapsto P(X; t).$$
(2) It has degree $2\dim(X)$ and its leading coefficient equals the number of irreducible components of maximal dimension of $X \times_F F^a$, where F^a is an algebraic closure of F. Thus we can read of the dimension of X from its class in the Grothendieck ring of varieties. In particular, $P(X; t) = 0$ if and only if X is empty.
(3) It specializes to the Euler characteristic at $t = 1$: we have $P(X; 1) = \chi(X)$.

When X is smooth and proper over F, then $P(X; t)$ is given explicitly by

$$P(X; t) = \sum_{j=0}^{2\dim(X)} (-1)^j b_j(X) t^j$$

where $b_j(X)$ is the j-th *Betti number* of X. For $F = \mathbb{C}$, the j-th Betti number is the dimension of the j-th singular cohomology space $H^j(X(\mathbb{C}), \mathbb{Q}) = H^j_c(X(\mathbb{C}), \mathbb{Q})$ of the compact space $X(\mathbb{C})$. For general F, we can again replace singular cohomology by étale cohomology. The definition for schemes that are not smooth and proper is much more subtle and requires Deligne's theory of *weights*, the key ingredient to the proof of the Riemann hypothesis over finite fields.

Exercise 9.1.1. Let F be any field. You may freely use that $b_0(\mathbb{P}^1_F) = b_2(\mathbb{P}^1_F) = 1$ and $b_1(\mathbb{P}^1_F) = 0$.

(1) For notational convenience, we write $\mathbb{L} = [\mathbb{A}^1_F]$. Show that, for every $n \geq 0$,
$$[\mathbb{P}^n_F] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n$$
in $\mathbf{K}_0(\mathrm{Var}_F)$.
(2) Deduce that $b_j(\mathbb{P}^n_F) = 1$ when j is an even element in $\{0, \ldots, 2n\}$ and $b_j(\mathbb{P}^n_F) = 0$ otherwise.

9.1.4. *The Hodge-Deligne polynomial.* Assume that F has characteristic zero. Then there is a unique way to attach to each F-scheme X of finite type a polynomial $\mathrm{HD}(X;u,v) \in \mathbb{Z}[u,v]$, called the *Hodge-Deligne polynomial*, such that the following properties are satisfied.

(1) The Hodge-Deligne polynomial is additive and multiplicative, so that it defines a ring morphism
$$\mathrm{HD}\colon \mathbf{K}_0(\mathrm{Var}_F) \to \mathbb{Z}[u,v],\ [X] \mapsto \mathrm{HD}(X;u,v).$$
(2) If X is smooth and proper over F, then
$$\mathrm{HD}(X;u,v) = \sum_{p,q\geq 0} (-1)^{p+q} h^{p,q}(X) u^p v^q$$
where
$$h^{p,q}(X) = \dim H^q(X, \Omega^p_{X/F})$$
is the (p,q)-th Hodge number of X.

Uniqueness follows from the fact that the Grothendieck ring of varieties over a field of characteristic zero is generated by the classes of smooth and proper schemes (we will prove this in Corollary 9.2.3 below). Existence follows from Deligne's theory of mixed Hodge structures, the complex analytic part of his theory of weights. It can also be deduced from Bittner's presentation of the Grothendieck ring (see below). Hodge theory implies that, when X is smooth and proper, we have
$$b_j(X) = \sum_{p+q=j} h^{p,q}(X)$$
for all $j \geq 0$, so that $P(X;t) = \mathrm{HD}(X;t,t)$ and $\chi(X) = \mathrm{HD}(X;1,1)$. The Hodge-Deligne polynomial is a refinement of the Betti polynomial that sees not only the Betti numbers but also the Hodge numbers, for smooth and proper F-schemes.

9.2. **Resolution of singularities and weak factorization.** All the above examples of additive invariants were constructed from various cohomology theories. To obtain even finer information, we will need two major results from birational geometry: resolution of singularities (which tells us that, by applying a specific form of birational surgery, we can remove all singularities from a scheme) and weak factorization (which explains how different resolutions of singularities are related). Both of these theorems are currently known only in characteristic zero (and in low dimension in positive characteristic). Resolution of singularities in positive characteristic is one of the big open problems in algebraic geometry.

Theorem 9.2.1 (Hironaka's resolution of singularities)**.** *Assume that F has characteristic* 0*. Let X be an integral F-scheme of finite type, and let Z be a strict closed subscheme of X. Then there exists a proper birational morphism $h\colon X' \to X$ such that*

(1) *the scheme X' is smooth over F;*
(2) *the inverse image $h^{-1}(Z)$ is a divisor with strict normal crossings on X': this means that its irreducible components are smooth, and they all intersect transversally;*
(3) *the morphism h is an isomorphism over the open subset of X where X is already smooth and Z is already a strict normal crossings divisor.*

We say that h is a resolution of singularities for the pair (X, Z). There exist explicit algorithms to construct such resolutions as compositions of *blow-ups*. Hironaka's original proof was several hundreds of pages long, and earned him a Fields medal in 1970. Since then, many people have streamlined the algorithm and generalized it to a wider class of schemes.

Corollary 9.2.2. *Assume that F has characteristic 0. Every smooth separated F-scheme X of finite type has a* snc-compactification*: a dense open embedding $X \to \overline{X}$ into a smooth and proper F-scheme $\overline{X}$ such that the boundary $\overline{X} \setminus X$ (with its induced reduced structure) is a strict normal crossings divisor.*

Proof. We may assume that X is connected, since we can deal with each connected component separately. Since X is separated, there exists a dense open embedding $X \to \overline{X}$ into a proper F-scheme $\overline{X}$. Applying resolution of singularities to the pair $(\overline{X}, \overline{X} \setminus X)$ we can arrange that $\overline{X}$ is smooth and that the boundary is a strict normal crossings divisor (note that this does not alter X because X is already smooth). □

Corollary 9.2.3. *Assume that F has characteristic 0. As an abelian group, $\mathbf{K}_0(\mathrm{Var}_F)$ is generated by the classes of connected smooth and proper F-schemes.*

Proof. Let X be a F-scheme of finite type. We need to write $[X]$ as a $\mathbb{Z}$-linear combination of classes of connected smooth and proper F-schemes. Partitioning X into smooth separated subschemes and applying the scissor relations, we reduce to the case where X is smooth and separated. Let $X \to \overline{X}$ be a snc-compactification and let D_i, $i \in I$ be the irreducible components of the boundary. For every non-empty subset J of I, we set

$$D_J = \bigcap_{j \in J} D_j.$$

Then by inclusion-exclusion, we can write

$$[X] = [\overline{X}] + \sum_{\emptyset \neq J \subset I} (-1)^{|J|} [D_J].$$

Since the boundary is a strict normal crossings divisor, all the schemes D_J are smooth and proper. Writing each $[D_J]$ as the sum of the classes of connected components of D_J, we obtain an expression of the desired form. □

Thus, we can use snc-compactifications to write classes in the Grothendieck ring in terms of classes of smooth and proper schemes. However, snc-compactifications are not unique, so that we get many different expressions for the same object. The following powerful result tells us that different snc-compactifications are connected by finitely many elementary steps.

Theorem 9.2.4 (Weak factorization)**.** *Assume that F has characteristic 0. Let X be a smooth and separated F-scheme of finite type, and let $X \to \overline{X}$ and $X \to \overline{X}'$ be two snc-compactifications of X. Then there exists a chain of proper birational morphisms*

$$\begin{tikzcd}
 & \overline{X}_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \overline{X}_r \arrow[dl] \arrow[dr] & \\
\overline{X} & & \overline{X}_2 & & \overline{X}_{r-1} & & \overline{X}'
\end{tikzcd}$$

such that each $\overline{X}_i$ is a snc-compactification of X and each morphism in the chain is an admissible blowup*: a blow-up of a smooth center contained in the boundary divisor that has transversal intersections with the boundary divisor.*

The weak factorization theorem was proved by Abramovich, Karu, Matsuki and Włodarczyk around 2000. The adjective *weak* refers to the existence of a stronger version of the statement, which is still an open problem, where one first performs a chain of blow-ups and then a chain of blow-downs (no alternations). I am not aware of any potential applications of the strong version that do not already follow from the weak version.

The key point of the weak factorization theorem is that admissible blow-ups are a very mild form of birational surgery, and we understand quite well what happens to the geometry of the scheme under such an operation. This gives us strong control over all possible snc-compactifications of a smooth separated F-scheme. In the next lecture, we will use this result to construct a new presentation for the Grothendieck ring of varieties, which leads to further examples of additive invariants.

10. Lecture 10 (21 November)

10.1. **Bittner's presentation.** We have proved in Corollary 9.2.3 that the Grothendieck ring of varieties over a field of characteristic zero is generated by the classes of connected smooth and proper schemes. The scissor relations, however, still involve non-proper schemes because the complement of a closed subscheme is only proper in trivial cases. Building upon the weak factorization theorem, Bittner constructed an alternative presentation of the Grothendieck ring, which only involves smooth and proper schemes.

Consider the abelian group $\mathbf{K}_0^{\mathrm{bl}}(\mathrm{Var}_F)$ defined by the following presentation:

- *generators:* isomorphism classes $[X]_{\mathrm{bl}}$ of connected smooth proper F-schemes X;
- *relations:*
  1. $[\emptyset]_{\mathrm{bl}} = 0$;
  2. for every connected smooth proper F-scheme X and every connected smooth closed subscheme Y of X, we have
     $$[\mathrm{Bl}_Y X]_{\mathrm{bl}} - [E]_{\mathrm{bl}} = [X]_{\mathrm{bl}} - [Y]_{\mathrm{bl}}$$
     where $\mathrm{Bl}_Y X$ is the blowup of X along Y and E is the exceptional divisor (the inverse image of Y in the blowup). These relations are called *blowup relations.*

The abelian group $\mathbf{K}_0^{\mathrm{bl}}(\mathrm{Var}_F)$ has a unique ring structure such that $[X]_{\mathrm{bl}} \cdot [X']_{\mathrm{bl}}$ is the sum of the classes of the connected components of $X \times_F X'$, for all connected smooth and proper F-schemes X and X'.

The geometric set-up of the blowup relations is summarized in the blowup square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Bl}_Y X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

All schemes in this diagram are connected, smooth and proper over F. If $Y = X$ then $\mathrm{Bl}_Y X$ and E are empty. Otherwise, the morphism $\mathrm{Bl}_Y X \to X$ is proper and birational; it restricts to an isomorphism

$$\mathrm{Bl}_Y X \setminus E \to X \setminus Y.$$

The exceptional divisor E in $\mathrm{Bl}_Y X$ a projective bundle over Y of rank one less than the codimension of Y in X.

Theorem 10.1.1 (Bittner)**.** *Assume that F has characteristic zero. There exists a unique ring morphism*

$$\alpha\colon \mathbf{K}_0^{\mathrm{bl}}(\mathrm{Var}_F) \to \mathbf{K}_0(\mathrm{Var}_F)$$

that maps $[X]_{\mathrm{bl}}$ to $[X]$ for every connected smooth proper F-scheme X. This morphism is an isomorphism.

Proof. Uniqueness of α is clear, because we have specified the images for a set of generators of the source. To prove existence of α as a morphism of groups, we need to show that the blowup relations follow from the scissor relations. This is true because in $\mathbf{K}_0(\mathrm{Var}_F)$ we can write

$$[\mathrm{Bl}_Y X] - [E] = [\mathrm{Bl}_Y X \setminus E] = [X \setminus Y] = [X] - [Y].$$

It follows immediately from the definitions of the ring structures that α is a morphism of rings.

The main point is proving that α is an isomorphism. We do this by constructing an inverse. We only give a sketch of the argument to show how the weak factorization theorem is used. Let X be a connected smooth separated F-scheme of finite type, and choose a snc-compactification $X \to \overline{X}$. Let $D_i, i \in I$ be the irreducible components of the boundary divisor. For each non-empty subset J of I, we write $[D_J]_{\mathrm{bl}}$ for the sum of the classes of the connected components of D_J in $\mathbf{K}_0^{\mathrm{bl}}(\mathrm{Var}_F)$. Then we define an element $\beta(X)$ in $\mathbf{K}_0^{\mathrm{bl}}(\mathrm{Var}_F)$ by

$$\beta(X) = [\overline{X}]_{\mathrm{bl}} + \sum_{\emptyset \neq J \subset I} [D_J]_{\mathrm{bl}}.$$

A direct calculation shows that this expression is invariant under admissible blowups of the snc-compactification $\overline{X}$, so that it follows from the weak factorization theorem that it only depends on X, and not on the choice of the snc-compactification.

If X is a F-scheme of finite type, then we choose a partition of X into connected smooth separated subschemes $X_1, \ldots, X_r$ and we set

$$\beta(X) = \sum_{i=1}^{r} \beta(X_i).$$

The most technical part of the proof is showing that this definition does not depend on the choice of the partition. Since any two such partitions have a common refinement, we may assume that X is itself connected, smooth and separated; then the invariance under partition is proved by carefully choosing an snc-compactification that is compatible with the partition in a suitable sense.

We obtain a ring morphism

$$\beta\colon \mathbf{K}_0(\mathrm{Var}_F) \to \mathbf{K}_0^{\mathrm{bl}}(\mathrm{Var}_F),\ [X] \mapsto \beta(X)$$

that is inverse to the morphism α (prove that this is indeed an inverse as an exercise). $\square$

Bittner's presentation is very useful for the construction of new additive invariants, since we now only need to define them on connected smooth proper F-schemes and check that the blowup relations are satisfied. In particular, this gives a proof of the existence of the Hodge-Deligne polynomial that does not use mixed Hodge theory: one only needs to verify the behaviour of Hodge numbers of smooth and proper schemes under blowups of smooth centers and in projective bundles, which is well understood. Here we will focus on a new kind of invariant: stable birational types.

10.2. **Birational types and stable birational types.** The ultimate goal of algebraic geometry is the classification of algebraic varieties. This problem is typically broken up into two parts: we first try to classify varieties up to *birational equivalence*, which means that we ignore pieces of lower dimension. More precisely, two integral F-schemes X and X' of finite type are called *birational* if they contain isomorphic dense open subschemes. From the viewpoint of this equivalence relation, the simplest schemes are those which are birational to a projective space $\mathbb{P}^n_F$; they are called *rational*.

Proving that a given integral F-scheme of finite type is rational is easy in principle: write down an isomorphism from a dense open subscheme to an open in a projective space. For instance, one shows that all smooth quadrics over an algebraically closed field are rational by projecting from a closed point onto a projective space. Of course, in more complicated examples, finding such a map can be very difficult.

But the hardest problem is proving that a given scheme is *not* rational, because (except in low dimensions) we do not know computable invariants that can always tell the difference between rational and non-rational schemes. To get more grip on the problem, algebraic geometers have defined various weaker forms of rationality; the most important are (from stronger to weaker) stable rationality, unirationality, rational connectedness, uniruledness. Uniruledness is the weakest property but also the best-behaved; at least conjecturally, it is controlled by a numerical invariant (the Kodaira dimension).

The property that is the most intimately connected to the Grothendieck ring of varieties is stable rationality. We say that two integral F-schemes X and X' of finite type are *stably birational* if $X \times_F \mathbb{P}^\ell_F$ is birational to $X' \times_F \mathbb{P}^m_F$ for some $\ell, m \geq 0$. We say that X is *stably rational* if it is stably birational to $\operatorname{Spec} F$; equivalently, if $X \times_F \mathbb{P}^\ell_F$ is rational for ℓ sufficiently large.

Exercise 10.2.1. Show that every rational integral F-scheme of finite type is also stably rational.

The converse implication is much more delicate. If F is algebraically closed, then rational is equivalent to stably rational for curves and surfaces (by Lüroth's theorem and Castelnuovo's rationality criterion, respectively). For a long time, it was an open problem to find an example of a stably rational variety over an algebraically closed field that is not rational; the first such example was given by Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer around 1985 (a carefully constructed hypersurface in $\mathbb{A}^4_{\mathbb{C}}$). Thus being rational is a strictly stronger property than being stably rational.

To explain the relation with the Grothendieck ring of varieties, we first construct the ring of birational types and its stable version. We denote by Bir_F the set of birational equivalence classes $\{X\}_{\mathrm{bir}}$ of integral F-schemes X of finite type. We denote by $\mathbb{Z}[\mathrm{Bir}_F]$ the free abelian group with basis Bir_F. For every F-scheme X of finite type, we set

$$\{X\}_{\mathrm{bir}} = \{X_1\}_{\mathrm{bir}} + \ldots + \{X_r\}_{\mathrm{bir}} \quad \in \mathbb{Z}[\mathrm{Bir}_F],$$

where $X_1, \ldots, X_r$ are the irreducible components of X. The abelian group $\mathbb{Z}[\mathrm{Bir}_F]$ has a unique ring structure such that

$$\{X\}_{\mathrm{bir}} \cdot \{X'\}_{\mathrm{bir}} = \{X \times_F X'\}_{\mathrm{bir}}$$

for all F-schemes X and X' of finite type. The resulting ring is called the *ring of birational types* over F. Note that we have not introduced any additional relations in this ring: by definition, two integral F-schemes of finite type are birational if and only if they have the same class in $\mathbb{Z}[\mathrm{Bir}_F]$.

We define the ring $\mathbb{Z}[\mathrm{SB}_F]$ of stable birational types in exactly the same way, taking stable birational equivalence classes instead of birational equivalence classes. The class in $\mathbb{Z}[\mathrm{SB}_F]$ of a F-scheme X of finite type is denoted by $\{X\}_{\mathrm{sb}}$.

10.3. **The theorem of Larsen and Lunts.** The following remarkable theorem shows that, even though the Grothendieck ring of varieties itself is a very complicated object (that we still do not fully understand), one can give a beautiful explicit description of the quotient of this ring by the ideal generated by the class of the affine line.

Theorem 10.3.1 (Larsen-Lunts)**.** *Assume that F has characteristic zero. There exists a unique ring morphism*

$$\mathrm{sb}\colon \mathbf{K}_0(\mathrm{Var}_F) \to \mathbb{Z}[\mathrm{SB}_F]$$

that maps $[X]$ to $\{X\}_{\mathrm{sb}}$ for every smooth and proper F-scheme X. This morphism is surjective and its kernel is the ideal generated by $[\mathbb{A}^1_F]$.

Proof. We use Bittner's presentation to construct the morphism sb. We need to check that the blow-up relation is satisfied:

$$\{\mathrm{Bl}_Y X\}_{\mathrm{sb}} - \{E\}_{\mathrm{sb}} = \{X\}_{\mathrm{sb}} - \{Y\}_{\mathrm{sb}}$$

whenever X is a connected smooth and proper F-scheme and Y is a connected smooth closed subscheme of X. If $X = Y$ then both sides are 0 and the equation is trivially satisfied. Otherwise, $\mathrm{Bl}_Y X$ is birational to X, and E is birational to $Y \times_F \mathbb{P}^{c-1}_F$ with $c = \mathrm{codim}_X Y$. In particular, $\mathrm{Bl}_Y X$ is stably birational to X, and E is stably birational to Y, which implies the blowup relation.

In order to compute the image of $[\mathbb{A}^1_F]$ under sb we express it in terms of classes of smooth and proper schemes: viewing the affine line as the projective line minus the point at infinity, we get $[\mathbb{A}^1_F] = [\mathbb{P}^1_F] - [\mathrm{Spec}\, F]$ and therefore

$$\mathrm{sb}([\mathbb{A}^1_F]) = \{\mathbb{P}^1_F\}_{\mathrm{sb}} - \{\mathrm{Spec}\, F\}_{\mathrm{sb}} = 0$$

since the projective line is stably birational to a point. It follows that sb factors through a ring morphism

$$\mathbf{K}_0(\mathrm{Var}_F)/([\mathbb{A}^1_F]) \to \mathbb{Z}[\mathrm{SB}_F]$$

and we show that this is an isomorphism by constructing an inverse.

By resolution of singularities, we can write each element in the basis SB_F of $\mathbb{Z}[\mathrm{SB}_F]$ as $\{X\}_{\mathrm{sb}}$ where X is a connected, smooth and proper F-scheme (pick an affine representative in the birational equivalence class, take its closure in projective space to make it proper, then resolve singularities to make it smooth). If X' is another such representative, one can deduce from resolution of singularities and the weak factorization theorem that X and X' are connected through a chain of blowups of smooth centers. If Y is a connected smooth closed subscheme of X of codimension $c > 0$, and E is the exceptional divisor in the blowup $\mathrm{Bl}_Y X$, then $[\mathrm{Bl}_Y X] - [X] = [E] - [Y]$ and E is a projective bundle of rank $c-1$ over Y. Locally on Y, such a bundle is a product: we can cover Y by open subschemes U such that $E \times_Y U$ is isomorphic with $U \times_F \mathbb{P}^{c-1}_F$. The scissor relations now imply that $[E] = [Y][\mathbb{P}^{c-1}_F]$. By Exercise 9.1.1, the element $[\mathbb{P}^{c-1}_F]$ is congruent to 1 modulo $[\mathbb{A}^1_F]$, so that $[E] - [Y] = 0$ in $\mathbf{K}_0(\mathrm{Var}_F)/([\mathbb{A}^1_F])$. We conclude that X and X' have the same class in $\mathbf{K}_0(\mathrm{Var}_F)/([\mathbb{A}^1_F])$, and sending $\{X\}_{\mathrm{sb}}$ to this class defines an inverse to the morphism

$$\mathbf{K}_0(\mathrm{Var}_F)/([\mathbb{A}^1_F]) \to \mathbb{Z}[\mathrm{SB}_F].$$

□

Remark 10.3.2. Beware that, when X is a F-scheme of finite type that is not smooth and proper, then $\mathrm{sb}([X])$ is usually *not* equal to $\{X\}_{\mathrm{sb}}$. For instance, we have seen that $\mathrm{sb}([\mathbb{A}^1_F]) = 0$.

Corollary 10.3.3. *Assume that F has characteristic zero. If two connected smooth and proper F-schemes have the same class in $\mathbf{K}_0(\mathrm{Var}_F)$, then they are stably birational.*

Proof. This follows directly from the theorem. □

The corollary tells us that the Grothendieck ring of varieties remembers the stable birational types of connected, smooth and proper schemes in characteristic zero. In particular, it also remembers all invariants of connected smooth and proper schemes that are constant on stable birational equivalence classes; important examples are the existence of a F-rational point, the fundamental group (topological for $F = \mathbb{C}$, étale otherwise), and the Albanese variety.

11. Lecture 11 (28 November)

In this lecture, we continue our quest for birational information in the Grothendieck ring of varieties. We will first discuss a class of varieties for which birational equivalence and stable birational equivalence coincide.

11.1. **Uniruled schemes.** Let F be an algebraically closed field of characteristic zero, and let X be a connected smooth proper F-scheme. If F is uncountable, then we call X *uniruled* if, for every closed point $x \in X$, we can find a non-constant morphism of F-schemes $\mathbb{P}^1_F \to X$ whose image contains x. If F is countable, then we say that X is uniruled if the previous property holds after base change to any uncountable algebraically closed field extension of F.

The property of being uniruled expresses that there exist many rational curves on X. The distinction between few and many rational curves turns out to be a crucial dichotomy in the birational classification of algebraic varieties. What is especially convenient about uniruledness is that, at least conjecturally, it is controlled by numerical invariants. A crucial source of information on the geometry of X is the *canonical line bundle* $\omega_{X/F} = \Omega^{\dim(X)}_{X/F}$. For every $n > 0$, the *n-th plurigenus* of X is the dimension of the F-vector space

$$H^0(X, \omega^{\otimes n}_{X/F}),$$

the space of global sections of the n-th tensor power of the line bundle $\omega_{X/K}$. The *Kodaira dimension* is a number in the set $\{-\infty, 0, 1, \ldots, \dim(X)\}$ that expresses how fast the plurigenera grow when n tends to infinity; this is the most important invariant in birational geometry. By definition, the Kodaira dimension is $-\infty$ if and only if all plurigenera are equal to zero, that is, if none of the powers $\omega^{\otimes n}_{X/K}$ have non-zero global sections. It is known that uniruled schemes have Kodaira dimension $-\infty$; Mumford conjectured that the converse implication also holds.

Conjecture 11.1.1 (Mumford)**.** *The scheme X is uniruled if and only if it has Kodaira dimension $-\infty$.*

The "if" implication has only been proved up to dimension 3; it is intimately related to other major conjectures in birational geometry (the famous minimal model programme). For our purposes, a useful feature of schemes that are *not* uniruled is that birational and stable birational equivalence coincide.

Theorem 11.1.2. *Let X and X' be connected smooth proper F-schemes of the same dimension and assume that at least one of them is not uniruled. If X and X' are stably birational, then they are also birational (and neither of them is uniruled).*

Proof. Let us sketch the rough idea of the proof. If X and X' are stably birational then there exists a birational map

$$f \colon X \times_F \mathbb{P}^{\ell}_F \dashrightarrow X' \times_F \mathbb{P}^{m}_F$$

for some $\ell, m \geq 0$. Since birational equivalence preserves dimension, and X and X' have the same dimension, we have $\ell = m$. For general X and X', we cannot deduce the existence of a birational map $X \dashrightarrow X'$ because f need not respect the product structure: it mixes up the geometry of X and $\mathbb{P}^{\ell}_F$. However, if X or X' is not uniruled, this mixup is not possible because the geometries are too different. In that case, one shows that f fits into a commutative diagram

$$\begin{tikzcd} X \times_F \mathbb{P}^{\ell}_F \arrow[r, dashed, "f"] \arrow[d] & X' \times_F \mathbb{P}^{\ell}_F \arrow[d] \\ X \arrow[r, dashed] & X' \end{tikzcd}$$

where the horizontal arrows are birational maps and the vertical arrows are the projection morphisms. $\square$

This result has interesting consequences for the structure of the Grothendieck ring of varieties. As a first application, we give a modified version of a result due to Poonen, who was the first to prove the existence of zero divisors in the Grothendieck ring of varieties over an algebraically closed field.

Proposition 11.1.3 (Poonen)**.** *There exist elliptic curves A and B over $\mathbb{C}$ such that $[A]^2 = [B]^2$ but $[A] \neq [B]$ in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$. Then $[A] - [B]$ and $[A] + [B]$ are non-trivial zero divisors in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$.*

Proof. Shioda constructed an example of elliptic curves A and B over $\mathbb{C}$ such that $A \times_{\mathbb{C}} A$ and $B \times_{\mathbb{C}} B$ are isomorphic, but A and B are not. Then A and B are not birational (for smooth proper curves, isomorphic and birational are equivalent) and therefore not stably birational (the only uniruled smooth proper curve over $\mathbb{C}$ is the projective line, by Lüroth's theorem). It follows from Corollary 10.3.3 that $[A] \neq [B]$ in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$.

We have

$$([A]-[B])([A]+[B]) = [A]^2 - [B]^2 = 0$$

and the element $[A]+[B] = [A \sqcup B]$ is non-zero by the existence of the Betti polynomial. Consequently, $[A]-[B]$ and $[A]+[B]$ are non-trivial zero divisors in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$. □

11.2. **Piecewise isomorphisms and Borisov's example.**

Definition 11.2.1. Let F be any field, and let X and X' be F-schemes of finite type. We say that X and X' are *piecewise isomorphic* if we can partition X and X' into subschemes $X_1, \ldots, X_r$ and $X'_1, \ldots, X'_r$, respectively, such that X_j is isomorphic to X'_j for every j in $\{1, \ldots, r\}$.

Note that, when X and X' are integral and piecewise isomorphic, then they are birational because one of the pieces of the partition must contain the generic point. Piecewise isomorphic F-schemes X and X' of finite type define the same class in the Grothendieck ring of F-varieties: using the scissor relations, we can write

$$[X] = [X_1] + \ldots + [X_r] = [X'_1] + \ldots + [X'_r] = [X'].$$

Larsen and Lunts posed the question whether this is the *only* case where X and X' define the same class in $\mathbf{K}_0(\mathrm{Var}_F)$. Somewhat suprisingly, the answer is *no*: the latest breakthrough in the study of the Grothendieck ring if varieties is the following result[7] published by Borisov in 2018.

Theorem 11.2.2 (Borisov)**.** *There exist smooth projective Calabi-Yau varieties X and X' of dimension 3 over $\mathbb{C}$ with the following properties:*

(1) *the schemes X and X' are not stably birational;*
(2) *we have $[X \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}] = [X' \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}]$ in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$.*

Proof. We only give a rough sketch of the argument. Borisov uses two explicit examples X and X' which were known to be non-birational. The Calabi-Yau property implies that $\omega_{X/\mathbb{C}} \cong \mathcal{O}_X$ so that

$$H^0(X, \omega_{X/\mathbb{C}}) = H^0(X, \mathcal{O}_X) = \mathbb{C};$$

this also applies to X'. By the known implication in Mumford's conjecture, X and X' are not uniruled and therefore also not stably birational.

To prove the desired equality in the Grothendieck ring, Borisov essentially constructs embeddings of $Y = X \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}$ and $Y' = X' \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}$ into some large $\mathbb{C}$-scheme Z of finite type such that the complements of these embeddings are piecewise isomorphic (to be precise, these are only piecewise embeddings, but that does not matter for the argument). Then we can write

$$[Y] = [Z] - [Z \setminus Y] = [Z] - [Z \setminus Y'] = [Y']$$

[7] What is particularly intriguing about Borisov's example is that it arose from mathematical physics (string theory): the theory of *mirror symmetry* suggested that the geometries of X and X' should be closely related because they have the same mirror partner.

in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$. $\square$

Corollary 11.2.3. *The class of* $\mathbb{A}^1_{\mathbb{C}}$ *is a zero-divisor in* $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$.

Proof. By Corollary 10.3.3, the schemes X and X' from Borisov's example define different classes in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$, but these classes become equal after multiplication with the sixth power of $[\mathbb{A}^1_{\mathbb{C}}]$. $\square$

Corollary 11.2.4. *There exist integral smooth* $\mathbb{C}$*-schemes* Y *and* Y' *of finite type such that* $[Y] = [Y']$ *in* $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$ *but* Y *and* Y' *are not stably birational (in particular, not piecewise isomorphic).*

Proof. Since the schemes X and X' in Borisov's example are not stably birational, the schemes $Y = X \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}$ and $Y' = X' \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}$ are not stably birational, and therefore not piecewise isomorphic. $\square$

We see that the question of Larsen and Lunts has a negative answer because subtraction gives a new source of equalities in the Grothendieck ring: schemes with embeddings in a common ambient scheme such that the complements are piecewise isomorphic. Borisov's example shows that the original schemes need not be piecewise isomorphic. We also see that the Grothendieck ring does *not* detect stable birational types of smooth schemes of finite type that are not proper. *A fortiori*, it also does not detect birational types of such schemes. It is still an open problem whether connected smooth proper $\mathbb{C}$-schemes with the same class in $\mathbf{K}_0(\mathrm{Var}_{\mathbb{C}})$ are birational. To circumvent this issue, we can pass to the Grothendieck ring of varieties graded by dimension.

11.3. **The graded Grothendieck ring detects birational types.** Let F be any field.

Theorem 11.3.1. *There exists a unique ring morphism*

$$\mathrm{bir}\colon \mathbf{K}_0^{\dim}(\mathrm{Var}_F) \to \mathbb{Z}[\mathrm{Bir}_F]$$

such that, for every integer $n \geq 0$ *and every* F*-scheme of finite type and of dimension at most* n*, the element* $\mathrm{bir}([X]_n)$ *is the sum of the birational equivalence classes of the* n*-dimensional irreducible components of* X*.*

This ring morphism is surjective, and its kernel is the ideal generated by $[\mathrm{Spec}\, F]_1$*.*

Proof. Uniqueness of bir is clear; we leave its existence as an **exercise**. Since $\mathrm{Spec}\, F$ has no one-dimensional irreducible components, $\mathrm{bir}([\mathrm{Spec}\, F]_1) = 0$, so that bir factors through a ring morphism

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_F)/([\mathrm{Spec}\, F]_1) \to \mathbb{Z}[\mathrm{Bir}_F].$$

We prove that this is an isomorphism by constructing an inverse. If X and X' are birational integral F-schemes of finite type, then they have the same dimension d. By definition, there exist dense open subschemes $U \subset X$ and $U' \subset X'$ such that U and U' are isomorphic. We endow the closed subsets $Y = X \setminus U$ and $Y' = X' \setminus U'$ with their induced reduced structures. These are schemes of dimension at most $d-1$, so that we can write

$$[X]_d - [X']_d = [U]_d + [Y]_d - [U']_d - [Y']_d = [\mathrm{Spec}\, F]_1([Y]_{d-1} - [Y']_{d-1})$$

in $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)$. It follows that the class of $[X]_d$ in $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)/([\mathrm{Spec}\, F]_1)$ only depends on $\{X\}_{\mathrm{bir}}$, and this defines an inverse ring morphism

$$\mathbb{Z}[\mathrm{Bir}_F] \to \mathbf{K}_0^{\dim}(\mathrm{Var}_F)/([\mathrm{Spec}\, F]_1),\ \{X\}_{\mathrm{bir}} \to [X]_{\dim(X)}.$$

□

Corollary 11.3.2. *Let X and X' be integral F-schemes of finite type and of dimension d. If $[X]_d = [X']_d$ in $\mathbf{K}_0^{\dim}(\mathrm{Var}_F)$ then X and X' are birational.*

Proof. This follows immediately from Theorem 11.3.1. □

In Borisov's example, we have

$$[X \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}]_9 \neq [X' \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}]_9$$

in $\mathbf{K}_0^{\dim}(\mathrm{Var}_{\mathbb{C}})$ because $X \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}$ and $X' \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}$ are not birational, but

$$[X \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}]_n = [X' \times_{\mathbb{C}} \mathbb{A}^6_{\mathbb{C}}]_n$$

for n sufficiently large (in the notation of the proof of Theorem 11.2.2, for $n \geq \dim(Z)$).

12. Lecture 12 (5 December)

12.1. **A refinement of the motivic volume.** As we have seen in the previous lecture, in order to pick up information about birational types it is convenient to pass to the graded Grothendieck ring of varieties. Our original construction of the motivic volume took values in the ordinary Grothendieck ring $\mathbf{K}_0(\mathrm{Var}_k)$, but it can be refined to take the grading into account by using the explicit formula from Theorem 8.2.4 as a definition.

Theorem 12.1.1 (N.-Ottem)**.** *There exists a unique morphism of graded rings*

$$\mathrm{Vol}\colon \mathbf{K}_0^{\dim}(\mathrm{Var}_K) \to \mathbf{K}_0^{\dim}(\mathrm{Var}_k)$$

such that, for every strictly semistable proper R-scheme $\mathscr{X}$ with $\mathscr{X} \times_R k = \sum_{i\in I} E_i$ and every integer $n \geq \dim(\mathscr{X} \times_R K)$, we have

$$\mathrm{Vol}([\mathscr{X} \times_R K]_n) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|-1} [E_J^o \times_k \mathbb{G}_{m,k}^{|J|-1}]_n.$$

Proof. This result can be deduced from Hrushovski and Kazhdan's description of the Grothendieck *semiring* of semialgebraic sets (where subtraction is not allowed). This also gives an extension of the refined volume to semialgebraic sets. But it is also possible to give a proof based on algebraic geometry. One can formulate a variant of Bittner's presentation (Theorem 10.1.1) for the graded Grothendieck ring of K-varieties. This shows in particular that this ring is generated by classes of the form $[X]_n$ with X a connected smooth and proper K-scheme and $n \geq \dim(X)$. Every such K-scheme X has a strictly semistable model (Corollary 8.2.3) so that Vol is uniquely characterized by the formula in the statement. To show its existence, one uses the weak factorization theorem to compare different strictly semistable models for the same connected smooth proper K-scheme and show that the formula for Vol is independent of the model; then one shows that Bittner's blow-up relations are respected by constructing a strictly semi-stable model that is compatible with the blow-up in a suitable sense. □

Our new morphism Vol is connected to the previous one by means of the commutative diagram

$$\begin{array}{ccc} \mathbf{K}_0^{\dim}(\mathrm{Var}_K) & \xrightarrow{\mathrm{Vol}} & \mathbf{K}_0^{\dim}(\mathrm{Var}_k) \\ \downarrow & & \downarrow \\ \mathbf{K}_0(\mathrm{VF}_K) & \xrightarrow[\mathrm{Vol}]{} & \mathbf{K}_0(\mathrm{Var}_k) \end{array}$$

where $\mathbf{K}_0^{\dim}(\mathrm{Var}_K) \to \mathbf{K}_0(\mathrm{VF}_K)$ is the ring morphism that sends $[X]_n$ to $[X(K)]$ for every K-scheme of X of finite type and every $n \geq \dim(X)$ (note that $X(K)$ is a semialgebraic set so that we can take its class in $\mathbf{K}_0(\mathrm{VF}_K)$.

Corollary 12.1.2. *There exist unique ring morphisms*

$$\begin{aligned} \mathrm{sp}\colon \mathbb{Z}[\mathrm{Bir}_K] &\to \mathbb{Z}[\mathrm{Bir}_k] \\ \mathrm{sp}\colon \mathbb{Z}[\mathrm{SB}_K] &\to \mathbb{Z}[\mathrm{SB}_k] \end{aligned}$$

such that, for every strictly semistable proper R-scheme $\mathscr{X}$ with $\mathscr{X} \times_R k = \sum_{i\in I} E_i$, we have

$$\begin{aligned} \mathrm{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{bir}}) &= \textstyle\sum_{\emptyset\neq J\subset I}\{E_J \times_k \mathbb{P}_k^{|J|-1}\}_{\mathrm{bir}}, \\ \mathrm{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{sb}}) &= \textstyle\sum_{\emptyset\neq J\subset I}\{E_J\}_{\mathrm{sb}}. \end{aligned}$$

Proof. Uniqueness again follows from the fact that every birational (resp. stably birational) equivalence class over K has a smooth and proper representative, so that it suffices to prove existence.

The morphism Vol in Theorem 12.1.1 maps $[\operatorname{Spec} K]_1$ to $[\operatorname{Spec} k]_1$ (apply the formula to $\mathscr{X} = \operatorname{Spec} R$) so that it descends to a ring morphism

$$\mathbf{K}_0^{\dim}(\mathrm{Var}_K)/([\operatorname{Spec} K]_1) \to \mathbf{K}_0^{\dim}(\mathrm{Var}_k)/([\operatorname{Spec} k]_1).$$

By Theorem 11.3.1 we can identify the source with $\mathbb{Z}[\mathrm{Bir}_K]$ and the target with $\mathbb{Z}[\mathrm{Bir}_k]$, and we get a ring morphism

$$\mathrm{sp}\colon \mathbb{Z}[\mathrm{Bir}_K] \to \mathbb{Z}[\mathrm{Bir}_k]$$

that maps $\{\mathscr{X} \times_R K\}_{\mathrm{bir}}$ to

$$\sum_{\emptyset\neq J\subset I} (-1)^{|J|-1}\{E_J^o \times_k \mathbb{G}_{m,k}^{|J|-1}\}_{\mathrm{bir}}$$

for each $\mathscr{X}$ as in the statement. The scheme $E_J^o \times_k \mathbb{G}_{m,k}^{|J|-1}$ is birational to $E_J \times_k \mathbb{P}^{|J|-1}$ so that these schemes define the same class in $\mathbb{Z}[\mathrm{Bir}_k]$. The version for stable birational types follows easily from the result for birational types (**exercise**). Note that we can omit the factors $\mathbb{P}_k^{|J|-1}$ in the formula because E_J is stably birational to $E_J \times_k \mathbb{P}_k^{|J|-1}$. □

The morphisms sp are called *specialization morphisms*. You should think of sp as taking a limit of a (stable) birational type over the Puiseux series field K for $t \to 0$; they control what happens to (stable) birational types when we specialize from the generic point of $\operatorname{Spec} R$ to the closed point of $\operatorname{Spec} R$.

Example 12.1.3. If $\mathscr{X}$ is a smooth and proper R-scheme and $n \geq \dim(\mathscr{X} \times_R K)$, then our formulas reduce to

$$\begin{array}{rcl} \mathrm{Vol}([\mathscr{X} \times_R K]_n) & = & [\mathscr{X} \times_R k]_n, \\ \mathrm{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{bir}}) & = & \{\mathscr{X} \times_R k\}_{\mathrm{bir}}, \\ \mathrm{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{sb}}) & = & \{\mathscr{X} \times_R k\}_{\mathrm{sb}}. \end{array}$$

Indeed, the connected components of $\mathscr{X} \times_R k$ are smooth (in particular irreducible), so that E_J^o is empty for all $J \subset I$ with $|J| \geq 2$ and $\mathscr{X} \times_R k$ is the disjoint union of the components $E_i = E_i^o$ with $i \in I$.

Even this simple example already has striking implications, which resolved a long-standing open problem in algebraic geometry.

12.2. Specialization of (stable) birational types.

Theorem 12.2.1 (Specialization of (stable) rationality)**.** *Let $\mathscr{X}$ be a smooth and proper R-scheme. If $\mathscr{X} \times_R K$ is rational (resp. stably rational), then $\mathscr{X} \times_R k$ is rational (resp. stably rational).*

Proof. We only prove the result for rationality, the stable version is proved analogously (or deduced from specialization of rationality by multiplying $\mathscr{X}$ with $\mathbb{P}^{\ell}_R$ for sufficiently large ℓ). By definition, the K-scheme $\mathscr{X} \times_R K$ is rational if and only if

$$\{\mathscr{X} \times_R K\}_{\mathrm{bir}} = \{\mathbb{P}^n_K\}_{\mathrm{bir}}$$

with $n = \dim(\mathscr{X} \times_R K)$. By Example 12.1.3, we have

$$\mathrm{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{bir}}) = \{\mathscr{X} \times_R k\}_{\mathrm{bir}}.$$

Applying the example to $\mathscr{X} = \mathbb{P}^n_R$, we find

$$\mathrm{sp}(\{\mathbb{P}^n_K\}_{\mathrm{bir}}) = \{\mathbb{P}^n_k\}_{\mathrm{bir}}.$$

It follows that

$$\{\mathscr{X} \times_R k\}_{\mathrm{bir}} = \{\mathbb{P}^n_k\}_{\mathrm{bir}}$$

so that $\mathscr{X} \times_R k$ is rational. □

This result is highly non-trivial because a birational map $\mathscr{X} \times_R K \dashrightarrow \mathbb{P}^n_K$ will usually *not* extend to a rational map $\mathscr{X} \dashrightarrow \mathbb{P}^n_R$ that can be restricted to a birational map between the special fibers. A basic example is the isomorphism

$$\mathbb{P}^1_K \to \mathbb{P}^1_K, [x_0, x_1] \mapsto [tx_0, x_1].$$

Note that this is indeed an isomorphism because t is invertible in K. It extends uniquely to a morphism

$$\mathbb{P}^1_R \setminus \{p\} \to \mathbb{P}^1_R$$

where p is the point $[1, 0]$ in the special fiber $\mathbb{P}^1_k$. The restriction of this morphism to the special fibers is the constant morphism

$$\mathbb{P}^1_k \setminus \{p\} \to \mathbb{P}^1_k,\ [x_0, x_1] \mapsto [0, 1]$$

which is clearly not birational.

Remark 12.2.2. The genesis of the above results is a bit different from the way we have presented them here. Evgeny Shinder and I first proved the stable versions of Corollary 12.1.2 and Theorem 12.2.1 by using the non-refined version of Vol and the theorem of Larsen and Lunts (Theorem 10.3.1). Maxim Kontsevich and Yuri Tschinkel realized shortly afterwards that one could upgrade the specialization morphism from stable birational types to birational types and deduce Theorem 12.2.1 for rationality instead of stable rationality. Both results were published in 2019. Finally, John Christian Ottem and I gave a common refinement of all these specialization morphisms by refining the motivic volume as in Theorem 12.1.1.

Theorem 12.2.1 is important because it allows us to control the locus of (stably) rational geometric fibers in smooth and proper families in characteristic zero. For every scheme S and every point $s \in S$, a *geometric point* based at s is a morphism $\operatorname{Spec} F \to S$ with image s where F is an algebraically closed field. This is the same thing as choosing an algebraically closed extension F of the residue field of S at s.

Theorem 12.2.3. *Let S be a Noetherian $\mathbb{Q}$-scheme and let $\mathscr{X} \to S$ be a smooth and proper morphism of schemes. Consider the sets*

$$\begin{array}{rcl} S_{\mathrm{rat}} & = & \{s \in S \mid \mathscr{X} \times_S \overline{s} \text{ is rational}\}, \\ S_{\mathrm{srat}} & = & \{s \in S \mid \mathscr{X} \times_S \overline{s} \text{ is stably rational}\} \end{array}$$

where $\overline{s}$ is any geometric point based at s (the definitions do not depend on the choice of $\overline{s}$). Then S_{rat} and S_{srat} are countable unions of closed subsets of S.

Note that the statement is false if we replace $\mathscr{X} \times_S \overline{s}$ by $\mathscr{X} \times_S s$; it is important to pass to an algebraic closure of the residue field at s to get well-behaved sets. There are many examples of non-rational schemes over a field that become rational over an algebraically closed extension; for instance, the affine plane curve over $\mathbb{R}$ defined by $x^2 + y^2 + 1 = 0$. This curve is not rational over $\mathbb{R}$ because it does not have any points with coordinates in $\mathbb{R}$, but it becomes rational over $\mathbb{C}$ (there it is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ minus two points).

Proof. We prove the result for S_{rat}; the statement for S_{srat} follows from a similar argument, or can be deduced from the result for S_{rat} by replacing $\mathscr{X}$ by $\mathscr{X} \times_S \mathbb{P}^{\ell}_S$ for all $\ell \geq 0$.

We will first show that S_{rat} is a countable union of *locally* closed subsets of S by means of an important technique that is often used in algebraic geometry. This part of the statement had been known for a long time. Theorem 12.2.1 will then allow us to replace *locally closed* by *closed*.

We may assume that S is connected, because we can treat each of the finitely many connected components of S separately. For simplicity, let us also assume that $\mathscr{X} \to S$ is projective instead of merely proper; the general case can be reduced to this one in a standard way.

Since S is connected, the fibers of the smooth and proper family $\mathscr{X} \to S$ have constant dimension; we denote this dimension by n. We denote by

$$\mathcal{H} \colon (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Sets})$$

the *relative Hilbert functor* of $\mathscr{X} \times_S \mathbb{P}^n_S$ over S, from the opposite category of S-schemes to the category of sets. The functor $\mathcal{H}$ sends each S-scheme T to the set of closed subschemes of $(\mathscr{X} \times_S \mathbb{P}^n_S) \times_S T$ that are flat over T. In particular, for every geometric point $\overline{s}\colon \operatorname{Spec} F \to S$, $\mathcal{H}(\overline{s})$ is the set of closed subschemes of

$$(\mathscr{X} \times_S \overline{s}) \times_F \mathbb{P}^n_F.$$

Grothendieck has proved that $\mathcal{H}$ is represented by a S-scheme, called the *relative Hilbert scheme*, and that this scheme is a countable disjoint union of S-schemes of finite type.

The idea is now to test rationality of geometric fibers of $\mathscr{X} \to S$ by searching the Hilbert scheme for graphs of birational maps. In each connected component C of $\mathcal{H}$, we can consider the geometric points $\overline{c}\colon \operatorname{Spec} F \to C$ such that the corresponding closed subscheme of

$$(\mathscr{X} \times_S \overline{c}) \times_F \mathbb{P}^n_F$$

is the graph of a birational map. The images of these geometric points in C form a constructible subset, and by Chevalley's theorem, the projection of this constructible subset to S is still constructible (that is, a finite union of locally closed subsets). Taking the union over the countably many connected components of $\mathcal{H}$, we find precisely the set S_{rat}.

To conclude the argument, it suffices to prove that, for every locally closed subset A of S_{rat}, the closure of A is still contained in S_{rat}. Let s be a point in this closure. Then, for a suitable algebraically closed field k of characteristic zero, one can construct a morphism $\operatorname{Spec} R \to S$ such that the closed point of $\operatorname{Spec} R$ is mapped to s and the generic point is mapped into A (this is analogous to constructing a sequence of points in A that converges to s). Now $\mathscr{X} \times_S \operatorname{Spec} R$ is a smooth and proper R-scheme with rational generic fiber (because A is contained in S_{rat}) and it follows from Theorem 12.2.1 that the special fiber of $\mathscr{X} \times_S \operatorname{Spec} R$ is also rational, which means that s lies in S_{rat}. □

Another interesting application of the specialization morphism for stable birational types is that we can use it as an *obstruction* to stable rationality of the generic fiber.

Corollary 12.2.4. *Let $\mathscr{X}$ be a strictly semistable proper R-scheme, with $\mathscr{X} \times_R k = \sum_{i\in I} E_i$. If*

$$\sum_{\emptyset \neq J \subset I} (-1)^{|J|-1}\{E_J\}_{\mathrm{sb}} \neq \{\operatorname{Spec} k\}_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_k]$, then $\mathscr{X} \times_R K$ is not stably rational (in particular, not rational).

Proof. If $\mathscr{X} \times_R K$ is stably rational then $\{\mathscr{X} \times_R K\}_{\mathrm{sb}} = \{\operatorname{Spec} K\}$, so that

$$\mathrm{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{sb}}) = \mathrm{sp}(\{\operatorname{Spec} K\}_{\mathrm{sb}}) = \{\operatorname{Spec} k\}_{\mathrm{sb}}.$$

□

In the next lecture, we will see a concrete application of this idea.

13. Lecture 13 (12 December)

In this lecture, we will use Corollary 12.2.4 to prove that very general quartic hypersurfaces in $\mathbb{P}^5_{\mathbb{C}}$ and $\mathbb{P}^6_{\mathbb{C}}$ are not stably rational.

13.1. **Very general hypersurfaces.** To explain what is meant by *very general*, we consider the universal family of degree d hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$, for any $d, n \geq 1$:

$$\mathscr{H}_{n,d} \subset \mathbb{P}^{n+1}_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d)) \to \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d)).$$

Here $\mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d))$ is the projectivization of the $\mathbb{C}$-vector space $H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d))$ of homogeneous polynomials P of degree d in $n+2$ variables, and the fiber of $\mathscr{H}_{n,d}$ over a point $[P]$ in this projective space is the hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}}$ defined by any representative P. The homogeneous polynomials defining a singular hypersurface form a strict closed subset $\Delta_{n,d}$ of $\mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d))$, defined by the vanishing of the discriminant, and by restricting our universal family over the complement

$$\mathscr{U}_{n,d} = \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d)) \setminus \Delta_{n,d}$$

we obtain the universal family

$$\mathscr{H}^{\mathrm{sm}}_{n,d} \to \mathscr{U}_{n,d}$$

of *smooth* degree d hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$. This is a smooth and proper morphism of schemes.

We say that a property is true over a *general* (resp. *very general*) degree d hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}}$ if it holds for all the hypersurfaces parameterized by the closed points in a non-empty Zariski-open subset of $\mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(d))$ (resp. a countable intersection of such opens). For instance, a general degree d hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}}$ is smooth.

So, to prove a property for a very general degree d hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}}$, we may exclude countably many strict subsets defined by polynomial conditions on the coefficients of an equation for the hypersurface. This is the correct notion for rationality questions, because the locus of (stably) rational geometric fibers in smooth and proper families is a countable union of closed subsets (Theorem 12.2.3).

13.2. **The Lüroth problem.** Before proving that very general quartic fourfolds and fivefolds over $\mathbb{C}$ are not stably rational, we give a brief historical overview to place this result in its proper context. An excellent survey is Beauville's article "The Lüroth problem" (arXiv:1507.02476).

Definition 13.2.1. Let F be a field, and let X be an integral F-scheme of finite type. We say that X is *unirational* if there exists a dominant rational map of F-schemes $\mathbb{P}^n_F \dashrightarrow X$ for some $n \geq 0$.

It is not hard to show that, if X is unirational, one can find a dominant rational map $\mathbb{P}^n_F \dashrightarrow X$ with $n = \dim(X)$. This notion expresses that points in a dense open subscheme of X can be parameterized by an open in $\mathbb{P}^n_F$ (but, unlike for rationality, the parameterization is not necessarily one-to-one).

Exercise 13.2.2. Show that rational $\Rightarrow$ stably rational $\Rightarrow$ unirational $\Rightarrow$ uniruled.

In 1876, Lüroth proved that every field L between F and the field of rational functions $F(x)$ is of the form $F(y)$, with $y \in F(x)$. Thus, either $F = L$ (when y lies in F) or L is isomorphic to the field of rational functions over F (when y does not lie in F). By the dictionary between dominant rational maps and extensions of function fields, this is equivalent to saying that every unirational curve over F is rational. Moreover, for curves, unirational and uniruled are equivalent by definition, Therefore, in dimension one, all the different flavours of rationality coincide.

In dimension 2, Castelnuovo (in characteristic 0) and Zariski (in general) proved a numerical criterion for rationality: a connected smooth proper surface X over an algebraically closed field F is rational if and only if the cohomology spaces $H^1(X,\mathcal{O}_X)$ and $H^0(X,\omega_{X/F}^{\otimes 2})$ are both zero (recall that the dimension of the latter space is called the second plurigenus of X). From this one can deduce that rationality and stable rationality are equivalent in dimension 2, and that they are also equivalent to unirationality if F has characteristic zero. In positive characteristic, there are examples of surfaces that are unirational but not rational (this is caused by the appearance of inseparable morphisms).

For a long time, it was an open problem to find examples in dimension ≥ 3 over an algebraically closed field of characteristic zero which are unirational but not rational; this became known as the Lüroth problem. It was solved independently around 1970 by three groups of people.

Clemens and Griffiths proved that smooth hypersurfaces of degree 3 in $\mathbb{P}^4_{\mathbb{C}}$ are never rational, while these were known to be unirational. They introduced a sophisticated new invariant based on Hodge theory, called the *intermediate Jacobian*, that forms an obstruction to rationality. Unfortunately, it is specific to dimension 3 and does not obstruct stable rationality; it is still unknown whether very general cubic threefolds are stably rational. Going up in dimension, it is expected that very general cubic fourfolds are not rational, but this is one of the main open problems in the field.

Iskovskikh and Manin proved that smooth hypersurfaces X of degree 4 in $\mathbb{P}^4_{\mathbb{C}}$ are never rational. Segre had constructed in 1960 an example of such a hypersurface that is unirational. To obstruct rationality, Iskovskikh and Manin proved that the group of birational maps from X to itself is finite; on the other hand, this group is clearly invariant under birational equivalence, and it is infinite (and quite complicated) for $\mathbb{P}^3_{\mathbb{C}}$. Only fairly recently, around 2015, Colliot-Thélène and Pirutka proved that very general degree 4 hypersurfaces in $\mathbb{P}^4_{\mathbb{C}}$ are also not stably rational. We do not know if they are unirational.

Artin and Mumford used the geometry of quartic double threefolds to construct the first example of a unirational scheme over $\mathbb{C}$ that is not stably rational. A quartic double threefold over $\mathbb{C}$ is an integral proper $\mathbb{C}$-scheme X with a morphism $X \to \mathbb{P}^3_{\mathbb{C}}$ that is two-to-one except over a quartic surface in $\mathbb{P}^3_{\mathbb{C}}$, where it is ramified. The invariant used by Artin and Mumford is the subgroup of torsion elements in the third cohomology group; they showed that this subgroup is invariant under *stable* birational equivalence (contrary to the invariants used by Clemens-Griffiths and Iskovskikh-Manin, which are only birational invariants). This group is trivial for $\mathbb{P}^3_{\mathbb{C}}$ but not for Artin and Mumford's example.

13.3. **Voisin's specialization method, Schreider's bound and quartic fivefolds.** The field received a new boost around 2015 with the introduction of Voisin's *specialization method*, which is based on the theory of algebraic cycles and Chow groups. The rough idea is to construct an obstruction to stable rationality that can be computed on sufficiently mild degenerations (similar in spirit to Corollary 12.2.4, but the obstruction is of a different nature and the class of degenerations is also different). This technique has led to further breakthroughs by many researchers and culminated in the following result by Schreieder, which is the strongest general statement currently available.

Theorem 13.3.1 (Schreieder)**.** *Let F be an algebraically closed field, and let n and d be integers with $n \geq 3$ and $d \geq \log_2 n + 2$. Then a very general degree d hypersurface in $\mathbb{P}^{n+1}_F$ is not stably rational.*

With the exception of the irrationality of the cubic threefold, this bound covered all the cases that were known at the time. It is remarkable for two reasons: there is no restriction on the characteristic of F, and the bound on d is logarithmic in n, while previous bounds by Kollár and Totaro were linear. Still, the bound is probably far from sharp: it is expected that smooth hypersurfaces of degree at least 4 are never stably rational. However, we currently do not know any value of d such that very general degree d hypersurfaces are non-stably rational in all dimensions.

Smooth hypersurfaces of dimension $n \geq 1$ and degree $d \in \{1, 2\}$ over an algebraically closed field are all rational, and one of the big results of 19th century algebraic geometry is the rationality of the cubic surface ($n = 2, d = 3$). The case of higher-dimensional cubics is much more complicated. For instance, for $n \geq 5$ odd, there is not a single example of a smooth cubic for which we know whether it is rational or not. For $n = 4$, many families of rational cubic fourfolds over $\mathbb{C}$ were constructed by Hassett, but we do not know any irrational example (note that the existence of one such example would imply that a very general one is irrational, by Theorem 12.2.1). This is symptomatic for the difficulty of rationality questions in algebraic geometry.

We will now prove the first new case that falls outside Schreieder's bound: quartic fivefolds ($d = 4$ and $n = 5$). The same argument applies to quartic fourfolds, but that case lies in Schreieder's range and was originally proved by Totaro.

Theorem 13.3.2 (N.-Ottem)**.** *Very general hypersurfaces of degree 4 in $\mathbb{P}^5_{\mathbb{C}}$ and $\mathbb{P}^6_{\mathbb{C}}$ are not stably rational.*

Proof. Let $n \in \{4, 5\}$. Applying Theorem 12.2.3 to the universal family

$$\mathscr{H}_{n,4} \to \mathscr{U}_{n,d}$$

of smooth hypersurfaces of degree 4 in $\mathbb{P}^{n+1}_{\mathbb{C}}$, we see that it suffices to find *one* algebraically closed field extension F of $\mathbb{C}$ and *one* smooth hypersurface of degree 4 in $\mathbb{P}^{n+1}_{\mathbb{C}}$ that is not stably rational. We will take $F = K$, the field of Puiseux series over $k = \mathbb{C}$, and will use Corollary 12.2.4 as an obstruction to stable rationality for smooth and proper K-schemes.

Let f be a very general member of the space of homogeneous polynomials in $k[z_0, \ldots, z_{n+1}]$ of degree 4 that are symmetric in z_n and z_{n+1}. We consider the graded R-algebra $R[z_0, \ldots, z_{n+1}, y]$ where the z-variables have degree 1 and the y-variable has degree 2. In this graded R-algebra, we have the homogeneous ideal $(ty - z_n z_{n+1}, y^2 - f)$ (the first generator is homogeneous of degree 2 and the second is homogeneous of degree 4). We set

$$\mathscr{X} = \operatorname{Proj} R[z_0, \ldots, z_{n+1}, y]/(ty - z_n z_{n+1}, y^2 - f).$$

This is a proper R-scheme; it is not strictly semistable, but it belongs to a larger class of models that still have mild singularities: the so-called *strictly toroidal* schemes. For such a scheme one can explicitly construct a resolution to make it strictly semistable, and the right-hand side in the formula for the specialization map is invariant under this resolution, so that it also applies to strictly toroidal proper R-schemes.

Over the field K, where the element t is invertible, we can use the equation $ty - z_n z_n + 1 = 0$ to eliminate the y-variable:

$$\mathscr{X} \times_R K \cong \operatorname{Proj} K[z_0, \ldots, z_{n+1}]/((z_n z_{n+1}/t)^2 - f).$$

This is a smooth hypersurface of degree 4 in $\mathbb{P}^{n+1}_K$ (smoothness can be deduced from the fact that we chose f to be very general). It suffices to prove that $\mathscr{X} \times_R K$ is not stably rational.

The special fiber

$$\mathscr{X} \times_R k = \operatorname{Proj} k[z_0, \ldots, z_{n+1}, y]/(z_n z_{n+1}, y^2 - f)$$

has two irreducible components E_1 and E_2, defined by $z_n = 0$ and $z_{n+1} = 0$, respectively. These components are isomorphic because f is symmetric in z_n and z_{n+1}. Their intersection is given by

$$E_{\{1,2\}} = \operatorname{Proj} R[z_0, \ldots, z_{n-1}, y]/(y^2 - f(z_0, \ldots, z_{n-1}, 0, 0))$$

which is a very general quartic double $(n-1)$-fold (a double cover of $\mathbb{P}^{n-1}_k$ ramified along a quartic hypersurface). For $n = 4$, Voisin used her specialization method and the Artin-Mumford result to deduce that such a scheme is not stably rational. For $n = 5$, this was proved by Hassett, Pirutka and Tschinkel.

Now the formula for the specialization of stable birational types (extended to strictly toroidal schemes) tells us that

$$\operatorname{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{sb}}) = \{E_1\}_{\mathrm{sb}} + \{E_2\}_{\mathrm{sb}} - \{E_{\{1,2\}}\}_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_k]$. The k-schemes E_1 and E_2 are isomorphic and therefore have the same stable birational equivalence class. We also have

$$\{E_{\{1,2\}}\}_{\mathrm{sb}} \neq \{\operatorname{Spec} k\}_{\mathrm{sb}}$$

because $E_{\{1,2\}}$ is not stably rational. Reducing our formula modulo 2, we find that

$$\operatorname{sp}(\{\mathscr{X} \times_R K\}_{\mathrm{sb}}) = \{E_{\{1,2\}}\}_{\mathrm{sb}} \neq \{\operatorname{Spec} k\}_{\mathrm{sb}}$$

in $\mathbb{F}_2[\mathrm{SB}_k]$, the $\mathbb{F}_2$-vector space with basis SB_k. Corollary 12.2.4 now tells us that $\mathscr{X} \times_R K$ is not stably rational. □

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