# Mode stability of blow up for wave maps

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# 1 Introduction

Wave maps are the hyperbolic analogue of harmonic maps. Let $\mathbb{R}^{n+1}$ denote Minkowski space with the usual Minkowski metric $\eta_{\alpha\beta} = \operatorname{diag}(-1, 1, \dots, 1)$. Let us further denote by $(M, g)$ an $m$-dimensional Riemannian manifold and $(\kappa, U)$ a coordinate chart on $M$. Then the intrinsic form of the wave maps equation is given by

$$\Box u_i + \sum_{j,k=1}^{m} \Gamma^i_{jk}(u)\, Q_0(u_j, u_k) = 0 \qquad i = 1, \dots, m \tag{1}$$

where the solution is assumed to be a map $u : \mathbb{R}^{n+1} \to \kappa(U) \subset \mathbb{R}^m$, the null form $Q_0$ is defined by $Q_0(f, g) = \partial_\alpha f\, \partial^\alpha g = -\partial_t f\, \partial_t g + \nabla_x f \cdot \nabla_x g$ and $\Gamma$ denotes the Christoffel symbols of $M$ in the chart $(\kappa, U)$. The scaling symmetry $u(t, x) \mapsto u(\mu t, \mu x)$, $\mu > 0$ of the equation implies that $s_c = \frac{n}{2}$ is the critical regularity, in the sense that the $\dot{H}^{s_c}$ seminorm is preserved by scaling. Klainerman and Selberg [1] and Klainerman and Machedon [2, 3, 4] have shown local well-posedness of the intrinsic form of the equation for initial data in $H^s \times H^{s-1}(\mathbb{R}^n)$, $s > \frac{n}{2}$ and $n \geq 2$. Tataru [5, 6] has shown well-posedness in the homogeneous critical 1-Besov space $\dot{B}^{\frac{n}{2}}_{2,1}$ for all $n \geq 2$. The proof of well-posedness in the critical homogeneous Sobolev space $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$ goes back among others to Tao [7, 8], Klainerman and Rodnianski [9], Nahmod, Stefanov and Uhlenbeck [10], Shatah and Struwe [11], Krieger [12, 13] and Tataru [14]. See also the review article by Krieger [15].

We will be interested in the case of space dimension three and target manifold $M = S^3$. In this case, there is an explicitly known self-similar blowup solution, whose existence was proven by a variational argument [16] before it was found in closed form [17]. The solution takes a particularly simple form when we choose the chart $\kappa$ to be stereographic projection from the sphere:

$$u_0 : [0, 1) \times \mathbb{R}^3 \to \mathbb{R}^3, \quad u_0(t, x) = \frac{x}{1-t} \tag{2}$$

By finite speed of propagation, a solution with compactly supported initial data which still blows up at $t = 1$ can be constructed from this. Numerical simulations [18] seem to indicate that $u_0$ may describe the *generic blowup behaviour* in the backward lightcone of the blowup point. In particular, one would expect the solution $u_0$ to be stable in a certain sense. Due to the equation's symmetries, $u_0$ generates a whole family of blowup solutions. The symmetries are important since we can only expect stability in the lightcone modulo symmetries. Consider a time shift for instance, which can either make the solution regular in the cone, or make the solution blow up at an earlier time.

A natural first step in the attempt to understand the stability of this solution is to study the linearization of the wave maps equation around $u_0$:

$$\Box\Psi + 2\Gamma(u_0) Q_0(u_0, \Psi) + \nabla\Gamma(u_0) \cdot \Psi\, Q_0(u_0, u_0) = 0 \tag{3}$$

It is then natural to introduce self-similar coordinates defined by

$$\tau = -\log(1-t), \quad y = \frac{x}{1-t} \tag{4}$$

In these coordinates, the blowup solution evidently becomes stationary. The truncated backward lightcone of the blowup point $C_{1,0} = \{(t, x) \in \mathbb{R}^4 : t > 0 \text{ and } |x| < 1 - t\}$

transforms into the half-infinite cylinder $(0,\infty)\times B_1(0)$ in self-similar coordinates. In order to understand the linear stability, a first step is to consider mode solutions. We say that $\Psi : B_1(0)\to\mathbb{R}^3$ is a mode solution to (3) with mode $\lambda\in\mathbb{C}$ if $e^{\lambda\tau}\Psi(y)$ solves the equation (3) transformed to self-similar coordinates. In this work, we characterize mode solutions which are smooth up to the boundary of the ball, that is, we assume $\Psi\in C^\infty(\overline{B}_1(0))$ throughout. Further we assume that $\operatorname{Re}\lambda\geq 0$, which is natural since we are mainly interested in understanding the neutral and unstable modes. Our main result is a characterization of the modes with $\operatorname{Re}\lambda\geq 0$ without the assumption of a special symmetry:

**Theorem 1.1.** *Assume* $\operatorname{Re}\lambda\geq 0$. *There are no nontrivial mode solutions* $\Psi$ *smooth up to the boundary of* $B_1(0)$ *with mode* $\lambda$ *unless* $\lambda\in\{0,1\}$. *The space of such solutions for* $\lambda=1$ *is four-dimensional and spanned by the functions* $\Psi_{10}$ *and* $\Psi_0^i$, $i=1,2,3$, *where*

$$\Psi_{10}(y)=y,\quad \Psi_0^i(y)=e_i,\ i=1,2,3$$

*The space of such solutions for* $\lambda=0$ *is nine-dimensional and spanned by the functions* $\Phi_0^i,\Psi_{11}^i,\Psi_{21}^i$, $i=1,2,3$, *where*

$$\Phi_0^i(y)=(r^2-3)e_i,\ i=1,2,3$$

$$\Psi_{11}^1(y)=\begin{pmatrix}0\\-y_3\\y_2\end{pmatrix},\quad \Psi_{11}^2(y)=\begin{pmatrix}-y_3\\0\\y_1\end{pmatrix},\quad \Psi_{11}^3(y)=\begin{pmatrix}-y_2\\y_1\\0\end{pmatrix}$$

$$\Psi_{21}^1(y)=\begin{pmatrix}-2y_1^2+y_2^2+y_3^2\\-3y_1y_2\\-3y_1y_3\end{pmatrix},\ \Psi_{21}^2(y)=\begin{pmatrix}-3y_1y_2\\y_1^2-2y_2^2+y_3^2\\-3y_2y_3\end{pmatrix},\ \Psi_{21}^3(y)=\begin{pmatrix}-3y_1y_3\\-3y_2y_3\\y_1^2+y_2^2-2y_3^2\end{pmatrix}$$

*where all the functions have been expressed in self-similar coordinates* (4) *and* $r=|y|$.

### Known results: the corotational case

A special class of solutions is given by the *corotational solutions*. In our setting of using stereographic projection from the sphere, the corotational solutions are most naturally represented in the form $u(t,x)=\tan\left(\frac{\varphi(t,r)}{2}\right)\frac{x}{r}$, where $r=|x|$. Under this assumption, the wave maps equation (1) reduces to the radial semilinear wave equation

$$\varphi_{tt}-\varphi_{rr}-\frac{2}{r}\varphi_r+\frac{\sin(2\varphi)}{r^2}=0 \tag{5}$$

Of all the symmetries of (1), see section 2.1, only the time shift symmetry remains in the corotational case. The solution $u_0=\frac{x}{1-t}$ now takes the form $\varphi=2\operatorname{atan}\left(\frac{r}{1-t}\right)$. In this case, mode stability has been shown by Costin, Donninger and Glogić using the quasisolution method [19], see also [20, Section 2.7]. Donninger, Schörkhuber and Aichelburg [21] showed that mode stability is sufficient to obtain linear stability. Donninger [22] also gave a proof of the nonlinear stability. Thus, the corotational case is completely understood and since mode stability has been shown to be sufficient for linear and nonlinear stability in the corotational case, there is reason to hope that the same applies without the assumption of symmetry.

## Notation and conventions

We denote by $\Delta$ the usual (negative-definite) Laplacian on $\mathbb{R}^n$ and by $\Box = -\partial_{tt} + \Delta$ the usual wave operator. We denote the Laplace-Beltrami operator on the sphere $S^2$ by $\Delta_{S^2}$. We also introduce a special notation concerning contractions of Christoffel symbols: If $\Gamma^i_{jk}(x)$ denotes the Christoffel symbols of a Riemannian manifold in a certain chart, and $Q(x) = (Q_{ij}(x))_{ij}$ is a matrix-valued function, then we often use the shorthand notation $\Gamma Q(x)$ to mean the vector-valued function whose $i$-th component is given by $(\Gamma Q)_i(x) = \sum_{jk} \Gamma^i_{jk}(x) Q_{jk}(x)$. This notation will simplify things in chapter 3. The set $\mathbb{N}$ denotes the natural numbers without 0 and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Whenever we have a vector $x \in \mathbb{R}^m$, we write $x = (x_1, \dots, x_m)$ for its components. Although this is a slight abuse of notation, we always denote radial variables by $r$, both in physical and in self-similar coordinates. It should be clear from the context what is meant. The tensor product $\otimes$ is always understood to be taken over the complex numbers.

# 2 Mode stability without symmetry

We consider the wave maps equation in three space dimensions with target $S^3$ in its intrinsic form:

$$\Box u_i + \sum_{j,k=1}^{m} \Gamma^i_{jk}(u)\, Q_0(u_j, u_k) = 0 \qquad i = 1, \dots, m$$

The coordinates we shall use are given by stereographic projection from the south pole. Explicitly, let $S = (0, 0, 0, -1) \in S^3$ and $\sigma : S^3 \setminus \{S\} \to \mathbb{R}^3, (\vec{x}, x_4) \mapsto \frac{1}{1+x_4}\vec{x}$. Then the pushforward of the standard round metric on $S^3$ is $\sigma^* g_{S^3}(x) = \frac{4}{(|x|^2+1)^2} g_{\mathbb{R}^3}(x)$ where $g_{\mathbb{R}^3}$ denotes the usual flat Euclidean metric. Then we can compute the Christoffel symbols:

$$\Gamma^i_{jk}(x) = \frac{2}{|x|^2+1}(x_i\delta_{jk} - x_j\delta_{ik} - x_k\delta_{ij})$$

As noted in the introduction, we chose the coordinate formulation of the wave maps equation and use stereographical projection due to the simplicity of the blowup solution in these coordinates. The linearisation of the equation around the blowup solution $u_0 = \frac{x}{1-t}$ then becomes

$$\Box\Psi + 2\Gamma(u_0)Q_0(u_0, \Psi) + \nabla\Gamma(u_0) \cdot \Psi\, Q_0(u_0, u_0) = 0 \tag{6}$$

or more explicitly, the $i$-th component of the equation reads

$$\Box\Psi_i + 2\sum_{j,k=1}^{3} \Gamma^i_{jk}(u_0)Q_0(u_{0j}, \Psi_k) + \sum_{j,k,l=1}^{3} \partial_l\Gamma^i_{jk}(u_0)\, \Psi_l\, Q_0(u_{0j}, u_{0k}) = 0$$

Without loss of generality, we study the stability of the blowup solution $u_0 = \frac{x}{1-t}$ which blows up at the spacetime point $(1, 0) \in \mathbb{R} \times \mathbb{R}^3$. To study the stability, we need to use a coordinate system where the blowup solution becomes stationary:

$$\tau = -\log(1-t), \quad y = \frac{x}{1-t}$$

with an obvious modification if we were to consider the blowup solution $u = \frac{x-x_0}{T-t}$. These coordinates are called *self-similar coordinates*. Before we write out the equation explicitly in self-similar coordinates, we study the symmetries of the wave maps equation.

## 2.1 Symmetries of the wave maps equation

The group of symmetries of the wave maps equation is generated by the symmetries of the free wave equation, i.e. translations in spacetime and Lorentz transformations, together with a scaling symmetry and the isometries of the target space, which in our case of $S^3$ yields an additional $O(4)$ symmetry. More explicitly, if we denote $X = (t, x) \in \mathbb{R}^4$ and $u : \mathbb{R}^4 \to \mathbb{R}^3$ denotes a solution to the wave maps equation, the following functions are again solutions:

| | | |
|---|---|---|
| Translations | $u(t+\alpha, x)$<br>$u(t, x+\alpha e_i), \quad i = 1, 2, 3$ | (7) |
| Scaling | $u(\alpha t, \alpha x) \quad (\alpha > 0)$ | (8) |
| Rotations | $u(t, R_i(\alpha)x), \quad i = 1, 2, 3$ | (9) |
| Lorentz boosts | $u(\Lambda_i(\alpha)X), \quad i = 1, 2, 3$ | (10) |
| Rotations on the sphere | $\sigma\mathbf{R}_j(\alpha)\sigma^{-1}u(t, x), \quad j = 1 \dots 6$ | (11) |

Here $\alpha \in \mathbb{R}$ is a parameter, $e_i$ denote the standard unit vectors in $\mathbb{R}^3$, $R_i(\alpha) = \exp(\alpha F_i)$ where $F_i$ generate the Lie algebra so(3), $\mathbf{R}_j(\alpha) = \exp(\alpha \mathbf{F}_j)$, where $\mathbf{F}_j$ generate the Lie algebra so(4) and $\Lambda_i(\alpha)$ are the Lorentz boosts along the $e_i$-axes with rapidity $\alpha$. The explicit form of all of these matrices is given in Appendix B.

Some care needs to be taken with the definition of (11) due to the use of coordinates. For any given solution $u$, it might happen that some rotation $\mathbf{R}_j(\alpha)$ maps $\sigma^{-1}u(t,x)$ to the south pole for some $(t,x) \in \mathbb{R}^4$, leading to (11) being ill-defined at such points $(t,x)$. In our case however, this does not happen since we are only interested in studying the ground state blowup solution $u_0 = \frac{x}{1-t}$ on the lightcone $C$. This means that $\operatorname{ran}(u_0|_C) = u_0(C) = B_1(0) \subset \mathbb{R}^3$ which implies that $\operatorname{ran}(\sigma^{-1}u_0|_C) = \{z \in S^3 : z_4 > 0\}$ is the upper half-sphere. Since we are projecting from the south pole, expression (11) is well-defined for $u = u_0$ on the cone $C$ for sufficiently small rotations.

Of course, also any combination of the above symmetry transformations produces new solutions, yielding a 17 parameter symmetry group. Our interest in the symmetries of the wave maps equation stems from the fact that the symmetries generate solutions to the linearised equation. As a general principle, whenever we have a one-parameter family of solutions $u_\alpha$ with the property that $u_\alpha|_{\alpha=0} = u_0$, our explicit blow-up solution from above, and sufficiently many derivatives in the space and time variables and $\alpha$ exist and commute, then $\partial_\alpha|_{\alpha=0}\, u_\alpha$ solves the equation that has been linearized around $u$. Since the spacetime translations can for instance shift the singularity out of the lightcone $C$, we expect them to generate unstable solutions of the linearised equation, i.e. eigenvalues with a positive real part. The scaling symmetry effectively becomes a time translation for our self-similar solution. On the other hand, the rotations on $S^3$ only act on the image space and hence leave the lightcone invariant. Similarly, the Lorentz transformations based at $(1,0)$ leave the cone invariant. Thus one might expect the corresponding solutions of the linearised equation to produce neutral eigenvalues, i.e. eigenvalues with real part 0.

### 2.1.1 Eigenfunctions generated by the symmetries

Now we follow the recipe outlined in the previous paragraph for computing solutions to the linearized equation. For convenience, we transform each solution to self-similar coordinates. To simplify the notation, we will use the symbol $u_\alpha$ or slight variations of it to denote each one-parameter family generated by symmetries. We will also assign names to each solution for ease of later reference. Note that the variable names might seem nonsensical now, but will make sense in section 2.2.1.

**Scaling.** Consider the one-parameter family given by scaling, $u_\alpha(t,x) = u_0(\alpha t, \alpha x) = \frac{\alpha x}{1-\alpha t} = u_0(t + 1 - \frac{1}{\alpha}, x)$. Observe that due to the self-similarity of our solution, scaling basically becomes a time shift. In this case we need to take the derivative at $\alpha = 1$ and obtain

$$\partial_\alpha|_{\alpha=1}\, u_\alpha = \frac{x}{1-t} + \frac{tx}{(1-t)^2} = \frac{x}{(1-t)^2} = e^\tau y.$$

which is a mode solution for mode $\lambda = 1$. We set $\Psi_{10}(y) := y$.

**Spacetime translations.** Consider the one-parameter family given by time translation, $u_\alpha(t,x) = u_0(t+\alpha, x) = \frac{x}{1-(t+\alpha)}$. Thus

$$\partial_\alpha|_{\alpha=0}\, u_\alpha = \frac{x}{(1-t)^2} = e^\tau y$$

Notice that this generates the same solution as the scaling symmetry which is natural in light of the fact that scaling is just a time shift (using a different parametrization) for our self-similar solution. For the one-parameter families generated by the space translations $u^i_\alpha(t,x) = u_0(t, x + \alpha e_i)$ we find the three solutions

$$\partial_\alpha|_{\alpha=0}\, u^i_\alpha = \frac{e_i}{1-t} = e^\tau e_i, \quad i = 1,2,3.$$

again for the mode $\lambda = 1$. Here we define the functions $\Psi^i_0(y) := e_i$ for $i = 1,2,3$.

**Lorentz boosts.** Consider the three one-parameter families generated by the Lorentz boosts $\Lambda_i$ and given by $u^i_\alpha(t,x) = u_0(\Lambda_i(\alpha)(t,x))$, $i = 1,2,3$. Then after some computation one finds

$$\partial_\alpha|_{\alpha=0}\, u^i_\alpha = e_i - y_i y =: \tilde{\Psi}^i_1(y), \quad i = 1,2,3.$$

**Rotations in $\mathbb{R}^3$.** Consider the three one-parameter families generated by the rotations in $\mathbb{R}^3$. They are given by $u^i_\alpha(t,x) = u_0(t, R_i(\alpha)\,x) = \frac{R_i(\alpha)\,x}{1-t} = \frac{\exp(\alpha F_i)\,x}{1-t}$ so that

$$\partial_\alpha|_{\alpha=0}\, u^i_\alpha = \frac{F_i\, x}{1-t} = F_i\, y =: \Psi^i_{11}(y), \quad i = 1,2,3.$$

or more explicitly, we obtain the three solutions expressed in self-similar coordinates

$$\Psi^1_{11}(y) = \begin{pmatrix} 0 \\ -y_3 \\ y_2 \end{pmatrix}, \quad \Psi^2_{11}(y) = \begin{pmatrix} -y_3 \\ 0 \\ y_1 \end{pmatrix}, \quad \Psi^3_{11}(y) = \begin{pmatrix} -y_2 \\ y_1 \\ 0 \end{pmatrix}.$$

**Rotations on the sphere $S^3$.** Consider the six one-parameter families generated by the rotations on the sphere as follows: $u^j_\alpha(t,x) = \sigma \mathbf{R}_j(\alpha)\sigma^{-1} u_0(t,x)$ for $j = 1, \dots 6$. We compute the partial derivative by $\alpha$:

$$\partial_\alpha|_{\alpha=0}\, u^j_\alpha = D\sigma|_{\sigma^{-1}(u_0)}\, \partial_\alpha|_{\alpha=0}\, \mathbf{R}_j(\alpha)\, \sigma^{-1} u_0$$

where $\partial_\alpha|_{\alpha=0}\, \mathbf{R}_j(\alpha) = \partial_\alpha|_{\alpha=0} \exp(\alpha \mathbf{F}_j) = \mathbf{F}_j$ and a computation gives

$$D\sigma|_{\sigma^{-1}(u)} = (1 + |u|^2) \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2}u_1 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2}u_2 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}u_3 \end{pmatrix}$$

Putting everything together, we find that for $j = 1,2,3$ we obtain the same solutions as those generated by the rotations in $\mathbb{R}^3$ again. For $j = 4,5,6$ we obtain the three new solutions

$$\tilde{\Psi}^1_2(y) := \partial_\alpha|_{\alpha=0}\, u^4_\alpha = \begin{pmatrix} -y_1^2 + y_2^2 + y_3^2 - 1 \\ -2y_1 y_2 \\ -2y_1 y_3 \end{pmatrix}, \quad \tilde{\Psi}^2_2(y) := \partial_\alpha|_{\alpha=0}\, u^5_\alpha = \begin{pmatrix} -2y_1 y_2 \\ y_1^2 - y_2^2 + y_3^2 - 1 \\ -2y_2 y_3 \end{pmatrix}$$

$$\tilde{\Psi}^3_2(y) := \partial_\alpha|_{\alpha=0}\, u^6_\alpha = \begin{pmatrix} -2y_1 y_3 \\ -2y_2 y_3 \\ y_1^2 + y_2^2 - y_3^2 - 1 \end{pmatrix}.$$

One can easily see that all of the above 13 solutions (counting the solutions generated by both rotations in $\mathbb{R}^3$ and rotations on the sphere and the one generated by both scaling and time translation only once) are linearly independent over $\mathbb{C}$. In addition, all of the solutions are of the form $e^{\lambda\tau}\Psi(y)$ for $\lambda \in \{0,1\}$. More specifically, the translations generate a four dimensional space of mode solutions for the mode $\lambda = 1$, whereas the rest of the symmetries generate a nine dimensional mode space for the mode $\lambda = 0$. Aside from these unstable and neutral eigenvalues generated by the symmetries, we expect no other eigenvalues with non-negative real part $\operatorname{Re}\lambda \geq 0$ and smooth eigenfunctions.

We make a change of basis for the eigenfunctions to $\lambda = 0$ generated by the Lorentz boosts and the rotations on the sphere. This will turn out to be convenient in section 2.2.1. For $i = 1, 2, 3$ we define

$$\Psi_{21}^i := \tilde{\Psi}_1^i + \tilde{\Psi}_2^i$$
$$\Phi_0^i := -2\tilde{\Psi}_1^i + \tilde{\Psi}_2^i$$

To summarize, we have now computed four independent mode solutions for $\lambda = 1$ given by

$$\Psi_{10}(y) = y, \quad \Psi_0^i(y) = e_i, \; i = 1,2,3$$

as well as nine solutions for $\lambda = 0$ given by

$$\Phi_0^i(y) = (r^2 - 3)e_i, \quad \Psi_{11}^i(y) = F_i\, y \qquad i = 1,2,3$$

$$\Psi_{21}^1(y) = \begin{pmatrix} -2y_1^2 + y_2^2 + y_3^2 \\ -3y_1y_2 \\ -3y_1y_3 \end{pmatrix}, \; \Psi_{21}^2(y) = \begin{pmatrix} -3y_1y_2 \\ y_1^2 - 2y_2^2 + y_3^2 \\ -3y_2y_3 \end{pmatrix}, \; \Psi_{21}^3(y) = \begin{pmatrix} -3y_1y_3 \\ -3y_2y_3 \\ y_1^2 + y_2^2 - 2y_3^2 \end{pmatrix}$$

Again we emphasize that the notation will make sense in section 2.2.1.

## 2.2 Symmetries of the linearized equation

We will momentarily work in the larger space $C^\infty(\bar{B}_1(0), \mathbb{R}^3) \subset L^2(B_1(0); \mathbb{C}^3)$. Let us recall the basic properties of representations of so(3) first, see [23] and [24, Chapter 17]. For every finite dimension $k \in \mathbb{N}_0$, there exists an irreducible representation of so(3) of dimension $k$ and it is unique up to isomorphism. The Casimir element of a representation $\rho$ of so(3) is defined by $C_\rho = -\sum_{i=1,2,3} \rho(F_i)^2$. This definition is in fact independent of the choice of generators of so(3). Let us express the dimension in the form $k = 2l + 1$ for $l \in \frac{1}{2}\mathbb{N}_0$. It can then be shown that if the irreducible representation has dimension $2l+1$, then $C_\rho = l(l+1)\,\mathrm{id}$.

Let us now recall the standard way to realize the irreducible representations of so(3) of odd dimension inside $L^2(S^2; \mathbb{C})$: We denote by $\Pi_1$ denote the standard infinite-dimensional unitary representation of SO(3) on $L^2(S^2; \mathbb{C})$ given by

$$\Pi_1 : \mathrm{SO}(3) \to U(L^2(S^2; \mathbb{C})) \tag{12}$$
$$R \mapsto (\psi(y) \mapsto \psi(R^T y)) \tag{13}$$

and denote the corresponding Lie algebra representation by $\pi_1$. A standard computation shows that for our choice of generators $F_1, F_2, F_3$ of so(3), see Appendix B, each $\pi_1(F_i)$ is an angular differential operator which on smooth functions is given by

$$\pi_1(F_1) = \cot(\theta)\cos(\phi)\partial_\phi + \sin(\phi)\partial_\theta \tag{14}$$
$$\pi_1(F_2) = \cot(\theta)\cos(\phi)\partial_\phi - \cos(\phi)\partial_\theta \tag{15}$$
$$\pi_1(F_3) = -\partial_\phi \tag{16}$$

where $(r, \theta, \phi)$ denote the usual spherical coordinates. Recall that the representation $\pi_1$ comes with an associated decomposition of $L^2(S^2;\mathbb{C})$ into orthogonal irreducible subspaces spanned by spherical harmonics. We shall write $L^2(S^2;\mathbb{C}) = \bigoplus_{l=0}^{\infty} H_l$, where $H_l$ is spanned by the spherical harmonics of degree $l$. The Casimir element of $\pi_1$ is then $C_{\pi_1} = -\Delta_{S^2}$, where $\Delta_{S^2}$ denotes the (negative-definite) Laplace Beltrami operator on the sphere. From the remark above we know that it is diagonal on each subspace: $\Delta_{S^2}|_{H_l} = -l(l+1)\,\mathrm{id}$.

We further define the representation $\Pi_2$ by setting

$$\Pi_2 : \mathrm{SO}(3) \to U(\mathbb{C}^3) \tag{17}$$
$$R \mapsto (z \mapsto Rz) \tag{18}$$

and again denote the corresponding so(3) representation by $\pi_2$. It is immediate that $\pi_2 = \mathrm{id}$, i.e. for any $F \in \mathrm{so}(3)$, $\pi_2(F)$ acts by multiplication by the matrix $F$ on $\mathbb{C}^3$. Here it is immediate that $\pi_2$ is irreducible and of dimension $3 = 2 \cdot 1 + 1$, so that the Casimir element is $C_{\pi_2} = 2\,\mathrm{id}$.

We are now interested in the tensor product representation $\Pi = \Pi_1 \otimes \Pi_2$, respectively the corresponding tensor product of the Lie algebra representations, $\pi = \pi_1 \otimes \mathrm{id} + \mathrm{id} \otimes \pi_2$. If we identify $L^2(S^2;\mathbb{C}^3) \cong L^2(S^2;\mathbb{C}) \otimes \mathbb{C}^3$, then $\Pi$ acts on a function $\Psi \in L^2(S^2;\mathbb{C}^3)$ as $\Pi\Psi(y) = R\Psi(R^T y)$. We now want to find a decomposition of $L^2(S^2;\mathbb{C}^3)$ into a direct sum of subspaces which are irreducible for $\pi$. Under the identification above, we find $L^2(S^2;\mathbb{C}^3) = \bigoplus_{l=0}^{\infty} H_l \otimes \mathbb{C}^3$. Note that now $(\pi, H_l \otimes \mathbb{C}^3)$ is a tensor product of an irreducible representation of dimension $2l+1$ and of dimension 3. By [23, Theorem C.1] we thus find that when $l > 0$, we may decompose $H_l \otimes \mathbb{C}^3 = \bigoplus_{m=l-1}^{l+1} W_{lm}$, a direct sum of three subspaces of $H_l \otimes \mathbb{C}^3$, each of which is irreducible under $\pi$ and such that $W_{lm}$ has dimension $2m+1$. The subspace $W_0 := H_0 \otimes \mathbb{C}^3$, which consists of the component-wise constant functions, is evidently irreducible for $\pi$. From the remark at the beginning of the section it now follows that the Casimir operator $C_\pi$ acts as $C_\pi|_{W_0} = 2\,\mathrm{id}$ and $C_\pi|_{W_{lm}} = m(m+1)\,\mathrm{id}$.

In total, we thus obtain the decomposition

$$L^2(B_1(0);\mathbb{C}^3) = L^2(([0,1], r^2 dr);\mathbb{C}) \otimes \left( W_0 \oplus \bigoplus_{l \geq 1, |m-l| \leq 1} W_{lm} \right). \tag{19}$$

The subspaces $W_{lm}$ are mutually orthogonal in $L^2$, which follows directly from the fact that the same is true of the $H_l$. The representations above naturally extend to $L^2(B_1(0);\mathbb{C}^3)$ by letting them act trivially on the radial part. The interest in this decomposition stems from the fact that evidently, the blowup solution $u_0(y) = y$ expressed in self-similar coordinates is invariant under $\Pi$. Since $\Pi$ (or rather its extension to $L^2(B_1(0);\mathbb{C}^3)$) is also a symmetry of the full wave maps equation (in either physical or self-similar coordinates, since the space variable for fixed time is just rescaled), we find that the linearised wave maps equation around $u_0$ is invariant under $\Pi$. This can also easily be checked explicitly, however since the actual proof does not depend on this fact, we will not reproduce the calculation here.

*Remark* 2.1. We adopt the convention of referring to the subspace $W_0$ as $W_{lm}$ with $l = 0$. Thus, whenever we say that $l = 0$, it is understood that we are referring to the subspace

$W_0$ or, later, the associated coefficients $f_0$, etc. and that $m$ does not actually have any relevance in this case.

### 2.2.1 Clebsch-Gordan basis for small $l$

The decomposition $H_l \otimes \mathbb{C}^3 = \bigoplus_{|l-m|\leq 1} W_{lm}$ can be made explicit by means of a Clebsch-Gordan basis. We want to compute this basis explicitly for the first few values of $l$ and $m$. This allows us to understand in which subspaces the solutions generated by the symmetries lie. We use the notation $Z^k_{lm}$, $m \in \{l-1, l, l+1\}$ and $k = 1, \ldots 2m+1$ to denote the Clebsch-Gordan basis for the space $H_l \otimes \mathbb{C}^3 = W_{l,l-1} \oplus W_{l,l} \oplus W_{l,l+1}$. For fixed $l, m$, the functions $Z^k_{lm}$, $k = 1, \ldots 2m+1$ span the subspace $W_{lm}$. The basis functions may be computed by consulting a table of Clebsch-Gordan coefficients [25, Chapter 44]. Expressed in cartesian coordinates, the basis functions $Z^k_{lm}$ are provided in the right-hand column of the following table:

| | | |
|---|---|---|
| $l = 0$ | / | $e_1, e_2, e_3$ |
| $l = 1$ | $m = 0$ | $\frac{1}{\lvert y\rvert} y$ |
| | $m = 1$ | $\frac{1}{\lvert y\rvert}\begin{pmatrix}0\\-y_3\\y_2\end{pmatrix}, \frac{1}{\lvert y\rvert}\begin{pmatrix}-y_3\\0\\y_1\end{pmatrix}, \frac{1}{\lvert y\rvert}\begin{pmatrix}-y_2\\y_1\\0\end{pmatrix}$ |
| | $m = 2$ | $\frac{1}{\lvert y\rvert}\begin{pmatrix}0\\y_2\\-y_3\end{pmatrix}, \frac{1}{\lvert y\rvert}\begin{pmatrix}0\\y_3\\y_2\end{pmatrix}, \frac{1}{\lvert y\rvert}\begin{pmatrix}y_3\\0\\y_1\end{pmatrix}, \frac{1}{\lvert y\rvert}\begin{pmatrix}-y_1\\y_2\\0\end{pmatrix}, \frac{1}{\lvert y\rvert}\begin{pmatrix}y_2\\y_2\\0\end{pmatrix}$ |
| $l = 2$ | $m = 1$ | $\frac{1}{\lvert y\rvert^2}\begin{pmatrix}-2y_1^2+y_2^2+y_3^2\\-3y_1y_2\\-3y_1y_3\end{pmatrix}, \frac{1}{\lvert y\rvert^2}\begin{pmatrix}-3y_1y_2\\y_1^2-2y_2^2+y_3^2\\-3y_2y_3\end{pmatrix}, \frac{1}{\lvert y\rvert^2}\begin{pmatrix}-3y_1y_3\\-3y_2y_3\\y_1^2+y_2^2-2y_3^2\end{pmatrix}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

We take the convention that in each row, the functions are listed in order provided by $k$, so that for instance, $Z^k_0(y) = e_k$, $k = 1, 2, 3$. By inspection of the basis provided above we can identify in which subspace each of the solution generated by symmetries lies. The names of the solutions computed above were chosen precisely such that it now holds for $k = 1, \ldots 2m+1$ that

$$\Psi^k_{lm}(y) = f_{lm}(r)\, Z^k_{lm}(\theta, \phi), \quad \text{for } (l, m) = (1,0), (1,1), (2,1) \tag{20}$$

$$\Psi^k_0(y) = f_0(r)\, Z^k_0(\theta, \phi), \quad \Phi^k_0(y) = g_0(r)\, Z^k_0(\theta, \phi) \tag{21}$$

where

$$f_0(r) = 1, \quad g_0(r) = r^2 - 3, \quad f_{10}(r) = r, \quad f_{11}(r) = r, \quad f_{21}(r) = r^2 \tag{22}$$

and we have again used spherical coordinates $(r, \theta, \phi)$ in the variable $y$. Thus the subspaces $W_{11}$ and $W_{21}$ contribute to the mode space with mode $\lambda = 0$, $W_{10}$ contributes to mode $\lambda = 1$ and $W_0$ contributes to both. At this point the reader hopefully feels that the choice of notation in section 2.1.1 was justified.

### 2.2.2 Computation of the Casimir element $C_\pi$

We recall that the Casimir element $C_\rho$ of any representation $\rho$ of so(3) may be computed as $C_\rho = -\sum_{i=1,2,3} \rho(F_i)^2$. Note our sign convention, which corresponds with the one typically used in the physics literature rather than the convention of not having a minus sign in the definition of the Casimir element more commonly employed in the mathematics community. If $\pi = \pi_1 \otimes \mathrm{id} + \mathrm{id} \otimes \pi_2$ as above, we see that

$$C_\pi = C_{\pi_1} \otimes \mathrm{id} + \mathrm{id} \otimes C_{\pi_2} - 2 \sum_i \pi_1(F_i) \otimes \pi_2(F_i)$$

If we insert the explicit form of $\pi_1, \pi_2$ from above, we find

$$C_\pi = -\Delta_{S^2} + 2 + 2K. \tag{23}$$

where we have abbreviated

$$K = \begin{pmatrix} 0 & \pi_1(F_3) & -\pi_1(F_2) \\ -\pi_1(F_3) & 0 & \pi_1(F_1) \\ \pi_1(F_2) & -\pi_1(F_1) & 0 \end{pmatrix} \tag{24}$$

with the angular differential operators $\pi_1(F_i)$ given in (14) above. With this information in hand, we can now proceed to decoupling the equation.

## 2.3 Self-similar coordinates and decoupling the equation

Recall that we introduced self-similar coordinates defined by

$$\tau = -\log(1-t), \quad y = \frac{x}{1-t}$$

Notice that the truncated backwards lightcone of the blowup point $(1, 0)$, given by $C_{1,0} = \{(t, x) \in \mathbb{R}^4 : t > 0 \text{ and } |x| < 1 - t\}$ becomes the half-infinite cylinder $(0, \infty) \times B_1(0)$ in self-similar coordinates. We will primarily be working in spherical coordinates, so as above we denote by $(r, \theta, \phi)$ spherical coordinates in the $y$ coordinate. We now want to transform the linearized wave maps equation into self-similar coordinates. We first compute the free wave operator

$$\Box = -\partial_{\tau\tau} - \partial_\tau - 2r\partial_{r\tau} - 2r\partial_r - r^2\partial_{rr} + \Delta_y \tag{25}$$

$$= -\partial_{\tau\tau} - \partial_\tau - 2r\partial_{r\tau} + \left(\frac{2}{r} - 2r\right)\partial_r + (1-r^2)\partial_{rr} + \frac{1}{r^2}\Delta_{S^2} \tag{26}$$

For the null form $Q_0$ we find

$$Q_0(f, g) = -\partial_\tau f\, \partial_\tau g + \nabla_y f\, \nabla_y g - \partial_\tau f\,(r\partial_r g) - \partial_\tau g\,(r\partial_r f) - r^2\partial_r f\, \partial_r g.$$

The term $\nabla\Gamma(u) \cdot \Psi\, Q_0(u_0, u_0) = \nabla\Gamma(y) \cdot \Psi\, Q_0(y, y)$ turns out to be a potential term,

$$\nabla\Gamma(y) \cdot \Psi\, Q_0(y, y) = \frac{2(1-r^2)}{1+r^2}\Psi$$

To compute $2\Gamma(u_0)Q_0(u_0,\Psi) = 2\Gamma(y)Q_0(y,\Psi)$, we note that

$$Q_0(y_i,\Psi_j) = e^{2\tau}(\partial_{y_i}\Psi_j - \partial_\tau\Psi_j y_i - y_i r\partial_r\Psi_j)$$

By contracting the first term with $2\Gamma(y)$ we find, modulo the factor of $e^{2\tau}$,

$$\frac{4}{1+r^2}\begin{pmatrix} -r\partial_r & \partial_\phi & \cot(\theta)\cos(\phi)\partial_\phi - \cos(\phi)\partial_\theta \\ -\partial_\phi & -r\partial_r & -\cot(\theta)\cos(\phi)\partial_\phi - \sin(\phi)\partial_\theta \\ -\cot(\theta)\cos(\phi)\partial_\phi + \cos(\phi)\partial_\theta & \cot(\theta)\cos(\phi)\partial_\phi + \sin(\phi)\partial_\theta & -r\partial_r \end{pmatrix}\Psi$$

The second and third term yield, respectively,

$$\frac{4r^2}{1+r^2}\partial_\tau\Psi \quad \text{and} \quad \frac{4r^3}{1+r^2}\partial_r\Psi$$

By rearranging some terms, we arrive at

$$2\Gamma(y)Q_0(y,\Psi) = -\frac{4}{1+r^2}K\Psi + \frac{4r^3-4r}{1+r^2}\partial_r\Psi + \frac{4r^2}{1+r^2}\partial_\tau\Psi$$

where the operator $K$ denotes

$$\begin{pmatrix} 0 & -\partial_\phi & -\cot(\theta)\cos(\phi)\partial_\phi + \cos(\phi)\partial_\theta \\ \partial_\phi & 0 & \cot(\theta)\cos(\phi)\partial_\phi + \sin(\phi)\partial_\theta \\ \cot(\theta)\cos(\phi)\partial_\phi - \cos(\phi)\partial_\theta & -\cot(\theta)\cos(\phi)\partial_\phi - \sin(\phi)\partial_\theta & 0 \end{pmatrix}$$

which can be seen to agree with (24) above. Since $K = \frac{1}{2}(C_\pi + \Delta_{S^2} - 2)$ according to equation (23), the term $2\Gamma(y)Q_0(y,\Psi)$ can then be expressed as

$$2\Gamma(y)Q_0(y,\Psi) = -\frac{2}{1+r^2}(\Delta_{S^2} + C_\pi)\Psi + \frac{4r^3-4r}{1+r^2}\partial_r\Psi + \frac{4r^2}{1+r^2}\partial_\tau\Psi + \frac{4}{1+r^2}\Psi$$

Thus equation (6) as a whole can be expressed in self-similar coordinates as

$$\begin{aligned} -\partial_{\tau\tau}\Psi - \frac{1-3r^2}{1+r^2}\partial_\tau\Psi - 2r\partial_{r\tau}\Psi + (1-r^2)\partial_{rr}\Psi + 2\frac{(1-r^2)^2}{r(1+r^2)}\partial_r\Psi \\ - \frac{2}{1+r^2}C_\pi\Psi + \frac{1-r^2}{r^2(1+r^2)}\Delta_{S^2}\Psi + \frac{6-2r^2}{1+r^2}\Psi = 0 \end{aligned} \tag{27}$$

As explained above, we know that $-\Delta_{S^2} = C_{\pi_1}$ which acts as multiplication by $l(l+1)$ on the subspace $W_{lm}$ when $l > 0$ and vanishes on $W_0$. Recall that we also know that $C_\pi$ acts as multiplication by $m(m+1)$ on $W_{lm}$ when $l > 0$ and as multiplication by 2 on $W_0$.

Before decoupling the equation using this representation, we also provide the form of the linearised equation in self-similar coordinates without using spherical coordinates for completeness. Here we write $\partial_0 = \partial_\tau$ and $\partial_i = \partial_{y_i}$. The equation may then also be represented as

$$\Box\Psi + \sum_{\mu=0}^{3} A^\mu\partial_\mu\Psi + \frac{2(1-r^2)}{1+r^2}\Psi = 0$$

where $A^k_{i,j} = 2\Gamma^i_{k,j}(y)$, $A^0_{i,j} = -2\sum_k \Gamma^i_{k,j}(y)y_k$. Explicitly these matrices are given by

$$A_0 = \frac{4r^2}{r^2+1}\mathrm{id}_{3\times 3}$$
$$A_1 = \frac{4}{r^2+1}\begin{pmatrix} -y_1 & -y_2 & -y_3 \\ y_2 & -y_1 & 0 \\ y_3 & 0 & -y_1 \end{pmatrix}$$
$$A_2 = \frac{4}{r^2+1}\begin{pmatrix} -y_2 & y_1 & 0 \\ -y_1 & -y_2 & -y_3 \\ 0 & y_3 & -y_2 \end{pmatrix}$$
$$A_3 = \frac{4}{r^2+1}\begin{pmatrix} -y_3 & 0 & y_1 \\ 0 & -y_3 & y_2 \\ -y_1 & -y_2 & -y_3 \end{pmatrix}$$

Now we want to decouple the equation using the form (27). Suppose we are given $\Psi \in C^\infty(\overline{B}_1(0))$ a smooth mode solution to (27) for some mode $\lambda \in \mathbb{C}$. Then we express $\Psi$ in spherical coordinates and develop in the orthogonal basis $Z^k_{lm}$ introduced in section 2.2.1:

$$\Psi(r,\theta,\phi) = \sum_{l>0,|l-m|\le 1} \sum_{k=1}^{2m+1} f^k_{lm}(r)\, Z^k_{lm}(\theta,\phi) + \sum_{k=1}^{3} f^k_0(r) Z^k_0(\theta,\phi)$$

Since the $Z^k_{lm}$ are mutually orthogonal, we can, up to some normalization constants compute the radial coefficients as

$$f^k_{lm}(r) = \int_{S^2} \Psi(r\omega) Z^k_{lm}(\omega)\, d\omega$$

and similarly for $f^k_0$. Note that the smoothness of $\Psi$ in cartesian coordinates implies that $\Psi(r,\theta,\phi) \in C^\infty([0,1]\times[0,\pi]\times[0,2\pi])$. An application of dominated convergence now implies that $f^k_{lm}, f^k_0 \in C^\infty([0,1])$. Since the Casimir operators are diagonal on each $W_{lm}$ and $W_0$, we find that each coefficient $f^k_{lm}$ respectively $f^k_0$ must be a smooth solution to

$$-2r\partial_{\tau r} f + (1-r^2)\partial_{rr} f + 2\frac{(1-r^2)^2}{r(1+r^2)}\partial_r f + \frac{-2r^4 + C_{lm}r^2 - l(l+1)}{r^2(1+r^2)} f - \lambda^2 f - \frac{1-3r^2}{1+r^2}\lambda f = 0$$

where the coefficient $C_{lm}$ is given by

$$C_{lm} = \begin{cases} 6 - 2m(m+1) + l(l+1) & \text{if } l \ge 1 \\ C_0 = 2 & \text{if } l = 0 \end{cases}$$

We can clean this equation up through the following transformation of the dependent variable: $f = (1+r^2)\varphi$. Notice that $1+r^2$ and its inverse are regular at every point in $[0,1]$, so that the smoothness properties of solutions are unchanged by this. Then $\varphi$ solves

$$(1-r^2)\partial_{rr}\varphi + \left(\frac{2}{r} - 2(\lambda+1)r\right)\partial_r\varphi - (\lambda^2 + \lambda + V_{lm})\varphi = 0 \tag{28}$$

where the potential $V_{lm}$ is now given by

$$V_{lm} = -\frac{(C_{lm}-10)r^4 + (C_{lm} - l(l+1) + 6)r^2 - l(l+1)}{r^2(1+r^2)^2} \tag{29}$$

which can be written out explicitly as

$$V_{lm} = \frac{(4+2m(m+1)-l(l+1))r^4 + (2m(m+1)-12)r^2 + l(l+1)}{r^2(1+r^2)^2} \tag{30}$$

when $l > 0$ and

$$V_0 = 8\frac{1-r^2}{(1+r^2)^2} \tag{31}$$

when $l = 0$. The solutions generated by the symmetries now transform into solutions to (28):

$$\varphi_{10}(r) = \frac{f_{10}(r)}{1+r^2} = \frac{r}{1+r^2},\ \varphi_{11}(r) = \frac{f_{11}(r)}{1+r^2} = \frac{r}{1+r^2},\ \varphi_{21}(r) = \frac{f_{21}(r)}{1+r^2} = \frac{r^2}{1+r^2} \tag{32}$$

$$\varphi_0^1(r) = \frac{f_0(r)}{1+r^2} = \frac{1}{1+r^2}, \quad \varphi_0^0(r) = \frac{g_0(r)}{1+r^2} = \frac{r^2-3}{1+r^2} \tag{33}$$

where as above, $\varphi_{10}$ and $\varphi_0^1$ solve the equation with mode $\lambda = 1$ and $\varphi_{11}, \varphi_{21}$ and $\varphi_0^0$ with mode $\lambda = 0$.

We finally note that if we set $l = 1, m = 0$, the equation agrees exactly with the one studied in [21], as would be expected from the form of the basis functions $Z_{10}^k$. Therefore we recover the corotational case as the case $l = 1, m = 0$ in our equations. We also remark that the fact that we only obtain the equation in the form studied in [21] after performing the transformation $f = (1+r^2)\varphi$ is natural under the light of the different conventions used: When expressed in our coordinates, Donninger et al. use the ansatz $\tan\left(\frac{f(r)}{2}\right)\frac{y}{r}$ whereas in our setting it is more natural to use $f(r)\frac{y}{r}$. In the linearization, this induces a factor of $\tan'(\arctan(r)) = 1 + r^2$.

## 2.4 Transforming the ODEs

We begin our analysis of the ordinary differential equations (28) by transforming the equations into a more useful form and by removing the solutions that are known. We will make extensive use of the Fuchs-Frobenius theory of ordinary differential equations in the complex plane, see for instance [26, Chapter 4].

### 2.4.1 SUSY trick

For $(l, m) = (1, 0), (2, 1), (1, 1)$ and $l = 0$, the symmetries generate certain solutions to the respective ordinary differential equation. To show that there are no other smooth solutions with $\operatorname{Re}\lambda \geq 0$ up to linear combinations of these functions, we project the known solutions away using the "supersymmetry" trick. In this way, we obtain a new equation, where we now need to show that there are no smooth solutions at all. This section follows [21, Section 3.5].

**Case $l > 0$.** Assume first that $l > 0$, i.e. $(l, m) = (1, 0), (1, 1), (2, 1)$. In this case we found a mode solution denoted above by $\varphi_{lm}$ for the mode $\lambda_{lm} \in \{0, 1\}$. We begin by bringing the mode equation (28) to normal form, i.e. eliminating the first derivative term. Here we write $\partial_r\varphi = \varphi'$ for short-hand. We define the new variable $\psi(r) = r(1-r^2)^{\frac{\lambda}{2}}\varphi(r)$ to obtain

$$-\psi'' + \frac{V_{lm}(1-r^2) + \lambda(\lambda-2)}{(1-r^2)^2}\psi = 0 \tag{34}$$

By setting $\lambda = \lambda_{lm}$ in the transformation, the solution $f_{lm}$ transforms into a solution $\psi_{lm}$ to this equation with $\lambda = \lambda_{lm}$. Formulated in another way, if we define the potential $\mathcal{V}_{lm} = \frac{V_{lm}(1-r^2)+\lambda_{lm}(\lambda_{lm}-2)}{(1-r^2)^2}$, we see that $\psi_{lm}$ solves $-\psi_{lm}'' + \mathcal{V}_{lm}\psi_{lm} = 0$. Define $\omega_{lm} = \frac{\psi_{lm}'}{\psi_{lm}}$, this is well-defined and smooth on $(0,1]$ since the solutions are explicitly known and vanish nowhere on $(0,1]$ as can be checked from the explicit form provided at the end of section (2.3). Then the differential operator from above may be factorised as $-\partial_{rr} + \mathcal{V}_{lm} = (-\partial_r - \omega_{lm})(\partial_r - \omega_{lm})$. Now we set $\tilde{\psi} = (\partial_r - \omega_{lm})\psi$ and apply the operator $\partial_r - \omega_{lm}$ to (34) after multiplying by $(1-r^2)^2$. We obtain

$$(\partial_r - \omega_{lm})\left((1-r^2)^2(-\partial_r - \omega_{lm})\tilde{\psi}\right) = (\lambda(2-\lambda) - \lambda_{lm}(2-\lambda_{lm}))\,\tilde{\psi}$$

By expanding this out we arrive at the equation

$$-(1-r^2)^2\tilde{\psi}'' + 4r(1-r^2)\tilde{\psi}' + (1-r^2)W_{lm}(r)\tilde{\psi} = (\lambda(2-\lambda) - \lambda_{lm}(2-\lambda_{lm}))\,\tilde{\psi}$$

where the potential $W$ is given by

$$W_{lm}(r) = (1-r^2)(\omega_{lm}^2(r) - \omega_{lm}'(r)) + 4r\omega_{lm}(r)$$

Applying another transformation by making the ansatz $\tilde{\psi} = r(1-r^2)^{\frac{\lambda}{2}-1}\tilde{\varphi}$ we find

$$(1-r^2)\tilde{\varphi}'' + \left(\frac{2}{r} - 2(\lambda+1)r\right)\tilde{\varphi}' - \lambda(\lambda+1)\tilde{\varphi} - \tilde{V}_{lm}\tilde{\varphi} = 0 \tag{35}$$

where the potential $\tilde{V}_{lm}$ is given by

$$\tilde{V}_{lm}(r) = W_{lm}(r) - 2 + \frac{\lambda_{lm}(2-\lambda_{lm})}{1-r^2} = (1-r^2)(\omega_{lm}^2(r) - \omega_{lm}'(r)) + 4r\omega_{lm}(r) - 2 + \frac{\lambda_{lm}(2-\lambda_{lm})}{1-r^2}$$

thus effectively leading to a change in the potential only. Now we can simply make this calculation explicit in each of the three cases to obtain

$$\tilde{V}_{10}(r) = \frac{6-2r^2}{r^2(1+r^2)}, \qquad \tilde{V}_{11}(r) = \frac{6-2r^4}{r^2(1+r^2)}, \qquad \tilde{V}_{21}(r) = \frac{12}{r^2(1+r^2)}$$

**Case $l = 0$.** Now let us assume that $l = 0$. Recall that in this case we have the two solutions $\varphi_0^0(r) = \frac{r^2-3}{1+r^2}$ corresponding to the mode $\lambda = 0$ and $\varphi_0^1(r) = \frac{1}{1+r^2}$ corresponding to $\lambda = 1$. Therefore we need to remove two mode solutions here. As before, we begin the calculation by setting $\psi(r) = r(1-r^2)^{\frac{\lambda}{2}}\varphi(r)$ and obtain the equation in normal form given by (34). The transformed solutions to (34) then are $\psi_0(r) = \frac{r(r^2-3)}{1+r^2}$ and $\psi_1(r) = \frac{r\sqrt{1-r^2}}{r^2+1}$. We begin by removing the mode $\lambda = 0$. Exactly as above, we define $\omega_0 = \frac{\psi_0'}{\psi_0}$ and define $\hat{\psi} = (\partial_r - \omega_0)\psi$ so that $\hat{\psi}$ solves

$$-(1-r^2)^2\hat{\psi}'' + 4r(1-r^2)\hat{\psi}' + (1-r^2)W_0(r)\tilde{\psi} = \lambda(2-\lambda)\hat{\psi} \tag{36}$$

with the potential

$$W_0(r) = \frac{2\left(r^6 - 9r^4 - 9r^2 + 9\right)}{r^2\left(r^2-3\right)^2}$$

Instead of transforming the equation directly back to our preferred form, we now bring equation (36) into normal form again by setting $\bar{\psi} = (1-r^2)\hat{\psi}$ to obtain

$$-\bar{\psi}'' + \frac{(W_0(r)-2)(1-r^2)+\lambda(\lambda-2)}{(1-r^2)^2}\bar{\psi} = 0$$

This equation again has a solution $\bar{\psi}_1$ for the mode $\lambda = 1$ we want to remove by the same trick. This solution is given by

$$\bar{\psi}_1 = (1-r^2)\hat{\psi}_1 = (1-r^2)(\partial_r - \omega_0)\psi_1 = \frac{r^2\sqrt{1-r^2}}{r^2-3}$$

As above we define $\omega_1 = \frac{\bar{\psi}_1'}{\bar{\psi}_1}$ and $\tilde{\psi} = (\partial_r - \omega_1)\bar{\psi}$. Once more, we carry out the same calculations as above to arrive at the equation

$$-(1-r^2)^2\tilde{\psi}'' + 4r(1-r^2)\hat{\psi}' + (1-r^2)W_1(r)\tilde{\psi} = \lambda(2-\lambda)\hat{\psi} \tag{37}$$

with the potential

$$W_1(r) = \frac{2r^4+5r^2-6}{r^2(r^2-1)}$$

Finally we make the ansatz $\tilde{\psi} = r(1-r^2)^{\frac{\lambda}{2}-1}\tilde{\varphi}$ to obtain as above

$$(1-r^2)\tilde{\varphi}'' + \left(\frac{2}{r} - 2(\lambda+1)r\right)\tilde{\varphi}' - \lambda(\lambda+1)\tilde{\varphi} - \tilde{V}_0\tilde{\varphi} = 0 \tag{38}$$

where

$$\tilde{V}_0(r) = W_1 - 2 + \frac{1}{1-r^2} = \frac{6}{r^2}$$

Note that in the case of $l = 0$, the equation has six regular singular points at $r = 0, \pm 1, \pm i, \infty$ before applying the supersymmetry trick, whereas we only have the four regular singular points $r = 0, \pm 1, \infty$ after the transformation. When using $z = r^2$ as a variable, as we will do, this means that the equation is a hypergeometric differential equation rather than a Heun equation.

**Lemma 2.2.** *Let $l = 0$ or $(l,m) \in \{(1,0),(2,1),(1,1)\}$ and let $\varphi : [0,1] \to \mathbb{C}$ denote a smooth solution to the mode equation (28) with mode $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geq 0$. Then the transformed function $\tilde{\varphi}$ constructed above in the various cases is a smooth solution to*

$$(1-r^2)\partial_{rr}\tilde{\varphi} + \left(\frac{2}{r} - 2(\lambda+1)r\right)\tilde{\varphi} - (\lambda^2+\lambda+\tilde{V}_{lm})\tilde{\varphi} = 0 \tag{39}$$

*where the potentials $\tilde{V}_{lm}$ have been defined above. Further we have that if $(l,m) \in \{(1,1),(2,1)\}$ then there exists some $c \in \mathbb{C}$ so that*

$$\tilde{\varphi} = 0 \implies \begin{cases} \varphi = 0 & \text{if } \lambda \neq 0 \\ \varphi = c\varphi_{lm} & \text{if } \lambda = 0 \end{cases}$$

*Similarly, when $(l,m) = (1,0)$,*

$$\tilde{\varphi} = 0 \implies \begin{cases} \varphi = 0 & \text{if } \lambda \neq 1 \\ \varphi = c\varphi_{10} = \frac{cr}{1+r^2} & \text{if } \lambda = 1 \end{cases}$$

*and finally when $l = 0$,*

$$\tilde{\varphi} = 0 \implies \begin{cases} \varphi = 0 & \text{if } \lambda \neq 0, 1 \\ \varphi = c\varphi_0^0 = \frac{c(r^2-3)}{1+r^2} & \text{if } \lambda = 0 \\ \varphi = c\varphi_0^1 = \frac{c}{1+r^2} & \text{if } \lambda = 1 \end{cases}$$

*Proof.* We first remark that it is clear that the transformed solution $\tilde{\varphi}$ is smooth on the interior $(0,1)$ and solves the transformed equation there. Thus the following discussion focuses on showing that $\tilde{\varphi}$ is indeed smooth at the endpoints $r = 0$ and $r = 1$.

**Claim.** *$\varphi(r) \simeq r^l$ as $r \to 0$ and $\varphi(r) = O(1)$ as $r \to 1$.*

To see this, we compute the Frobenius indices of the untransformed mode equation (28). At $r = 0$ we find $\{l, -(l+1)\}$ and at $r = 1$ we have $\{0, 2 - \lambda - \lambda^2\}$. Thus at $r = 0$ there exists a fundamental system of the form

$$\varphi_1(r) = r^l f(r) \tag{40}$$

$$\varphi_2(r) = r^{-l-1} g(r) + C \log(r) \varphi_1(r) \tag{41}$$

with $f, g$ analytic in a neighbourhood of $r = 0$ such that $f(0) = g(0) = 1$ and a constant $C \in \mathbb{C}$ (which might be zero). Since we demand $\varphi$ to be smooth at $r = 0$ we see that $\varphi(r) \simeq r^l$ as $r \to 0$. Now first assume that $2 - \lambda - \lambda^2 = n \in \mathbb{Z}$. If $n \geq 0$, a fundamental system is given by

$$\tilde{\varphi}_1(r) = (1-r)^n \tilde{f}(r) \tag{42}$$

$$\tilde{\varphi}_2(r) = \tilde{g}(r) + C \log(1-r) \tilde{\varphi}_1(r) \tag{43}$$

where $\tilde{f}, \tilde{g}$ are analytic around $r = 1$ with $\tilde{f}(1) = \tilde{g}(1) = 1$ and $C \in \mathbb{C}$ might be zero except when $n = 0$. If $n < 0$, a fundamental system is given by

$$\tilde{\varphi}_1(r) = \tilde{f}(r) \tag{44}$$

$$\tilde{\varphi}_2(r) = (1-r)^n \tilde{g}(r) + C \log(1-r) \tilde{\varphi}_1(r) \tag{45}$$

If $2 - \lambda - \lambda^2 \notin \mathbb{Z}$, a fundamental system is given by

$$\tilde{\varphi}_1(r) = \tilde{f}(r) \tag{46}$$

$$\tilde{\varphi}_2(r) = (1-r)^{2-\lambda-\lambda^2} \tilde{g}(r) \tag{47}$$

In all cases it follows that $\varphi(r) \simeq 1$ as $r \to 1$, or better, i.e. $\varphi(r) \simeq (1-r)^n$ for some $n > 0$.

Now we want to use this information to show that $\tilde{\varphi}$ is smooth on $[0,1]$. First assume $l > 0$. Then one can check that $\omega_{lm}$ as defined above is given by

$$\omega_{10}(r) = \frac{r^4 + 3r^2 - 2}{r(r^4 - 1)}, \quad \omega_{11}(r) = \frac{2}{r(1+r^2)}, \quad \omega_{21}(r) = \frac{3 + r^2}{r(1+r^2)}$$

Thus we find $\omega_{lm}(r) \simeq \frac{1}{r}$ as $r \to 0$ in all cases, $\omega_{10}(r) \simeq \frac{1}{1-r}$ and $\omega_{11}(r), \omega_{21}(r) \simeq 1$ as $r \to 1$. By tracing all of the transformations and simply estimating the action of $\partial_r$ as reducing the order by one, we find that $\tilde{\varphi}(r) = O(r^{l-1})$ as $r \to 0$ and $\tilde{\varphi}(r) = O(\varphi(r)) = O(1)$ as $r \to 1$. Thus $\tilde{\varphi}$ is smooth on $[0,1]$. In the case $l = 0$ we find

$$\omega_0(r) = \frac{r^4 + 6r^2 - 3}{r(r^4 - 2r^2 - 3)}, \quad \omega_1(r) = \frac{r^4 - 9r^2 + 6}{r(1-r^2)(3-r^2)}$$

so that $\omega_0(r), \omega_1(r) \simeq \frac{1}{r}$ as $r \to 0$ and $\omega_0(r) \simeq 1$ and $\omega_1(r) \simeq \frac{1}{1-r}$ as $r \to 1$. By tracing through the transformations and again simply estimating the action of $\partial_r$ as reducing the order by one, we find that $\tilde{\varphi}(r) = O(r^{-2})$ as $r \to 0$ and $\tilde{\varphi}(r) = O(\varphi(r)) = O(1)$ as $r \to 1$. The Frobenius indices of the transformed equation at $r = 0$ are computed to be $\{2, -3\}$. Therefore, even though our crude estimate only gives us $\tilde{\varphi}(r) = O(r^{-2})$ as $r \to 0$, it excludes the possibility of our transformed solution to behave like $\tilde{\varphi}(r) \simeq \frac{1}{r^3}$ so that we must in fact have $\tilde{\varphi}(r) \simeq r^2$ as $r \to 0$. Thus, $\tilde{\varphi}$ is also smooth on $[0, 1]$ in this case.

Now we want to study the kernel of our transformation. First note that all of the functions we multiply with during the course of our transformations are non-vanishing on $(0, 1)$. Assume for the rest of the proof that $\varphi$ is a smooth solution to equation (28) with mode $\lambda$.

Let us assume first that $l > 0$, then $\tilde{\varphi} = 0$ is equivalent to $(\partial_r - \omega_{lm})\psi = 0$. This first-order ODE has the general solution $\psi = c\psi_{lm}$, $c \in \mathbb{C}$. Since we assume that $\varphi$ is a solution to (28), $\psi$ solves equation (34), which we can reformulate as

$$0 = (-\partial_r - \omega_{lm}) \underbrace{(\partial_r - \omega_{lm})\psi}_{=0} = \frac{\lambda(2-\lambda) - \lambda_{lm}(2-\lambda_{lm})}{(1-r^2)^2}\psi$$

In case $(l, m) = (1, 0)$ we have $\lambda_{10} = 1$ so that the numerator on the right becomes $-(1-\lambda)^2$. Since $\varphi$ is assumed to be a nontrivial solution, also $\psi \neq 0$ so that it follows that $\lambda = 1$ in this case. Thus we find $\varphi(r) = c\varphi_{10}$ as claimed. If $(l, m) = (1, 1)$ or $(l, m) = (2, 1)$, we have $\lambda_{lm} = 0$ so that the numerator takes the form $\lambda(2 - \lambda)$. Since $\psi \neq 0$, we find that $\lambda \in \{0, 2\}$ in this case. Thus we obtain two solutions corresponding to the two different values of $\lambda$, namely

$$\varphi_0(r) = c\varphi_{lm}(r), \quad \varphi_2(r) = c\frac{\varphi_{lm}(r)}{1-r^2}$$

where $\varphi_0$ solves equation (28) with $\lambda = 0$ whereas $\varphi_2$ solves (28) with $\lambda = 2$. See the following remark after the proof for the explicit form of the solutions. Note that since $\varphi_{lm} \simeq 1$ as $r \to 1$ in both cases, $\varphi_2$ is not actually a smooth solution. Thus we may exclude it by our assumption on $\varphi$ and conclude that $\lambda = 0$ and $\varphi(r) = \varphi_0(r) = c\varphi_{lm}(r)$, as claimed.

For the case $l = 0$ we distinguish the case where $\bar{\psi} = 0$ and $\bar{\psi} \neq 0$. If we assume that $\bar{\psi} = 0$ it follows that $\hat{\psi} = 0$ which by definition means $(\partial_r - \omega_0)\psi = 0$. Thus as above, we find that $\psi = c\psi_0$ and we can rewrite the equation that $\psi$ solves as

$$0 = (-\partial_r - \omega_0) \underbrace{(\partial_r - \omega_0)\psi}_{=0} = \frac{\lambda(2-\lambda)}{(1-r^2)^2}\psi$$

so that since $\varphi \neq 0 \implies \psi \neq 0$ it follows that $\lambda \in \{0, 2\}$, similar to above. Thus we obtain two solutions of equation (28) with $l = 0$ given by

$$\varphi_0(r) = c\varphi_0^0(r), \quad \varphi_2(r) = c\frac{\varphi_0^0(r)}{1-r^2}$$

where as before, $\varphi_0$ solves equation (28) with $\lambda = 0$ and $\varphi_2$ similarly solves the equation with $\lambda = 2$. Since we demand $\varphi$ to be smooth, we can exclude the solution $\varphi_2$ and conclude that $\lambda = 0$ and $\varphi = c\varphi_0^0(r)$ in this case.

If we assume $\bar{\psi} \neq 0$, then from $\tilde{\varphi} = 0$, or equivalently $\tilde{\psi} = (\partial_r - \omega_1)\bar{\psi} = 0$ it follows that $\bar{\psi} = c\bar{\psi}_1$. This is equivalent to saying that $\hat{\psi} = (\partial_r - \omega_0)\psi = c(\partial_r - \omega_0)\psi_1$. This first-order ODE has general solution given by $\psi = a\psi_0 + b\psi_1$. We must have that $b \neq 0$, since otherwise $\bar{\psi} = 0$. As above, we can express the equation that $\bar{\psi}$ solves as

$$0 = (-\partial_r - \omega_1)\underbrace{(\partial_r - \omega_1)\bar{\psi}}_{=0} = \frac{\lambda(2-\lambda)-1}{(1-r^2)^2}\bar{\psi}$$

Since we assume that $\bar{\psi} \neq 0$, it follows that $(1-\lambda)^2 = 0$ so that $\lambda = 1$. Therefore we can compute the general solution $\varphi$ in this case as

$$\varphi(r) = a\frac{r^2-3}{(1+r^2)\sqrt{1-r^2}} + b\frac{1}{1+r^2}$$

Since we demand $\varphi$ to be smooth, we obtain $a = 0$ so that we find $\varphi = c\varphi_0^1$ and $\lambda = 1$ as required. This concludes the proof. □

*Remark* 2.3. In the proof of the previous lemma we have obtained three new solutions $\tilde{\varphi}_{lm}$ to equation (28) for the mode $\lambda = 2$, one for $(l, m) = (1, 1)$, one for $(l, m) = (2, 1)$ and one for $l = 0$. These are given by

$$\tilde{\varphi}_0(r) = \frac{1}{1-r^4}, \quad \tilde{\varphi}_{11}(r) = \frac{r}{1-r^4}, \quad \tilde{\varphi}_{21}(r) = \frac{r^2}{1-r^4}$$

These solutions blow up like $(1-r)^{-1}$ around $r = 1$, so that they are not contained in the Sobolev space $H^s(0,1)$ when $s > \frac{3}{2}$. In fact they are not even in $L^2(\frac{1}{2}, 1)$. Thus it seems reasonable to believe that these solutions should not play a role in the linear stability.

*Remark* 2.4. We make a small remark on the effect observed in the proof of the previous lemma, where we can only obtain a rough estimate of the regularity of the solution, so that it might seem initially that it is not smooth, but salvage this by considering the Frobenius indices of the transformed equation. This effect can be observed explicitly in the computation for $l = 0$ for $\hat{\psi}_1$ above: the solution starts out as $\varphi_1 \simeq 1$ as $r \to 0$, so that $\psi_1(r) \simeq r$, and we can only say that $\hat{\psi}_1(r) = (\partial_r - \omega_0)\psi_1 = O(1)$. But we can explicitly compute that $\hat{\psi}_1 = \frac{r^2}{(1-r^2)^{1/2}(r^2-3)} \simeq r^2$ as $r \to 0$.

When $l = 0$ or $(l, m) = (1, 0), (2, 1), (1, 1)$, this result now leads us to study the transformed equations (39) instead of the original equations. To save on notation, from here on out we replace the old definition of $V_{lm}$ by $\tilde{V}_{lm}$ in these four cases. For all other values of $l$ and $m$, where no smooth solution is expected, this trick does of course not need to be executed and the definition of $V_{lm}$ remains unchanged.

### 2.4.2 Heun and hypergeometric standard form

After having removed the solutions generated by the symmetries if necessary, we now want to transform the resulting equations into a certain standard form through certain changes of variable. It turns out that for $l = 0$ we obtain a hypergeometric differential equation whereas for all other cases, we obtain a Heun equation. Let us briefly describe these two types of equations. An overview can be found in [27, Chapters 15 and 31].

The Heun's equation in canonical form is a second-order linear differential equation in the complex plane given by

$$\frac{d^2u}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right]\frac{du}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}u = 0 \tag{48}$$

where we assume $|a| \geq 1, a \neq 0$ and $\varepsilon = \alpha + \beta - \gamma - \delta + 1$ to ensure that the point $z = \infty$ is a regular singular point. This equation has regular singularities at $z = 0, 1, a, \infty$. Any linear second-order equation in the complex plane with four regular singular points may be brought to this form through a Möbius transformation of the independent variable and s-homotopic transformations of the dependent variable. The Frobenius indices of this equation at the points $z = 0, 1, a, \infty$ are $\{0, 1-\gamma\}, \{0, 1-\delta\}, \{0, 1-\varepsilon\}, \{\alpha, \beta\}$ respectively. The simpler relative of the Heun's equation with only three regular singular points at $z = 0, 1, \infty$ is the hypergeometric differential equation which in canonical form reads

$$\frac{d^2u}{dz^2} + \left(\frac{c}{z} + \frac{1+a+b-c}{z-1}\right)\frac{du}{dz} + \frac{ab}{z(z-1)}u = 0 \tag{49}$$

Similarly to the Heun's equation, any linear second-order ODE in the complex plane with three regular singular points can be brought to this form using the same transformations. The exponents at $z = 0, 1, \infty$ are given by $\{0, 1-c\}, \{0, c-a-b\}, \{a, b\}$ respectively.

First recall the form of the equations (28) obtained above

$$(1-r^2)\partial_{rr}\varphi + \left(\frac{2}{r} - 2(\lambda+1)r\right)\partial_r\varphi - \lambda(\lambda+1)\varphi - V_{lm}\varphi = 0 \tag{50}$$

and keep in mind that we have modified some of the potentials by the SUSY trick without reflecting this in the notation for $V_{lm}$. Note that the potentials are actually rational functions of $r^2$. This equation has six regular singular points at $r = 0, \pm 1, \pm i, \infty$ whenever $l > 0$ and four regular singular points at $r = 0, \pm 1, \infty$ when $l = 0$. We thus introduce the new variable $z = r^2$ and obtain

$$\frac{d^2\varphi}{dz^2} + \left(\frac{3}{2z} + \frac{\lambda}{z-1}\right)\frac{d\varphi}{dz} + \frac{\lambda(\lambda+1) + V_{lm}(z)}{4z(z-1)}\varphi = 0$$

This equation now only has 4 regular singular points at $z = 0, \pm 1, \infty$ when $l > 0$ respectively 3 regular singular points at $z = 0, 1, \infty$ when $l = 0$. We now distinguish:

**Case $l = 0$:** In order to bring the equation into canonical hypergeometric form (49), we need to compute the Frobenius indices at each singular point $z \in \mathbb{C}$ and then apply an appropriate transformation of the dependent variable. Once the Frobenius indices are known, the transformation can be determined from the fact that if the equation has Frobenius indices $\{\alpha, \beta\}$ at the singular point $z = a$, then the equation solved by $\psi(z) = (z-a)^\gamma\varphi(z)$ has indices $\{\alpha+\gamma, \beta+\gamma\}$ at the point $z = a$. Choosing the parameter $\gamma \in \{-\alpha, -\beta\}$ and doing this for every finite singular point brings the equation into canonical form. We make the ansatz $\varphi(z) = z\psi(z)$ and obtain the equation

$$\frac{d^2\psi}{dz^2} + \left(\frac{7}{2z} + \frac{\lambda}{z-1}\right)\frac{d\psi}{dz} + \frac{\lambda^2+5\lambda+6}{4(z-1)z}\psi = 0 \tag{51}$$

The coefficients are read off from this to be $a = \frac{3+\lambda}{2}, b = \frac{2+\lambda}{2}, c = \frac{7}{2}$.

**Case $l > 0$:** In order to study the existence (or rather absence) of smooth solutions in this case, we will later apply the quasi-solution method (see Section 2.6). This requires us to shift the singular points such that the only singular points inside the unit disk are $z = 0$ and $z = 1$. We thus apply the Möbius transformation $z \mapsto \frac{2z}{1+z}$. This sends

$(0,1,-1,\infty) \mapsto (0,1,\infty,2)$. Note carefully that any solution that is smooth at $z=0$ and $z=1$ remains so after this transformation. Denoting the new variable again by $z$, we obtain the equation

$$\frac{d^2\varphi}{dz^2} + \left(\frac{3}{2z} + \frac{\frac{1}{2}-\lambda}{z-2} + \frac{\lambda}{z-1}\right)\frac{d\varphi}{dz} + \frac{\lambda^2+\lambda+V_{lm}\left(\frac{z}{2-z}\right)}{2(z-2)^2(z-1)z}\varphi = 0$$

In order to bring this equation into canonical Heun form (48), we carry out the same procedure as above for each of the four singular points. We again make the ansatz $\varphi(z) = h(z)\psi(z)$ and provide our choice of $h$ in the following table:

| $(l,m)$ | $(1,0)$ | $(2,1)$ | $(1,1)$ | other values |
|---|---|---|---|---|
| $h(z)$ | $z(2-z)^{\frac{\lambda}{2}}$ | $z^{\frac{3}{2}}(2-z)^{\frac{\lambda}{2}}$ | $z(2-z)^{\frac{\lambda-1}{2}}$ | $z^{\frac{l}{2}}(2-z)^{\frac{\lambda}{2}}$ |

The equation for $\psi$ is then in canonical Heun form and given by

$$\frac{d^2\psi}{dz^2} + p_{lm}\frac{d\psi}{dz} + q_{lm}\psi = 0 \tag{52}$$

where the terms $p_{lm}, q_{lm}$ are given by

$$p_{lm} = \begin{cases} \frac{7}{2z} + \frac{\lambda}{z-1} + \frac{1}{2(z-2)} & \text{if } (l,m)=(1,0) \\ \frac{7}{2z} + \frac{\lambda}{z-1} - \frac{1}{2(z-2)} & \text{if } (l,m)=(1,1) \\ \frac{9}{2z} + \frac{\lambda}{z-1} + \frac{1}{2(z-2)} & \text{if } (l,m)=(2,1) \\ \frac{2l+3}{2z} + \frac{\lambda}{z-1} + \frac{1}{2(z-2)} & \text{other cases} \end{cases} \tag{53}$$

and

$$q_{lm} = \begin{cases} \frac{z\left(\lambda^2+6\lambda+8\right)-\lambda^2-12\lambda-12}{4(z-2)(z-1)z} & \text{if } (l,m)=(1,0) \\ \frac{z(\lambda^2+4\lambda+3)-\lambda^2-12\lambda-7}{4(z-2)(z-1)z} & \text{if } (l,m)=(1,1) \\ \frac{z\left(\lambda^2+8\lambda+15\right)-\lambda^2-16\lambda-27}{4(z-2)(z-1)z} & \text{if } (l,m)=(2,1) \end{cases} \tag{54}$$

and in the remaining cases,

$$q_{lm} = \frac{z\left((\lambda-2)(\lambda+4)+l^2+2\lambda l+2l\right)-l^2-4\lambda l-2l-2m^2-2m-\lambda^2-4\lambda+12}{4(z-2)(z-1)z}$$

The coefficients of the standard form (48) can be easily read off from this. We summarize the discussion above in the following

**Lemma 2.5.** *If $\varphi(r) \in C^\infty([0,1])$ is a solution to (50), then $\psi(z)$, as constructed above is also smooth on $[0,1]$ and solves (52) if $l>0$ and (51) if $l=0$.* □

## 2.5 Mode stability for $l = 0$

Here we show the mode stability in the case $l = 0$. In the previous section we showed that it is enough to demonstrate that the equation

$$\frac{d^2\psi}{dz^2} + \left(\frac{7}{2z} + \frac{\lambda}{z-1}\right)\frac{d\psi}{dz} + \frac{\lambda^2 + 5\lambda + 6}{4(z-1)z}\psi = 0 \tag{55}$$

possesses no solutions smooth on $[0, 1]$ when $\operatorname{Re}\lambda \geq 0$. Since this is a hypergeomtric equation, this is very easy to see:

**Lemma 2.6.** *Let* $\operatorname{Re}\lambda \geq 0$. *Then there are no solutions to equation (55) which are smooth on* $[0, 1]$.

*Proof.* We begin by noting that the Frobenius indices at $z = 0$ are $\{0, -\frac{5}{2}\}$ so that by the Fuchs-Frobenius theorem [26, Theorem 4.5] there is a unique (up to multiplication by constants) solution smooth at $z = 0$ given by the standard hypergeometric function

$$\psi(z) = {}_2F_1\left(\frac{3+\lambda}{2}, \frac{2+\lambda}{2}, \frac{7}{2}; z\right)$$

We recall the definition of the Gauss hypergeometric series, $a, b, c \in \mathbb{C}$, $c \notin -\mathbb{N}_0$,

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

when $|z| < 1$ and by analytic continuation elsewhere. Here $(a)_n = a(a+1)\ldots(a+n-1)$ denotes the Pochhammer symbol. Then it is clear that unless either $a \in -\mathbb{N}_0$ or $b \in -\mathbb{N}_0$, the coefficients of the series never vanish. In this case, we may compute the radius of convergence using the ratio test and find that the ratio of successive coefficients is given by

$$\frac{(a+n)(b+n)}{(c+n)(n+1)} \xrightarrow[n\to\infty]{} 1$$

Since in our case $a = \frac{3+\lambda}{2}$ and $b = \frac{2+\lambda}{2}$ and $\operatorname{Re}\lambda \geq 0$ we see that certainly $\operatorname{Re} a, \operatorname{Re} b \geq \frac{1}{2}$. Since $\psi(z)$ solves the hypergeometric differential equation in standard form, it can only be non-analytic at $z = 0, 1, \infty$. Thus $\psi(z)$ cannot be smooth at $z = 1$. $\square$

Let us remark that this is precisely the result we will also be able to obtain for the case where $l > 0$, where we will have to deal with the much more difficult Heun equation. One could carry out a much more elaborate analysis using the explicitly known connection coefficients [28, Corollary 2.3.3] for the hypergeometric equation. However, since no such argument is available for the other cases $l > 0$, there is not much point in improving the result for $l = 0$ at the moment.

## 2.6 Mode stability for $l > 0$

We use the *quasi-solution method* to show that there are no smooth solutions to the equations (28)

$$(1-r^2)\varphi'' + \left(\frac{2}{r} - 2(\lambda+1)r\right)\varphi' - \lambda(\lambda+1)\varphi - V_{lm}\varphi = 0$$

when $l > 0$, $m \in \{l-1, l, l+1\}$ and $\operatorname{Re}\lambda \geq 0$, where the potential $V_{lm}$ has been modified by the supersymmetry trick as explained above if $(l,m) = (1,0), (1,1), (2,1)$. The corotational case $(l,m) = (1,0)$ has been completely resolved by Costin, Donninger and Glogić using this method in [19], see also [20, Section 2.7.3]. In fact, they establish mode stability of the corresponding ground state blowup solution for all space dimensions at least three in the corotational case. We also remark that in the corotational case, there also exists an independent proof by Costin, Donninger and Xia [29]. This is based on a more direct rigorous numerical argument. The quasi-solution method has also been successfully applied to prove mode stability for several other problems, such as supercritical Yang-Mills [30] and stability of the ODE blowup solution to the supercritical cubic wave equation [31]. See also [32, 20]. This section closely follows the presentation in [20, 19, 31].

We begin the discussion by recalling some necessary properties of the Heun equation [27, Chapter 31]. Consider a general Heun equation in its canonical form given by

$$\frac{d^2\psi}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right]\frac{d\psi}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}\psi = 0$$

and recall that the Frobenius indices at $z = 0$ are given by $\{0, 1-\gamma\}$. We assume as is standard that $|a| \geq 1$. Let us assume here that $1 - \gamma \notin \mathbb{N}_0$. It can be directly checked from the coefficients $p_{lm}$ in (53) that this is indeed true for our case. This implies that there is a unique Frobenius solution smooth around $z = 0$, namely the one corresponding to the index 0. Let $\psi(z)$ be this Fuchs-Frobenius solution corresponding to the Frobenius index 0 at $z = 0$. Then $\psi$ has the following expansion

$$\psi(z) = \sum_{n=0}^{\infty} x_n z^n$$

which is valid at least for $|z| < 1$. The coefficients $x_n \in \mathbb{C}$ satisfy $x_0 = 1$, $a\gamma x_1 - q = 0$ and the following three term recurrence relation

$$R_n x_{n+1} - (Q_n + q)x_n + P_n x_{n-1} = 0, \quad n \geq 1 \tag{56}$$

with coefficients given by

$$P_n = (n-1+\alpha)(n-1+\beta), \qquad R_n = a(n+1)(\gamma+n) \tag{57}$$

$$Q_n = n((n-1+\gamma)(1+a) + a\delta + \varepsilon) \tag{58}$$

From now on we restrict ourselves to studying the Heun's equations obtained in section 2.4.2 above. In order to simplify notation, we suppress the dependence on $l$ and $m$ of the coefficients most of the time from now on. Since we are interested in studying the convergence radius of this expansion, we define

$$r_n = \frac{x_{n+1}}{x_n}.$$

From (56) one immediately finds that $r_n$ satisfies

$$r_0 = \frac{q}{a\gamma} \quad \text{and} \quad r_n = A_n + \frac{B_n}{r_{n-1}}, \quad n \geq 1 \tag{59}$$

with the coefficients

$$A_n = \frac{Q_n + q}{R_n}, \quad B_n = -\frac{P_n}{R_n}$$

In our case of the Heun equations computed in 2.4.2, we find the coefficients to be:

$$A_n = \begin{cases} \dfrac{\lambda^2 + 12\lambda + 12n^2 + 8(\lambda+4)n + 12}{4\left(2n^2 + 9n + 7\right)} & l = 1, m = 0 \\ \dfrac{\lambda^2 + 12\lambda + 12n^2 + 8\lambda n + 28n + 7}{8n^2 + 36n + 28} & l = 1, m = 1 \\ \dfrac{\lambda^2 + 16\lambda + 12n^2 + 8\lambda n + 44n + 27}{8n^2 + 44n + 36} & l = 2, m = 1 \end{cases} \tag{60}$$

and in the remaining cases,

$$A_n = \frac{\lambda^2 + 4\lambda + l^2 + 2l(2\lambda + 6n + 1) + 2m^2 + 2m + 12n^2 + 8\lambda n + 8n - 12}{4(n+1)(2l + 2n + 3)}$$

For the coefficients $B_n$ one finds

$$B_n = \begin{cases} -\dfrac{(\lambda + 2n)(\lambda + 2n + 2)}{4(n+1)(2n+7)} & l = 1, m = 0 \\ -\dfrac{(\lambda + 2n - 1)(\lambda + 2n + 1)}{4(n+1)(2n+7)} & l = 1, m = 1 \\ -\dfrac{(\lambda + 2n + 1)(\lambda + 2n + 3)}{4(n+1)(2n+9)} & l = 2, m = 1 \\ -\dfrac{(\lambda + l + 2n - 4)(\lambda + l + 2n + 2)}{4(n+1)(2l + 2n + 3)} & \text{other values} \end{cases} \tag{61}$$

Wwe note the agreement of the coefficients for $(l, m) = (1, 0)$ with those in [19]. The aim of the quasi-solution method is to prove that $r_n \to 1$ in each case. Since the solution $\psi$ (considered as a function on $\mathbb{C}$) can only be non-smooth at $z = 0, 1, 2$ by Fuchs-Frobenius theory, it follows from $r_n \to 1$ that the solution cannot be smooth at $z = 1$, which is the desired result. The first step thus consists in determining the existence and possible values of $\lim_{n\to\infty} r_n$ for each of the possible cases of different values of $l$ and $m$.

### 2.6.1 Determining the possible convergence radii

To determine the possible values of $\lim_{n\to\infty} r_n$, we use Poincaré's theorem on recurrence relations. Consider the $k$-th order linear difference equation with complex coefficients $p_n^i \in \mathbb{C}$ given by

$$x_{n+k} + p_n^1 x_{n+k-1} + \cdots + p_n^k x_n = 0. \tag{62}$$

We assume that for each coefficient, $\lim_{n\to\infty} p_n^i = p^i \in \mathbb{R}$ exists and is real. Then the characteristic equation associated to (62) is given by

$$t^k + p_1 t^{k-1} + \cdots + p_k = 0$$

and its roots $t_1, \dots, t_k \in \mathbb{C}$ are called the characteristic roots of equation (62). Under these assumptions, the following theorem [33, Theorem 8.9] holds:

**Theorem 2.7** (Poincaré's recurrence theorem)**.** *Let $x_n$ be a solution to (62). If $|t_i| \neq |t_j|$ for $i \neq j$, it holds that either*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = t_i$$

*for some characteristic root $t_i$ or $x_n = 0$ eventually.*

**Corollary 2.8.** *Let $l > 0$ and $m \in \{l-1, l, l+1\}$. Then the sequence $r_n$ defined as above is convergent and $\lim_{n\to\infty} r_n \in \{\frac{1}{2}, 1\}$.*

*Proof.* We begin by showing that there cannot exist a $N$ so that for $n \geq N$, $x_n = 0$. Suppose there did. Since the coefficients in the recursion relation never vanish, we find that if $x_{n+1}, x_n = 0$ also $x_{n-1} = 0$. By backwards induction we thus conclude that $x_0 = 0$ which is a contradiction since $x_0 = 1$. Thus we have a nowhere vanishing solution. Next we note that the limits of the coefficients of the recursion relation in all cases are

$$\lim_{n\to\infty} A_n = \frac{3}{2}, \quad \lim_{n\to\infty} B_n = -\frac{1}{2}$$

which can be seen directly by looking at the terms proportional to $n^2$ in the numerator and denominator. Therefore the characteristic equation of the recursion relation becomes

$$t^2 - \frac{3}{2}t + \frac{1}{2} = 0$$

which has the solutions $t = 1$ and $t = \frac{1}{2}$. Now an application of the Poincaré recurrence theorem proves the claim. □

It is clear that (59) does not have an explicit solution. So to prove that $\lim_{n\to\infty} r_n = 1$ we will instead construct an approximative solution or quasisolution $\tilde{r}_n$ so that

$$\lim_{n\to\infty} \tilde{r}_n = 1, \quad \text{and} \quad \limsup_{n\to\infty} \left| \frac{r_n}{\tilde{r}_n} - 1 \right| < \frac{1}{2}$$

These two properties of course imply that $r_n \to 1$. We now proceed to construct this approximation in each of the cases.

### 2.6.2 Construction of the approximations

We give a brief outline on how to construct the approximations. The process is discussed in more detail in [19] for instance. Note also that the actual process of constructing the approximations is entirely irrelevant to the proof itself.

We begin by studying the behavior of the sequence $r_n$ as $\lambda \to \infty$. By looking at the first few values, one observes that $r_n$ is a rational function in $\lambda$ of order two. Thus one suspects that $r_n \sim_{n,l,m} \lambda^2$ as $\lambda \to \infty$. Indeed by dividing equation (59) by $\lambda^2$ it is easy to see that $\lambda^{-2} r_n \sim \lambda^{-2} A_n$ as $\lambda \to \infty$. This provides the coefficient of $\lambda^2$ in the approximation. Similarly, one can obtain the term linear in $\lambda$. In case that $l > 0$, $m = l - 1$, $m = l$ or $m = l + 1$ and $l \geq 3$, $l \geq 2$ or $l \geq 1$ respectively, we also need to repeat this procedure to obtain the behavior of $r_n$ as $l \to \infty$ as well. To complete the approximation, we set $\lambda = 0$ and (if applicable) $l$ to the smallest relevant value (i.e. $l = 3, 2, 1$ in each of three cases respectively). In doing so, we can now numerically compute the sequence $r_n$ for these special values of $\lambda$ and $l$ for $n = 1, \dots 50$. The final term is obtained by fitting a rational function in $n$ with integer coefficients to this sequence.

The choice of this fit is somewhat tricky since it directly interferes with the size of the coefficients $a_n, b_n$ defined later, making it harder or easier to obtain appropriate upper bounds. For this reason, we had to separate the case $(l, m) = (l, l)$, $l \geq 2$ into two subcases, namely $(2, 2)$ and $(l, l)$ with $l \geq 3$ and find separate approximations for each case. Similarly, we had to split the case $(l, m) = (l, l+1)$, $l \geq 1$ into the three subcases $(1, 2)$, $(2, 3)$ and $(l, l+1)$ with $l \geq 3$. In principle it might of course be possible to avoid this by a better choice of approximation and bound. We now provide the approximations we computed as well as the known approximation for $(l, m) = (1, 0)$.

**Case $l = 1, m = 0$:** The approximation computed in [19] is

$$\tilde{r}_n = \frac{\lambda^2}{8n^2 + 36n + 28} + \frac{\lambda(2n+3)}{2n^2 + 9n + 7} + \frac{2n+4}{2n+7}$$

**Case $l = 1, m = 1$:**

$$\tilde{r}_n = \frac{\lambda^2}{8n^2 + 36n + 28} + \frac{\lambda(2n+3)}{2n^2 + 9n + 7} + \frac{15n+15}{15n+40}$$

**Case $l = 1, m = 2$:**

$$\tilde{r}_n = \frac{\lambda^2}{8n^2 + 28n + 20} + \frac{\lambda(2n+2)}{2n^2 + 7n + 5} + \frac{2n+12}{2n+14}$$

**Case $l = 2, m = 1$:**

$$\tilde{r}_n = \frac{\lambda^2}{8n^2 + 44n + 36} + \frac{\lambda(2n+4)}{2n^2 + 11n + 9} + \frac{2n+9}{2n+12}$$

**Case $l = 2, m = 2$:**

$$\tilde{r}_n = \frac{\lambda^2}{8n^2 + 20n + 2(8n+8) + 12} + \frac{\lambda(2n+3)}{2n^2 + 5n + 2(2n+2) + 3} + \frac{6n+30}{6n+35}$$

**Case $l = 2, m = 3$:**

$$\tilde{r}_n = \frac{\lambda^2}{8n^2 + 20n + 2(8n+8) + 12} + \frac{\lambda(2n+3)}{2n^2 + 5n + 2(2n+2) + 3} + \frac{4n+42}{4n+47}$$

**Case $l \geq 3, m = l - 1$:**

$$\tilde{r}_n = \frac{\lambda^2}{l(8n+8) + 8n^2 + 20n + 12} + \frac{\lambda(l+2n+1)}{l(2n+2) + 2n^2 + 5n + 3} + \frac{3(l-3)}{8n+8} + \frac{6n+11}{6n+20}$$

**Case $l \geq 3, m = l$:**

$$\tilde{r}_n = \frac{\lambda^2}{l(8n+8) + 8n^2 + 20n + 12} + \frac{\lambda(l+2n+1)}{l(2n+2) + 2n^2 + 5n + 3} + \frac{3(l-2)}{8n+8} + \frac{n+4}{n+6}$$

**Case $l \geq 3, m = l + 1$:**

$$\tilde{r}_n = \frac{\lambda^2}{l(8n+8) + 8n^2 + 20n + 12} + \frac{\lambda(l+2n+1)}{l(2n+2) + 2n^2 + 5n + 3} + \frac{3(l-1)}{8n+8} + \frac{2n+11}{2n+15}$$

### 2.6.3 Error estimates

We now want to control the size of the relative error

$$e_n = \frac{r_n}{\tilde{r}_n} - 1.$$

First we note the following that if $r_n$ satisfies (59) then for any choice of $\tilde{r}_n \in \mathbb{C} \setminus \{0\}$ the relative error satisfies the recurrence relation

$$e_n = a_n + b_n \frac{e_{n-1}}{1 + e_{n-1}}, \qquad n \geq 1$$

with

$$a_n = \frac{A_n \tilde{r}_{n-1} + B_n}{\tilde{r}_{n-1} \tilde{r}_n} - 1, \quad b_n = -\frac{B_n}{\tilde{r}_{n-1} \tilde{r}_n}. \tag{63}$$

To see this, first note that

$$\frac{e_{n-1}}{1 + e_{n-1}} = 1 - \frac{\tilde{r}_{n-1}}{r_{n-1}} \tag{64}$$

$$\implies \frac{1}{r_{n-1}} = \frac{1 - \frac{e_{n-1}}{1+e_{n-1}}}{\tilde{r}_{n-1}} \tag{65}$$

Now use that $r_n = A_n + \frac{B_n}{r_{n-1}}$ so that

$$e_n = \frac{A_n}{\tilde{r}_n} + \frac{B_n}{\tilde{r}_n r_{n-1}} - 1 = \frac{A_n}{\tilde{r}_n} - 1 + \frac{B_n}{\tilde{r}_n \tilde{r}_{n-1}} \left( 1 - \frac{e_{n-1}}{1 + e_{n-1}} \right) \tag{66}$$

$$= \frac{A_n}{\tilde{r}_n} + \frac{B_n}{\tilde{r}_n \tilde{r}_{n-1}} - 1 - \frac{B_n}{\tilde{r}_n \tilde{r}_{n-1}} \frac{e_{n-1}}{1 + e_{n-1}} = a_n + b_n \frac{e_{n-1}}{1 + e_{n-1}}. \tag{67}$$

The crucial point now is of course our ability to control the size of the coefficients $a_n$ and $b_n$. The explicit form of the coefficients is quite complicated, so we do not provide it in text form here. Instead, we provide it in digital form, see A.3 for further details.

**Lemma 2.9.** *For all* $\operatorname{Re}\lambda \geq 0$ *and* $n \geq n_0$ *we have* $|a_n| \leq a$, $|b_n| \leq b$ *and* $|e_{n_0}| \leq u$ *where all of* $a, b$ *depend on* $l, m$ *and* $n$ *and* $n_0$ *and* $u$ *depend on* $l, m$ *and are chosen to be*

| $(l,m)$ | $(1,1)$ | $(1,2)$ | $(2,1)$ | $(2,2)$ | $(2,3)$ |
|---|---|---|---|---|---|
| $a$ | $\frac{72+125n}{300(-3+5n)}$ | $\frac{75n+266}{150(6n+1)}$ | $\frac{-71+100n}{300(-5+4n)}$ | $\frac{125n+482}{300(5n+2)}$ | $\frac{5n+12}{60n}$ |
| $b$ | $\frac{-11+16n}{4(-1+8n)}$ | $\frac{25n-11}{50(n+1)}$ | $\frac{-37+50n}{25(-1+4n)}$ | $\frac{400n-179}{100(8n+11)}$ | $\frac{125n-96}{50(5n+1)}$ |
| $n_0$ | 2 | 4 | 2 | 3 | 2 |

| $(l,m)$ | $(3,2)$ | $(3,3)$ | | | |
|---|---|---|---|---|---|
| $a$ | $\frac{125n-121}{300(5n-7)}$ | $\frac{800n-443}{600(16n-19)}$ | | | |
| $b$ | $\frac{400n-319}{100(8n-3)}$ | $\frac{104n-133}{8(26n-27)}$ | | | |
| $n_0$ | 2 | 3 | | | |

| $(l,m)$ | $(l,l-1), l \geq 4$ | $(l,l), l \geq 4$ | $(l,l+1), l \geq 3$ |
|---|---|---|---|
| $a$ | $\frac{-1016+272l+125n}{300(-23+5l+5n)}$ | $\frac{-2810+887l+512n}{48(-515+128l+128n)}$ | $\frac{-27+10l+4n}{12(-15+4l+4n)}$ |
| $b$ | $\frac{-63+11l+20n}{20(-9+2l+2n)}$ | $\frac{-9842+2071l+2800n}{200(-113+28l+28n)}$ | $\frac{-29+5l+13n}{2(-45+13l+13n)}$ |
| $n_0$ | 2 | 2 | 2 |

*The value of* $u$ *is set to be* $u = \frac{3}{10}$ *except when* $(l,m) = (1,2)$, *where we set* $u = \frac{1}{3}$.

*Proof.* The first step of the argument consists in showing that the coefficients and the relative error for our choice of $n_0$ are analytic functions of the mode in the relevant domain.

**Claim.** $\tilde{r}_n^{-1}$ *is analytic as a function of* $\lambda$ *in the domain* $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ *for all* $n \geq 1$.

This follows simply by observing that $\tilde{r}_n$ is a quadratic polynomial in $\lambda$ for fixed $n, l, m$, computing the roots of this polynomial and checking that they lie in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ for each $n$ and for each of the different parameters $l$ and $m$. Indeed, both roots are negative real numbers in all cases. Since the explicit expressions are a bit cumbersome, we carry the calculation out in Appendix A.1.

**Claim.** $r_{n_0}$ *is analytic as a function of* $\lambda$ *in the domain* $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.

By explicitly computing $r_{n_0}$ we observe that it is a rational function of $\lambda$. Thus we need to show that its denominator has its roots contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. This is done by using Wall's criterion, which is explained and carried out in Appendix A.2.

From the explicit form (60), (61) of $A_n$ and $B_n$ provided above it is clear that $\lambda$ only occurs in the numerator and it does so polynomially. In particular $A_n$ and $B_n$ are analytic as functions of $\lambda \in \mathbb{C}$. It thus follows immediately that $a_n$ and $b_n$ are analytic as functions of $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. Similarly, the analyticity of $r_{n_0}$ and $\tilde{r}_n^{-1}$ implies that of $e_{n_0}$. According to the Phragmen-Lindelöf principle, it thus suffices to show the required bounds for $\lambda$ on the imaginary line $\{z \in \mathbb{C} : \operatorname{Re} z = 0\}$ in order to obtain them for the domain $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.

Now suppose $\lambda = it$, $t \in \mathbb{R}$. We demonstrate the argument on the example of $a_n$ for the case $(l, m) = (l, l-1)$, $l \geq 4$, but the argument is the exact same for $b_n$ and $e_{n_0}$ and the other cases. We explicitly compute $|a_n|^2(it)$ and note that this is a rational function of $t^2$ whose coefficients are polynomials in $n$ and $l$ with only integer coefficients, say $|a_n|^2(it) = \frac{F(t^2)}{G(t^2)}$. Observe that all of our bounds are also rational functions of $n$ and $l$ with integer coefficients. Thus to show $|a_n| \leq \frac{x}{y}$, say, we can equivalently show $Fy^2 - Gx^2 \leq 0$. This expression is again a polynomial in $t^2$ with coefficients that are polynomials in $n$ and $l$. When we shift the variables $n \mapsto n+2$ and $l \mapsto l+4$, we then find that all of the non-zero coefficients appearing here are negative integers, thus demonstrating the negativity of $Fy^2 - Gx^2$ for $n \geq 2$ and $l \geq 4$.

In all other cases the only difference in the argument lies in the shift in $n$ and $l$ (if it appears at all) is used. See Appendix A.3 for a discussion of the shifts that have been used in each case. There is one special case, when $(l, m) = (1, 2)$ in the proof of the bound for $e_{n_0}$. Here, not all coefficients are negative. It is however still very easy to see that the polynomial is always negative, we discuss this in Appendix A.3.

Carrying this procedure out is now a simple matter of computation, since everything is completely explicit. However, since the resulting expressions are very cumbersome, we have decided to provide the results of this calculation in a digital format rather than in text form [34]. See Appendix A.3 for an explanation of what exactly is provided. □

**Corollary 2.10.** *$|e_n| \leq \frac{1}{3}$ for all $n \geq n_0$ and $\operatorname{Re} \lambda \geq 0$.*

*Proof.* Note that for $x \in \mathbb{C}$ with $|x| \leq y < 1$

$$\left| \frac{x}{1+x} \right| \leq \frac{|x|}{1-|x|} \leq \frac{y}{1-y}$$

since the function $y \mapsto \frac{y}{1-y}$ is strictly increasing on the interval $[0, 1)$. Let us assume abstractly that we have made a choice of $0 < y < 1$ and that we have obtained bounds of the form $|a_n| \leq a$ and $|b_n| \leq b$. Then in order to close the argument we see that the crucial property is that we may choose $a, b$ and $y$ such that

$$a + b\frac{y}{1-y} \leq y \iff y^2 + (b-a-1)y + a < 0.$$

The reader may now convince himself that in every of the above cases for $l$ and $m$, the value $y = \frac{1}{3}$ will satisfy this condition for all $n \geq n_0$ and all values of $l$ and our choice of bounds $a$ and $b$. □

This result concludes the proof of Theorem (1.1).

# 3 Outlook: Linear stability

## 3.1 Well-posedness of the wave maps equation

In this section we discuss the proof of well-posedness of the wave maps equation locally around any given solution in the Sobolev space $H^s(\mathbb{R}^n)$ when $s > \frac{n}{2} + 1$. In this case, the proof is by standard methods and does not necessitate the much more refined methods from [1, 2, 3, 4] to reduce the regularity threshold to $s > \frac{n}{2}$ for instance. We put an emphasis on demonstrating explicitly that the time-development map is $C^1$ in the Fréchet sense. In the case $s > \frac{n}{2} + 1$ considered here the proof does in fact not use any special properties of the wave maps equation. In particular, in contrast to the more refined methods necessary for proving well-posedness at lower regularity, we are ignoring the null structure of the equation completely. The results contained here are all nicely explained in [35, Chapter 6 and Appendix A]. We assume $s > \frac{n}{2} + 1$ and $n \geq 2$ throughout this subsection.

Consider $X_T = C([0,T), H^s(\mathbb{R}^n;\mathbb{R}^n)) \cap \dot{C}^1([0,T), H^{s-1}(\mathbb{R}^n;\mathbb{R}^n))$ and initial data in the space $(u_0, u_1) \in Y = H^s(\mathbb{R}^n;\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n;\mathbb{R}^n)$. As above, we rewrite the wave maps equation in Duhamel form as a fixed point equation as $u = F(u)$ by defining for any given initial data

$$F(u) = \tilde{u} + \Box^{-1} N(u) \tag{68}$$

where $N(u) = \Gamma(u) Q_0(u,u)$ and $\tilde{u}$ solves the homogeneous wave equation with initial data $(u_0, u_1)$. Let $\hat{u} \in X_{T^*}$ be a solution to the wave maps equation which exists up to a given time $T^*$. We use the notation $u[t] = (u(t), \partial_t u(t))$.

**Lemma 3.1** (Product lemma)**.** *If $s \geq 0$, then*

$$\|fg\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{L^\infty(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)}$$

*for all $f, g \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In particular, if $s > \frac{d}{2}$ we get the algebra property*

$$\|fg\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)}$$

**Lemma 3.2.** *Let $f \in H^s(\mathbb{R}^d, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^n)$ for some $s \geq 0$. Let $k = \lceil s \rceil$ and $F \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ such that $F(0) = 0$. Then $F(f) \in H^s$ as well, with*

$$\|F(f)\|_{H^s(\mathbb{R}^d)} \lesssim_{F, \|f\|_\infty} \|f\|_{H^s(\mathbb{R}^d)}$$

We can consider the nonlinearity $N$ to be a nonlinear spatial differential operator acting on the pair $(u, \partial_t u)$. For brevity, we will write $N(u)$ instead of $N(u, \partial_t u)$ and instead of $(u, \partial_t u) \in H^s \times H^{s-1}$ we will often just write $u \in H^s \times H^{s-1}$. Then we have:

**Lemma 3.3.** *The nonlinearity $N$ maps $Y = H^s(\mathbb{R}^n;\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n;\mathbb{R}^n)$ into $H^{s-1}(\mathbb{R}^n;\mathbb{R}^n)$ and $N \in C^1(Y, H^{s-1}(\mathbb{R}^n;\mathbb{R}^n))$ in the Fréchet sense. Its derivative is given by*

$$DN(u)\phi = 2\Gamma(u) Q_0(u, \varphi) + \nabla\Gamma(u) \cdot \varphi \, Q_0(u,u)$$

*In addition, $N$ is Lipschitz on every bounded subset of $Y$.*

*Proof.* We begin by noting that $\Gamma \in C_b^\infty(\mathbb{R}^n)$ and that $\Gamma(0) = 0$. Let $u, v \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$. Then since $s - 1 > \frac{n}{2}$, we find

$$\|Q_0(u,u) - Q_0(v,v)\|_{H^{s-1}} \lesssim \|(\partial_t u)^2 - (\partial_t v)^2\|_{H^{s-1}} + \|(\nabla u)^2 - (\nabla v)^2\|_{H^{s-1}} \tag{69}$$

$$\lesssim \sum_{\alpha=0}^{n} \left(\|\partial_\alpha u\|_{H^{s-1}} + \|\partial_\alpha v\|_{H^{s-1}}\right) \|\partial_\alpha u - \partial_\alpha v\|_{H^{s-1}} \tag{70}$$

after an application of Lemma (3.1). Thus if $u, v \in Y$ and $Q_0(u,u)$ is now interpreted as the matrix $Q_0(u,u)_{ij} = Q_0(u_i, u_j)$ we thus find

$$\|Q_0(u,u) - Q_0(v,v)\|_{H^{s-1}} \lesssim (\|u\|_Y + \|v\|_Y)\|u - v\|_Y \tag{71}$$

In an entirely similar fashion one shows

$$\|Q_0(u,v)\|_{H^{s-1}} \lesssim \|u\|_Y \|v\|_Y \tag{72}$$

Next we note that for $u, v \in Y$, Lemma (3.2) implies

$$\|\Gamma(u+v) - \Gamma(u)\|_{H^{s-1}} \lesssim \|v\|_{H^{s-1}} \le \|v\|_Y$$

and similarly

$$\|\Gamma(u+v) - \Gamma(u) - D\Gamma(u)v\|_{H^{s-1}} \lesssim \|v\|_{H^{s-1}}^2 \le \|v\|_Y^2$$

Thus putting everything together we find

$$\|N(u) - N(v)\|_{H^{s-1}} \lesssim \|\Gamma(u) - \Gamma(v)\|_{H^{s-1}} \|u\|_{H^{s-1}}^2 + \|\Gamma(v)\|_{H^{s-1}} \|Q_0(u,u) - Q_0(v,v)\|_{H^{s-1}} \tag{73}$$

$$\lesssim \|u - v\|_Y \|u\|_Y^2 + \|v\|_Y \left(\|u\|_Y + \|v\|_Y\right) \|u - v\|_Y \tag{74}$$

$$\le \left(\|u\|_Y + \|v\|_Y\right)^2 \|u - v\|_Y \tag{75}$$

which shows both that $N$ maps $Y$ into $H^{s-1}(\mathbb{R}^n; \mathbb{R}^n)$ (by setting $v = 0$ and noting that $N(0) = 0$) and that $N$ is locally Lipschitz. To see that $N$ is indeed $C^1$, consider

$$\begin{aligned} &N(u+v) - N(u) - 2\Gamma(u)Q_0(u,v) - \nabla\Gamma(u) \cdot v\, Q_0(u,u) \\ &= \underbrace{\left(\Gamma(u+v) - \Gamma(u) - \nabla\Gamma(u) \cdot v\right) Q_0(u,u)}_{\text{I}} \\ &\quad + \underbrace{2\left(\Gamma(u+v) - \Gamma(u)\right) Q_0(u,v)}_{\text{II}} + \underbrace{\Gamma(u+v)Q_0(v,v)}_{\text{III}} \end{aligned} \tag{76}$$

Each of these terms can now be easily estimated in the $H^{s-1}$ norm using Lemma (3.1) and the estimates computed above:

$$\|\text{I}\|_{H^{s-1}} \lesssim \|v\|_Y^2 \|u\|_Y^2 \tag{77}$$

$$\|\text{II}\|_{H^{s-1}} \lesssim \|v\|_Y^2 \|u\|_Y \tag{78}$$

$$\|\text{III}\|_{H^{s-1}} \lesssim \|v\|_Y^2 \|u + v\|_Y \tag{79}$$

so that we find

$$\|N(u+v) - N(u) - 2\Gamma(u)Q_0(u,v) - \nabla\Gamma(u) \cdot v\, Q_0(u,u)\|_{H^{s-1}} = o(\|(v, \partial_t v)\|_Y) \tag{80}$$

We finally note that for a fixed $u \in Y$, the map $v \mapsto 2\Gamma(u)Q_0(u,v) + \nabla\Gamma(u) \cdot v\, Q_0(u,u)$ is a bounded operator from $Y$ into $H^{s-1}(\mathbb{R}^n;\mathbb{R}^n)$: Note $\nabla\Gamma(u) \in H^{s-1}$ by an application of Lemma (3.2) and $Q_0(u,u) \in H^{s-1}$ by the above estimates. Thus by Lemma (3.1), the potential term $\nabla\Gamma(u)Q_0(u,u) \in H^{s-1}$ and further,

$$\|\nabla\Gamma(u) \cdot v\, Q_0(u,u)\|_{H^{s-1}} \lesssim \|\nabla\Gamma(u)\, Q_0(u,u)\|_{H^{s-1}} \|v\|_Y$$

Similarly, for the first term we know from the above estimates that

$$\|Q_0(u,v)\|_{H^{s-1}} \lesssim \|u\|_Y \|v\|_Y$$

$\Gamma(u) \in H^{s-1}$ so that $\|\Gamma(u)Q_0(u,v)\|_{H^{s-1}} \lesssim \|\Gamma(u)\|_{H^{s-1}} \|u\|_Y \|v\|_Y$ proving the boundedness of the operator. Thus, we find that indeed $N \in C^1$ with

$$DN(u)\phi = 2\Gamma(u)Q_0(u,\varphi) + \nabla\Gamma(u) \cdot \varphi\, Q_0(u,u)$$

This concludes the proof. □

**Lemma 3.4.** *Let $r, T$ be chosen such that $0 < T < T^*$ and $T(r + 2\|\hat{u}\|_Y)^2 \leq \frac{1}{2}$. If the initial data satisfy $\|(u_0,u_1) - \hat{u}[0]\|_{H^s \times H^{s-1}} < \frac{r}{2}$ then $F(B^X(r,\hat{u})) \subset B^X(r,\hat{u})$ and $F|_{B^X(r,\hat{u})}$ is Lipschitz with Lipschitz constant bounded from above by $T(r + 2\|\hat{u}\|_Y)^2$.*

*Proof.* For brevity we write $X = X_T$. Let $u \in B^X(r,\hat{u}) \subset X$. Then $v = F(u)$ solves the equation $\Box v + N(u) = 0$ and $w = v - \hat{u}$ solves $\Box w + N(u) - N(\hat{u}) = 0$. We thus have the classical energy estimate for the inhomogeneous wave equation:

$$\|v - \hat{u}\|_X \leq \|u_0 - \hat{u}(0)\|_{H^s(\mathbb{R}^d)} + \|u_1 - \partial_t \hat{u}(0)\|_{H^{s-1}(\mathbb{R}^d)} + \|N(u) - N(\hat{u})\|_{L^1_t H^{s-1}_x}$$

The sum of the first two terms is by assumption bounded from above by $\frac{r}{2}$. For the third term we estimate

$$\|N(u) - N(\hat{u})\|_{L^1_t H^{s-1}_x} \leq T\, \|N(u) - N(\hat{u})\|_{C^0_t H^{s-1}_x}$$

Now we may apply the previous lemma, noting that the $C^0_t(H^s_x \times H^{s-1}_x)$ norm of $(u, \partial_t u)$ is just the $X$ norm to get

$$T\, \|N(u) - N(\hat{u})\|_{C^0_t H^{s-1}_x} \lesssim T(r + 2\|\hat{u}\|_Y)^2 \|u - \hat{u}\|_X \leq \frac{1}{2}\|u - \hat{u}\|_X \leq \frac{r}{2}$$

It follows that by our assumption on $r$ and $T$, $\|F(u) - \hat{u}\|_X \leq r$. Note that it is clear that $F(u) \in X$: From the previous estimate applied with $\hat{u} = 0$ it follows in particular that $F$ indeed maps $X$ into itself.

To show that $F|_{B^X(r,\hat{u})}$ is Lipschitz, let $u, v \in B^X(r,\hat{u})$. Note $F(u) - F(v) = \Box^{-1}(N(u) - N(v))$ which is a solution $w$ to $\Box w = N(u) - N(v)$ with vanishing initial data. Thus we apply the classical energy estimate as above

$$\|F(u) - F(v)\|_X \leq T\, \|N(u) - N(v)\|_{C^0_t H^{s-1}_x} \leq T(r + 2\|\hat{u}\|_Y)^2 \|u - v\|_X$$

where the estimate we used is the exact same as above, which completes the proof. □

**Theorem 3.5** (Local well-posedness)**.** *Let $s > \frac{d}{2} + 1$ and consider a solution to the wave maps equation which exists up to time $T^*$, $\hat{u} \in C([0,T^*), H^s(\mathbb{R}^n)) \cap \dot{C}^1([0,T^*), H^{s-1}(\mathbb{R}^n))$. Let $r, T$ be chosen such that $0 < T < T^*$ and $T(r + 2\|\hat{u}\|_Y)^2 \le \frac{1}{2}$. Then if*

$$\|(u_0, u_1) - \hat{u}[0]\|_{H^s \times H^{s-1}} < \frac{r}{2},$$

*there exists a unique solution $u \in X = C([0,T), H^s(\mathbb{R}^d)) \cap \dot{C}^1([0,T), H^{s-1}(\mathbb{R}^d))$ of the wave maps equation with initial data $(u_0, u_1)$. Further the solution is a Lipschitz function of the initial data.* □

Next we want to show that for a chosen time $0 < t < T$, the time-$t$ evolution map is of class $C^1$ and its derivative is given by the time-$t$ evolution for the linearised system.

**Theorem 3.6.** *The solution map $S : Y \supset B^Y(r, (u_0, u_1)) \to X$ is $C^1$ in the Fréchet sense with $DS(u_0, u_1)$ given by the time evolution of the linearised wave maps equation around $u = S(u_0, u_1)$.*

*Proof.* Let $u = S(u_0, u_1)$. Then recall from (6) the form of the linearisation of the wave maps equation around a solution $u$:

$$\Box\varphi + 2\Gamma(u) Q_0(u, \varphi) + \nabla\Gamma(u) \cdot \varphi\, Q_0(u, u) = 0$$

Since this is a standard linear wave equation, the well-posedness in the space $Y$ is clear as long as we choose the time small enough so that the solution $\hat{u}$ is regular (consider the blowup solution from above for instance). Now let $(\phi_0, \phi_1) \in B^Y(r, 0)$ and denote by $\phi \in X$ the corresponding solution of the linear system. Similarly, denote by $u^\phi = S(u_0 + \phi_0, u_1 + \phi_1)$. The classical energy estimate together with equation (80) yields

$$\|u^\phi - u - \phi\|_X \le \|N(u^\phi) - N(u) - 2\Gamma(u)Q_0(u,\varphi) - \nabla\Gamma(u) \cdot \varphi\, Q_0(u,u)\|_{L^1_t H^{s-1}_x} \tag{81}$$

$$\le T \sup_{0 \le t \le T} \|N(u^\phi) - N(u) - 2\Gamma(u)Q_0(u,\varphi) - \nabla\Gamma(u) \cdot \varphi\, Q_0(u,u)\|_{H^{s-1}_x} \tag{82}$$

$$= T \sup_{0 \le t \le T} o(\|(\phi, \partial_t\phi)\|_Y) = o(\|\phi\|_X) \tag{83}$$

which concludes the proof. □

**Corollary 3.7.** *For $0 < t < T$ consider $S(t) : B^Y(r, (u_0, u_1)) \to Y$ the time-$t$ evolution map. Then $S(t)$ is $C^1$ in the Fréchet sense and $DS(t)(u_0, u_1)$ is the time-$t$ evolution for the linearised wave maps equation around $u = S(u_0, u_1)$.*

*Proof.* The time-$t$ evolution may be written as the composition $K_t \circ S$ where for $0 < t < T$ we denote by $K_t : X_T \to Y$ the bounded operator which evaluates a function and its first time derivative at time $t$, i.e. $K_t(u) = (u(t), \partial_t u(t))$. Then the statement follows immediately from Theorem (3.6), the boundedness of the operator $K_t$ and the chain rule. □

## 3.2 Outlook and future directions

Here we give a brief overview of the next possible steps that could be taken here. One next possible step would be to restrict the evolution to the backwards lightcone, which is possible due to finite speed of propagation. Then one would show that any eigenvalue of $DS(1)$, considered as a bounded operator on $H^s \times H^{s-1}$ must in fact be a mode with a corresponding mode solution that is not necessarily smooth but in $H^s(B_1(0))$. Interior elliptic regularity would then immediately show that the eigenfunction must be smooth in the interior of $B_1(0)$. However, it is at present not clear to the author whether one could obtain smoothness at the boundary. Either way, Theorem (1.1) shows at least that there cannot be solutions $\Psi \in H^s(B_1(0))$ whenever $2 - \operatorname{Re}\lambda - (\operatorname{Re}\lambda)^2 + \operatorname{Im}\lambda > 1$ due to a simple Frobenius index analysis. Supposing that one could indeed exclude eigenvalues of $DS(1)$ with non-negative real part other than $\lambda = 0, 1$, a next step might be to define a suitable projection in order to restrict time evolution to a stable direction. A different approach might be to employ a semigroup formulation, not unlike the approach taken in [22]. In addition, the Fredholm spectrum of $DS(1)$ needs to be shown to be strictly contained within the unit disk.

# A Calculations

## A.1 Roots of $\tilde{r}_n$

In this section we provide the explicit formulas for the roots of the approximations $\tilde{r}_n$ as required in the first step of the proof of Lemma (2.9). For fixed $n, l$ and $m$, $\tilde{r}_n$ is a quadratic polynomial in $\lambda$. Thus its roots are readily computed. We do this in each case:

**Case $l \geq 3, m = l-1$:** Here the roots are directly computed to be located at

$$\lambda = \frac{l(8n+8)+8n^2+20n+12}{4ln+4l+4n^2+10n+6} \times \left(-(l+2n+1) \pm \sqrt{\frac{6l^2n+20l^2+30ln^2+199ln+162l+48n^3+262n^2+313n+218}{8(3n+10)}}\right)$$

Both of these values are in fact negative real numbers. To see this first observe that the expression in the square root is clearly positive for $n \geq 1$. Thus we only need to check that the last expression in the first line is greater in absolute value than the square root. We thus calculate the difference of the square of the last expression from the first line and the expression inside the square root. We obtain

$$\frac{18l^2n+60l^2+66ln^2+169ln-2l+48n^3+154n^2+31n-138}{8(3n+10)}$$

which is easily verified to be positive for all $l \geq 3$ and $n \geq 2$ by shifting the variables and expanding. For the remaining cases we will simply provide the formula and omit the proof of negativity, as it is a similarly unenlightening computation.

**Case $l \geq 3, m = l$:**

$$\lambda = \frac{l(8n+8)+8n^2+20n+12}{4ln+4l+4n^2+10n+6} \times \left(-(l+2n+1) \pm \sqrt{\frac{2l^2n+12l^2+10ln^2+95ln+50l+16n^3+132n^2+106n+60}{8(n+6)}}\right)$$

**Case $l \geq 3, m = l+1$:**

$$\lambda = \frac{l(8n+8)+8n^2+20n+12}{4ln+4l+4n^2+10n+6} \times \left(-(l+2n+1) \pm \sqrt{\frac{4l^2n+30l^2+20ln^2+208ln+19l+32n^3+300n^2+116n-9}{8(2n+15)}}\right)$$

**Case $l = 1, m = 1$:**

$$\lambda = \frac{8n^2+36n+28}{14+18n+4n^2}\left(-(3+2n) \pm \sqrt{\frac{6n^3+35n^2+75n+51}{3n+8}}\right)$$

**Case $l = 1, m = 2$:**

$$\lambda = \frac{8n^2 + 28n + 20}{10 + 14n + 4n^2}\left(-(2+2n) \pm \sqrt{\frac{-2 + 13n + 17n^2 + 2n^3}{n+7}}\right)$$

**Case $l = 2, m = 1$:**

$$\lambda = \frac{8n^2 + 44n + 36}{18 + 22n + 4n^2}\left(-(4+2n) \pm \sqrt{\frac{4n^3 + 40n^2 + 107n + 111}{2n+12}}\right)$$

**Case $l = 2, m = 2$:**

$$\lambda = \frac{8n^2 + 36n + 28}{4n^2 + 18n + 14}\left(-(3+2n) \pm \sqrt{\frac{12n^3 + 98n^2 + 162n + 105}{6n+35}}\right)$$

**Case $l = 2, m = 3$:**

$$\lambda = \frac{8n^2 + 36n + 28}{4n^2 + 18n + 14}\left(-(3+2n) \pm \sqrt{\frac{8n^3 + 116n^2 + 194n + 129}{4n+47}}\right)$$

## A.2 Analyticity of $r_{n_0}$

Here we verify the required analyticity of $r_{n_0}$ as a function of $\lambda$ in the region $\operatorname{Re}\lambda \geq 0$. Recall that the initial value $n_0$ we have chosen depends on $l$ and $m$. To do this, we use Wall's criterion [36]. The method can be described as follows:

First we note that $r_n$ is a rational function of $\lambda$, so let $d_n$ be its denominator, which is now a polynomial in $\lambda$ of degree $2n$. To show analyticity of $r_n$ as a function $\lambda$ when $\operatorname{Re}\lambda \geq 0$, we need to show that $d_n$ only vanishes in the region where $\operatorname{Re}\lambda < 0$. Let $\hat{d}_n$ denote the polynomial obtained from $d_n$ by setting all coefficients of even powers of $\lambda$ to be zero (i.e. we delete the leading order term and every second one after that). By successive polynomial division (using $\lambda$ as the variable), we can obtain a continued fraction expansion of the quotient $\frac{\hat{d}_n}{d_n}$. Wall's criterion now says that this expansion takes the form:

$$\frac{\hat{d}_n}{d_n} = \frac{1}{1 + x_1\lambda +}\,\frac{1}{x_2\lambda +}\,\frac{1}{x_3\lambda +}\cdots\frac{1}{x_{2n}\lambda}$$

with all coefficients $x_i > 0$ if and only if $d_n$ only has roots in the region $\operatorname{Re}\lambda < 0$. Below, we provide the form of $r_{n_0}$ and the values of the coefficients in the continued fractions expansion.

**Case $l = 1, m = 1$:** In this case $r_2$ is given by

$$r_2(\lambda) = \frac{\lambda^6 + 60\lambda^5 + 1201\lambda^4 + 10152\lambda^3 + 37851\lambda^2 + 55580\lambda + 19635}{132\left(\lambda^4 + 32\lambda^3 + 266\lambda^2 + 592\lambda + 245\right)}$$

so that $d_2$ is a polynomial of degree 4. Here we find

$$x_1 = \frac{1}{32}, \quad x_2 = \frac{64}{495}, \quad x_3 = \frac{49005}{110944}, \quad x_4 = \frac{55472}{24255}$$

**Case $l = 1, m = 2$:** Here $r_4$ has the numerator given by

$$\lambda^{10} + 120\lambda^9 + 5655\lambda^8 + 138560\lambda^7 + 1969418\lambda^6 + 17090160\lambda^5$$
$$+92390286\lambda^4 + 310928256\lambda^3 + 641783397\lambda^2 + 787540056\lambda + 488363755$$

and denominator

$$d_4 = 260(\lambda^8 + 80\lambda^7 + 2356\lambda^6 + 33584\lambda^5 + 256238\lambda^4$$
$$+1088432\lambda^3 + 2600580\lambda^2 + 3504848\lambda + 2391129)$$

For the continued fractions expansion we find

$$x_1 = \frac{1}{80}, \quad x_2 = \frac{400}{9681}, \quad x_3 = \frac{13388823}{162909760}, \quad x_4 = \frac{16587243689536}{113962657805643}$$
$$x_5 = \frac{4753120429870333534188 7}{191285306692074662805504}, \quad x_6 = \frac{172337198359411247530042356 20352}{40133868683257984044230012780567}$$
$$x_7 = \frac{48411126941324704457689920984461825 44921}{64823461301871772375727010696692891 81184}$$
$$x_8 = \frac{141331078934075674653925376}{166370224608919186154542269}$$

**Case $l = 2, m = 1$:** In this case $r_2$ is given by

$$r_2(\lambda) = \frac{\lambda^6 + 72\lambda^5 + 1813\lambda^4 + 20400\lambda^3 + 108019\lambda^2 + 251784\lambda + 194103}{156\,(\lambda^4 + 40\lambda^3 + 458\lambda^2 + 1688\lambda + 1701)}$$

so that $d_2$ is a polynomial of degree 4. Here we find

$$x_1 = \frac{1}{40}, \quad x_2 = \frac{200}{2079}, \quad x_3 = \frac{22869}{83840}, \quad x_4 = \frac{16768}{18711}$$

**Case $l = 2, m = 2$:** In this case $r_3$ has the numerator

$$\lambda^8 + 96\lambda^7 + 3456\lambda^6 + 60912\lambda^5 + 574976\lambda^4$$
$$+2974208\lambda^3 + 8253120\lambda^2 + 11432704\lambda + 6432768$$

and denominator

$$d_3 = 208\left(\lambda^6 + 60\lambda^5 + 1216\lambda^4 + 10488\lambda^3 + 39936\lambda^2 + 65984\lambda + 40704\right)$$

Here we find the following coefficients for the expansion:

$$x_1 = \frac{1}{60}, \quad x_2 = \frac{150}{2603}, \quad x_3 = \frac{6775609}{53687060}, \quad x_4 = \frac{21617253085827}{80716560627608}$$
$$x_5 = \frac{300487895237088557 78}{514430039637433083 57}, \quad x_6 = \frac{6388007832023}{4930439162424}$$

**Case $l = 3, m = 3$:** Here we find that $r_3$ has numerator

$$\lambda^8 + 112\lambda^7 + 4804\lambda^6 + 103408\lambda^5 + 1229214\lambda^4$$
$$+8329808\lambda^3 + 31803380\lambda^2 + 63649104\lambda + 52369065$$

and denominator

$$d_3 = 240\left(\lambda^6 + 72\lambda^5 + 1813\lambda^4 + 20400\lambda^3 + 109011\lambda^2 + 272264\lambda + 260055\right)$$

Here we obtain

$$x_1 = \frac{1}{72}, \quad x_2 = \frac{216}{4589}, \quad x_3 = \frac{21058921}{212658048}, \quad x_4 = \frac{117769389008256}{605966643139039}$$
$$x_5 = \frac{17436580563514884792601}{45956710403723331293184}, \quad x_6 = \frac{27661586227277824}{34339650333737805}$$

**Case $l \geq 3, m = l-1$:** In this case $r_2$ has numerator

$$\lambda^5 + 38\lambda^4 + 440\lambda^3 + 1816\lambda^2 + 2096\lambda + 27l^5 + 81\lambda l^4 + 306l^4 + 90\lambda^2 l^3 + 804\lambda l^3 + 1224l^3$$
$$+46\lambda^3 l^2 + 744\lambda^2 l^2 + 2792\lambda l^2 + 2104l^2 + 11\lambda^4 l + 284\lambda^3 l + 2008\lambda^2 l + 3984\lambda l + 1264l$$

and denominator

$$d_2 = 12(2l+7)\left(\lambda^3 + 18\lambda^2 + 68\lambda + 9l^3 + 15\lambda l^2 + 46l^2 + 7\lambda^2 l + 64\lambda l + 52l\right)$$

so that $d_2$ is a polynomial of degree 3. Here we find

$$x_1 = \frac{1}{7l+18}, \quad x_2 = \frac{(7l+18)^2}{8\left(12l^3 + 84l^2 + 197l + 153\right)}, \quad x_3 = \frac{96l^3 + 672l^2 + 1576l + 1224}{(7l+18)\left(9l^3 + 46l^2 + 52l\right)}$$

**Case $l \geq 4, m = l$:** Here $r_2$ has numerator

$$\lambda^6 + 36\lambda^5 + 364\lambda^4 + 936\lambda^3 - 1536\lambda^2 - 4192\lambda + 27l^6 + 108\lambda l^5 + 360l^5 + 171\lambda^2 l^4$$
$$+1236\lambda l^4 + 1524l^4 + 136\lambda^3 l^3 + 1632\lambda^2 l^3 + 4424\lambda l^3 + 1944l^3 + 57\lambda^4 l^2 + 1032\lambda^3 l^2$$
$$+4704\lambda^2 l^2 + 4440\lambda l^2 - 1440l^2 + 12\lambda^5 l + 312\lambda^4 l + 2168\lambda^3 l + 3432\lambda^2 l - 3104\lambda l - 3360l$$

and denominator given by

$$d_2 = 12(2l+7)(\lambda^4 + 16\lambda^3 + 32\lambda^2 - 136\lambda + 9l^4 + 24\lambda l^3 + 52l^3 + 22\lambda^2 l^2$$
$$+112\lambda l^2 + 24l^2 + 8\lambda^3 l + 76\lambda^2 l + 56\lambda l - 120l)$$

In this case, $d_2$ is again a polynomial of degree four in $\lambda$ and we find

$$x_1 = \frac{1}{8l+16}, \quad x_2 = \frac{8(l+2)^2}{19l^3 + 106l^2 + 177l + 81},$$
$$x_3 = \frac{\left(19l^3 + 106l^2 + 177l + 81\right)^2}{8(l+2)\left(48l^6 + 496l^5 + 1880l^4 + 2956l^3 + 955l^2 - 1962l - 1377\right)},$$
$$x_4 = \frac{8\left(48l^6 + 496l^5 + 1880l^4 + 2956l^3 + 955l^2 - 1962l - 1377\right)}{l\left(9l^3 + 52l^2 + 24l - 120\right)\left(19l^3 + 106l^2 + 177l + 81\right)}$$

Note that even though it is not immediately apparent, one can easily see that the coefficients $x_3, x_4$ are positive for $l \geq 2$ by inserting $l+2$ for $l$ and expanding. Then all coefficients become positive.

**Case $l \geq 2, m = l + 1$:** Here $r_2$ has numerator

$$\lambda^6 + 36\lambda^5 + 376\lambda^4 + 1224\lambda^3 + 160\lambda^2 - 2112\lambda + 27l^6 + 108\lambda l^5 + 468l^5 + 171\lambda^2 l^4$$
$$+1524\lambda l^4 + 2832l^4 + 136\lambda^3 l^3 + 1896\lambda^2 l^3 + 7112\lambda l^3 + 7112l^3 + 57\lambda^4 l^2 + 1128\lambda^3 l^2$$
$$+6456\lambda^2 l^2 + 12120\lambda l^2 + 6464l^2 + 12\lambda^5 l + 324\lambda^4 l + 2552\lambda^3 l + 6616\lambda^2 l + 4256\lambda l + 576l$$

and denominator

$$d_2 = 12(2l + 7)(\lambda^4 + 16\lambda^3 + 40\lambda^2 - 72\lambda + 9l^4 + 24\lambda l^3 + 76l^3$$
$$+22\lambda^2 l^2 + 144\lambda l^2 + 144l^2 + 8\lambda^3 l + 84\lambda^2 l + 152\lambda l - 24l)$$

For the continued fraction expansion of $d_2$ we find

$$x_1 = \frac{1}{8l + 16}, \quad x_2 = \frac{8(l + 2)^2}{19l^3 + 110l^2 + 189l + 89},$$
$$x_3 = \frac{\left(19l^3 + 110l^2 + 189l + 89\right)^2}{8(l + 2)\left(48l^6 + 560l^5 + 2424l^4 + 4732l^3 + 3723l^2 + 86l - 801\right)},$$
$$x_4 = \frac{8\left(48l^6 + 560l^5 + 2424l^4 + 4732l^3 + 3723l^2 + 86l - 801\right)}{l\left(9l^3 + 76l^2 + 144l - 24\right)\left(19l^3 + 110l^2 + 189l + 89\right)}$$

Here it is again not immediately apparent that $x_3$ and $x_4$ are positive. This can be remedied by shifting $l$ to $l + 1$. Then all coefficients become positive.

## A.3 Bounds for $|a_n|, |b_n|$ and $|e_{n_0}|$

As discussed above, the proof of the bounds for the coefficients $a_n$ and $b_n$ controlling the evolution of the relative error and the bounds for the initial value of the relative error $e_{n_0}$ boils down to a lengthy but straightforward computation. In order to increase accessibility, we provide the results of these computations together with the explicit form of all the necessary coefficients in a digital format in a Github repository [34]. Reproducing the calculations in text only would serve almost no purpose and hinder the ability of the reader to check the correctness of the author's results more than anything. Here we describe the organization and contents of the online repository. Aside from a `README.txt` file, the repository contains the ten files

`11.txt`, `12.txt`, `21.txt`, `22.txt`, `23.txt`, `32.txt`, `33.txt`, `l1.txt`, `l2.txt` and `l3.txt`

The files `l1.txt`, `l2.txt` and `l3.txt` correspond to the cases $(l, m) = (l, l-1)$ with $l \geq 4$, $(l, m) = (l, l)$ with $l \geq 4$, respectively $(l, m) = (l, l + 1)$ with $l \geq 3$. The other files named in the form `lm.txt` correspond to the case $(l, m)$ given in the name.

The meaning of the variables $l, n$ and $t$ occuring in the files is in principle the same as here, with the exception that sometimes the values of $l$ and $n$ are shifted, see the section below for when this happens. The mode variable $\lambda$ has been replaced for practical reasons with the variable $x$ in the files. Each file is structured in the same way. The following shows the structure of each file, where instead of the actual values we give a description of what each line contains on the right side of the equality sign.

`A=` Coefficient $A_n$ as defined in (60)

`B=` Coefficient $B_n$ as defined in (61)

`n0=` Value of $n_0$, the point at which we start the induction argument

`r_`$n_0$`=` Explicit form of $r_{n_0}$

`rtilde=` Explicit form of the approximation $\tilde{r}_n$ from section 2.6.2

`a=` Explicit form of the coefficient $a_n$ defined in (63)

`b=` Explicit form of the coefficient $b_n$ defined in (63)

`esta=` Contains the result of the computation necessary to prove the claimed bound for $|a_n|$ on the imaginary line. We recall from above: One finds that $|a_n|^2(it) = \frac{F(t^2)}{G(t^2)}$ for certain polynomials $F$ and $G$. Every coefficient of a power of $t^2$ is again a polynomial in $n$ and $l$. To show the bound $|a_n|(it) \leq \frac{x}{y}$ we compute the polynomial $Fy^2 - Gx^2$. In this polynomial, we always shift $n \mapsto n + n_0$. In the files `l1.txt`, `l2.txt` respectively `l3.txt` where also $l$ appears as a variable, we also shift $l \mapsto l + l_0$ where $l_0 = 4, 4, 3$ respectively. The result of this computation is then provided as `esta`. Since all non-zero coefficients are negative integers, this demonstrates the validity of the bound.

`estb=` Contains the result of the computation necessary to prove the claimed bound for $|b_n|$ on the imaginary line. The same remarks apply as for `esta` above.

`esterror=` Contains the result of the computation necessary to prove the claimed bound for $|e_{n_0}|$. The procedure is the same as above, with the difference that now $n$ does not appear as a variable. In the file `l1.txt` we do not shift the value of $l$, in `l2.txt` we shift $l \mapsto l + 4$ and in `l3.txt` we shift $l \mapsto l + 2$. In the other files all coefficients are already numbers. Again, all non-zero coefficients are negative integers except in the case $(l, m) = (1, 2)$, see the discussion below.

Thus, by consulting the provided files, the reader may verify the validity of the claimed bounds on the coefficients and the initial value of the error. As remarked above, in every computation except one, the negativity of the polynomial is immediately clear from the fact that every non-zero coefficient is a negative integer. In the case $(l, m) = (1, 2)$ the error estimate gives us the polynomial

$$\begin{aligned}\texttt{esterror} = &-32225063143731938775 + 12297032267828497 04t^2\\ &- 15043717515054126 69t^4 - 384997689421455888t^6\\ &- 27845993942231878t^8 - 896965123990016t^{10}\\ &- 13901512438618t^{12} - 99535465456t^{14}\\ &- 299917523t^{16} - 340648t^{18} - 121t^{20}\end{aligned}$$

so that the negativity is not immediately apparent. However, it can be easily seen that when $|t| \leq 5$, the sum of the first two terms is negative. Thus we may assume that $|t| > 5$. Define $x = t^2$ and write the polynomial as a function of $x$. Then shifting the variable

$x \mapsto x + 25$ and expanding the polynomial one arrives at

$$\begin{aligned} &-121x^{10} - 370898x^9 - 379966448x^8 - 167410425056x^7 - 37025801827168x^6 \\ &-4568086848705216x^5 - 333353001325840128x^4 - 14651780151181618688x^3 \\ &-378945095134180006144x^2 - 5287882315793456042496x - 30642585141535849676800 \end{aligned}$$

which is evidently negative when $x \geq 0$, i.e. when $|t| \geq 5$.

# B Explicit form of generators

Here we provide the explicit form of the rotation and Lorentz boost matrices used in the computations in Chapter 2.1. We fix a set of generators of the Lie algebra so(3) here:

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The generators of the Lie algebra so(4) we choose are the following

$$\mathbf{F}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{F}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{F}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{F}_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{F}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \mathbf{F}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The Lorentz boosts are given by

$$\Lambda_1(\alpha) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda_2(\alpha) = \begin{pmatrix} \cosh\alpha & 0 & \sinh\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\alpha & 0 & \cosh\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_3(\alpha) = \begin{pmatrix} \cosh\alpha & 0 & 0 & -\sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}$$

## References


[1] Sergiu Klainerman and Sigmund Selberg. Remark on the optimal regularity for equations of wave maps type. *Communications in Partial Differential Equations*, 22(5-6):99–133, 1997.

[2] Sergiu Klainerman and Matei Machedon. Smoothing estimates for null forms and applications. *Duke Math. J*, 81(1):99–133, 1996.

[3] S. Klainerman and M. MacHedon. Space-time estimates for null forms and the local existence theorem. *Communications on Pure and Applied Mathematics*, 46(9):1221–1268, 1993.

[4] Sergiu Klainerman and Matei Machedon. On the regularity properties of a model problem related to wave maps. *Duke Math. J.*, 87(3):553–589, 04 1997.

[5] Daniel Tataru. Local and global results for wave maps i. *Communications in Partial Differential Equations*, 23(9-10):1781–1793, 1998.

[6] Daniel Tataru. On global existence and scattering for the wave maps equation. *American Journal of Mathematics*, 123(1):37–77, 2001.

[7] Terence Tao. Multilinear weighted convolution of $L^2$ functions, and applications to non-linear dispersive equations. *arXiv Mathematics e-prints*, page math/0005001, Apr 2000.

[8] Terence Tao. Global regularity of wave maps ii: Small energy in two dimensions. *Communications in Mathematical Physics*, 224(2):443–544, 2001.

[9] Sergiu Klainerman and Igor Rodnianski. On the global regularity of wave maps in the critical sobolev norm. *International Mathematics Research Notices*, 2001(13):655–677, 2001.

[10] Andrea Nahmod, Atanas Stefanov, and Karen Uhlenbeck. On the well-posedness of the wave map problem in high dimensions. *arXiv preprint math/0109212*, 2001.

[11] Jalal Shatah and Michael Struwe. The cauchy problem for wave maps. *International Mathematics Research Notices*, 2002(11):555–571, 2002.

[12] Joachim Krieger. Global regularity of wave maps from r 3+ 1 to surfaces. *Communications in mathematical physics*, 238(1-2):333–366, 2003.

[13] Joachim Krieger. Global regularity of wave maps from r 2+ 1 to h 2. small energy. *Communications in mathematical physics*, 250(3):507–580, 2004.

[14] Daniel Tataru. Rough solutions for the wave maps equation. *American Journal of Mathematics*, 127(2):293–377, 2005.

[15] Joachim Krieger. Global regularity and singularity development for wave maps. *Surveys in differential geometry*, XII:167–201, 2007.

[16] Jalal Shatah. Weak solutions and development of singularities of the su (2) $\sigma$-model. *Communications on pure and applied mathematics*, 41(4):459–469, 1988.

[17] Neil Turok and David Spergel. Global texture and the microwave background. *Physical Review Letters*, 64(23):2736, 1990.

[18] Piotr Bizon, Tadeusz Chmaj, and Zbislaw Tabor. Dispersion and collapse of wave maps. *Nonlinearity*, 13(4):1411–1423, Jul 2000.

[19] Ovidiu Costin, Roland Donninger, and Irfan Glogić. Mode stability of self-similar wave maps in higher dimensions. *Communications in Mathematical Physics*, 351(3):959–972, 2017.

[20] Irfan Glogic. *On the existence and stability of self-similar blowup in nonlinear wave equations*. PhD thesis, The Ohio State University, 2018.

[21] Roland Donninger, Birgit Schörkhuber, and Peter C. Aichelburg. On stable self-similar blow up for equivariant wave maps: The linearized problem. *Annales Henri Poincaré*, 13(1):103–144, Feb 2012.

[22] Roland Donninger. On stable self-similar blowup for equivariant wave maps. *Communications on Pure and Applied Mathematics*, 64(8):1095–1147.

[23] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[24] Brian C Hall. *Quantum theory for mathematicians*, volume 267. Springer, 2013.

[25] M. et al. Tanabashi. Review of particle physics. *Phys. Rev. D*, 98:030001, Aug 2018.

[26] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140. American Mathematical Soc., 2012.

[27] Frank WJ Olver, Daniel W Lozier, Ronald F Boisvert, and Charles W Clark. *NIST handbook of mathematical functions hardback and CD-ROM*. Cambridge university press, 2010.

[28] George E Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71. Cambridge university press, 1999.

[29] Ovidiu Costin, Roland Donninger, and Xiaoyue Xia. A proof for the mode stability of a self-similar wave map. *Nonlinearity*, 29(8):2451, 2016.

[30] Ovidiu Costin, Roland Donninger, Irfan Glogić, and Min Huang. On the stability of self-similar solutions to nonlinear wave equations. *Communications in Mathematical Physics*, 343(1):299–310, 2016.

[31] Irfan Glogić and Birgit Schörkhuber. Co-dimension one stable blowup for the supercritical cubic wave equation. *arXiv preprint arXiv:1810.07681*, 2018.

[32] Roland Donninger and Irfan Glogić. On the existence and stability of blowup for wave maps into a negatively curved target. *Analysis & PDE*, 12(2):389–416, 2018.

[33] S. Elaydi. *An Introduction to Difference Equations*. Undergraduate Texts in Mathematics. Springer New York, 2005.

[34] Max Weissenbacher. Github repository "wavemaps-modestability". https://github.com/maxweissenbacher/wavemaps-modestability, 2019.

[35] Terence Tao. *Nonlinear dispersive equations: local and global analysis*. Number 106. American Mathematical Soc., 2006.

[36] HS Wall. Polynomials whose zeros have negative real parts. *The American Mathematical Monthly*, 52(6):308–322, 1945.