Abstracts

A geometric R-matrix for the Hilbert scheme of points on a general surface

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In [4], it is explained how to use the representation theory of quantum groups to describe structures in the enumerative geometry of Nakajima quiver varieties. A key ingredient is an R-matrix acting in equivariant cohomology, constructed geometrically using correspondences called stable envelopes. The R-matrix gives rise to a quantum group called the Yangian; it is shown in [4] that this Yangian's Baxter subalgebras can be identified with the operators of quantum multiplication by tautological divisors in the corresponding quiver variety.

In this project, we extend a portion of this package beyond the setting of Nakajima quiver varieties. The stable envelope construction of the R-matrix in [4] requires that the underlying variety carries both a symplectic form and a large automorphism group, which need not be the case for a general variety. However, for the Hilbert scheme of points on $\mathbb{C}^2$, an alternative construction of the R-matrix is given in [4, Ch. 13] using the language of conformal field theory. We adapt this construction to produce R-matrices for the Hilbert scheme of points on a general surface. We also show that classical multiplication by the tautological divisor in the Hilbert scheme coincides with a Baxter subalgebra of the associated Yangian.

1. Construction

Let S be a surface that is either proper, or admits an action of a non-trivial torus T such that the fixed locus S^T is proper. We then equip $H_T^*(S)$ with the structure of a Frobenius algebra over $\mathrm{Frac}(H_T^*(\mathrm{pt}))$, with trace given by $\epsilon(\gamma) := -\int_S \gamma$.

Let $\mathrm{Hilb}(S) = \sqcup_{n\geq 0}\mathrm{Hilb}_n(S)$ and let $\alpha_m(\gamma)$, for $m \in \mathbb{Z}$ and $\gamma \in H_T^*(S)$, be the generators of the Heisenberg algebra of [2] and [5] acting on $H_T^*(\mathrm{Hilb}(S))$. These operators are given by correspondences and satisfy the supercommutator relations

$$[\alpha_n(\gamma), \alpha_{n'}(\gamma')] = \delta_{n+n'} n\epsilon(\gamma\gamma').$$

Let F_S denote the Heisenberg module $H_T^*(\mathrm{Hilb}(S))$. We regard F_S a Fock space with lowest weight vector $|\varnothing\rangle \in H^0(\mathrm{Hilb}_0 S)$.

As defined, the Heisenberg action leaves the action of the "zero modes" $\alpha_0(\gamma)$ ambiguous. Taking advantage of this ambiguity, we introduce a formal parameter u, we let $F_S(u)$ denote $F_S \otimes \mathbb{C}(u)$ where $\alpha_0(\gamma)$ scales F_S by $-u\epsilon(\gamma)$.

The desired R-matrix will be defined in terms of the modified generators

$$\alpha_n^-(\gamma) := \frac{1}{\sqrt{2}}(\alpha_n(\gamma)\otimes 1 - 1 \otimes \alpha_n(\gamma))$$

acting in $F_S(u_1)\otimes F_S(u_2) := F(S)^{\otimes 2} \otimes \mathbb{C}(u_1, u_2)$ and will be a function of $u_1 - u_2$.

We use the Feigin-Fuchs construction to obtain a Virasoro algebra action on $F_S(u_1) \otimes F_S(u_2)$. Adjoin a parameter κ to the ground field. Then, if $\Delta\gamma = \sum_i \gamma_i^{(1)} \otimes \gamma_i^{(2)}$, set

$$L_n(\gamma, \kappa) = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_i : \alpha_m^-(\gamma_i^{(1)}) \alpha_{n-m}^-(\gamma_i^{(2)}) : - \frac{\kappa}{\sqrt{2}} n \alpha_n^-(\gamma) - \delta_n \frac{\kappa^2}{4} \epsilon(\gamma).$$

By [3, Thm 3.3], the operators $L_n(\gamma, \kappa)$ satisfy the following Virasoro relation

$$[L_n(\gamma, \kappa), L_{n'}(\gamma', \kappa)] = (n - n') L_{n+n'}(\gamma\gamma', \kappa) + \delta_{n+n'} \frac{n^3 - n}{12} \epsilon(\gamma\gamma'(c_2(S) - 6\kappa^2)). \tag{1}$$

For generic κ and $u_1 - u_2$, the Virasoro operators applied to $|\varnothing\rangle \otimes |\varnothing\rangle \in F_S(u_1) \otimes F_S(u_2)$ generate an irreducible lowest weight module with

$$L_0(\gamma, \kappa) |\varnothing\rangle \otimes |\varnothing\rangle = \frac{1}{4}(2(u_2 - u_1)^2 - \kappa^2) \epsilon(\gamma) |\varnothing\rangle \otimes |\varnothing\rangle. \tag{2}$$

The argument of ϵ in (1) plays the role of the central charge, while the scalar in (2) plays the role of lowest weight. Observe that both of these quantities are even functions of κ. We may therefore define $R(u_1 - u_2)$ to be the unique operator in $F_S(u_1) \otimes F_S(u_2)$ which fixes the vacuum $|\varnothing\rangle \otimes |\varnothing\rangle$, and satisfies

$$R(u_1 - u_2) L_n(\gamma, \kappa) = L_n(\gamma, -\kappa) R(u_1 - u_2),$$
$$R(u_1 - u_2)(\alpha_n(\gamma) \otimes 1 + 1 \otimes \alpha_n(\gamma)) = (\alpha_n(\gamma) \otimes 1 + 1 \otimes \alpha_n(\gamma)) R(u_1 - u_2)$$

for all n and γ.

2. Results

Theorem 1. *The operator $R(u)$ satisfies the Yang-Baxter equation with spectral parameter.*

This result is proved for $S = \mathbb{C}^2$ in [4, Thm. 14.3.1]; the proof for general S uses this special case. The quantum inverse scattering method then produces a Yangian Y_S with an action on $\oplus_i F_S^{\otimes i}$.

The matrix elements of $R(u)$ also encode multiplication by Chern classes of the tautological bundle.

Theorem 2. *For $n \geq 0$, let $x_1, \ldots, x_n$ be the Chern roots of the tautological bundle $\mathcal{O}^{[n]}$ on $\mathrm{Hilb}_n(S)$. Then, the vacuum matrix element*

$$|\varnothing\rangle \otimes H_T^*(\mathrm{Hilb}_n(S)) \to |\varnothing\rangle \otimes H_T^*(\mathrm{Hilb}_n(S))$$

of the normalized operator $R(u/\sqrt{2})$ is equal to multiplication by

$$\prod_{i=1}^n \frac{u - x_i}{u - \kappa - x_i}.$$

Given a non-trivial line bundle $\mathcal{L}$ on S, one can modify the action of the zero modes in the construction of $R(u)$ to a produce a new operator which does not solve the Yang-Baxter equation, but does satisfy an analog of Theorem 2 where $\mathcal{O}^{[n]}$ is replaced by $\mathcal{L}^{[n]}$.

3. Open questions

(1) What portion of the quantum cohomology of $\mathrm{Hilb}(S)$ is controlled by the Yangian Y_S? This question seems most tractable when S is a K3 surface.

(2) Taking inspiration from the level of generality of [1], note that the constructions in Section 1.1 can still be carried out if $H_T^*(S)$ is replaced by the cohomology of a higher-dimensional variety or, more generally, a graded supercommutative Frobenius algebra A. Does Theorem 1.1 still hold in this more general setting? Either an affirmative or a negative answer would be interesting.

References

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