LTCC ADVANCED COURSE EXAM: BEILINSON'S CONJECTURES

1) Use the analytic class number formula to verify that the class number of $\mathbb{Q}(i)$ is 1.

2) In this question, let $G \approx 0.915965594177219015054\ldots$ denote Catalan's constant.
   (a) Let $D : \mathbb{C}\backslash\{0,1\} \to \mathbb{R}$ be the Bloch-Wigner function. Show that
   $$D(i) = G\,.$$
   (b) Let $\zeta_{\mathbb{Q}(i)}(s)$ be the Dedekind zeta function of $\mathbb{Q}(i)$. Show that
   $$\zeta_{\mathbb{Q}(i)}(2) = \frac{\pi^2 G}{6}\,.$$

3) Let L/K be a finite extension of fields. Consider the pullback map
   $$\mathrm{CH}^i(K,n) \to \mathrm{CH}^i(L,n)\,.$$
   (a) Show that the pullback map is injective after tensoring with $\mathbb{Q}$.

   (b) Give an example where the pullback map is not injective integrally.

4) Let k be a field. Compute the motivic cohomology $H^i_{\mathcal{M}}(\mathbb{P}^m_k, \mathbb{Z}(n))$ of projective m-space $\mathbb{P}^m_k$.

5) Let E be an elliptic curve over $\mathbb{Q}$. Show that the Beilinson conjectures predict $H^3_{\mathcal{M}/\mathbb{Z}}(E, \mathbb{Q}(2))$ to be trivial.

6) Show that
   $$\xi := 2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right]$$
   is a non-trivial element in the Bloch group $\mathcal{B}(\mathbb{Q}(\sqrt{-7}))$ of $\mathbb{Q}(\sqrt{-7})$.

7) Let X be a smooth projective irreducible curve over $\mathbb{C}$. In this question I shall write down an explicit expression for the Beilinson regulator
   $$r_{\mathcal{B}} : H^2_{\mathcal{M}}(C, \mathbb{Q}(2)) \to H^2_{\mathcal{D}}(C, \mathbb{R}(2))\,.$$
   It turns out that $H^2_{\mathcal{M}}(C, \mathbb{Z}(2))$ is a subgroup of
   $$K_2(\mathbb{C}(C)) := \frac{\mathbb{C}(C)^* \otimes_{\mathbb{Z}} \mathbb{C}(C)^*}{\langle f \otimes (1-f) \,|\, f \in \mathbb{C}(C)^* \rangle}\,.$$
   Given two invertible meromorphic functions $f, g \in \mathbb{C}(C)^*$ on C, let $\{f, g\} \in K_2(\mathbb{C}(C))$ denote the class of $f \otimes g$. Let $S_{f,g} := \mathrm{supp}(\mathrm{div}(f)) \cup \mathrm{supp}(\mathrm{div}(g))$

denote the set of zeros and poles of either f or g. Define a real-analytic 1-form on $X \setminus S_{f,g}$ by

$$\eta(f,g) := \log|f| d\arg(g) - \log|g| d\arg(f)$$

where $d\arg(h) := \mathrm{Im}(d\log(h))$.

(a) Show that $\eta(f,g)$ is closed.
(b) Show that $\eta(f, 1-f)$ is exact (Hint: compute $dD(f)$ where $D(z)$ is the Bloch-Wigner function.)
(c) Let $z \mapsto \overline{z}$ be complex conjugation. Show that if f and g are such that $f(\overline{z}) = \overline{f(z)}$ and $g(\overline{z}) = \overline{g(z)}$, then $\eta(f,g) \mapsto -\eta(f,g)$ under the action of complex conjugation on C.

Given an element $\alpha = \displaystyle\sum_{i=1}^{n} \{f_i, g_i\} \in K_2(\mathbb{C}(C))$, we consider the real-analytic 1-form

$$\eta(\alpha) := \sum_{i=1}^{n} \eta(f_i, g_i)\,.$$

Then $\eta(\alpha)$ is defined on $C' := C \setminus \displaystyle\bigcup_{i=1}^{n} S_{f_i,g_i}$, which is C minus a finite set of points. It turns out that if $\alpha \in H^2_{\mathcal{M}}(C, \mathbb{Z}(2))$ then $\eta(\alpha)$ has trivial residue at every point of C, and hence the class $[\eta(\alpha)] \in H^1(C', \mathbb{R})$ extends to a class $[\eta(\alpha)] \in H^1(C, \mathbb{R})$. This class does not depend on the decomposition $\alpha = \displaystyle\sum_{i=1}^{n} \{f_i, g_i\} \in K_2(\mathbb{C}(C))$ because of (a) and (b). The Beilinson regulator is then

$$\begin{array}{rcl} r_{\mathcal{B}} : H^2_{\mathcal{M}}(C, \mathbb{Q}(2)) & \to & H^2_{\mathcal{D}}(C, \mathbb{R}(2)) \cong H^1(C, \mathbb{R}) \\ \alpha & \mapsto & [i \cdot \eta(\alpha)] \end{array}$$

Note that if C is defined over a number field, then

$$r_{\mathcal{B}} : H^2_{\mathcal{M}}(C, \mathbb{Q}(2)) \xrightarrow{\text{base change}} H^2_{\mathcal{M}}(C_{\mathbb{C}}, \mathbb{Q}(2)) \xrightarrow{\alpha \mapsto [i\cdot\eta(\alpha)]} H^2_{\mathcal{D}}(C_{\mathbb{C}}, \mathbb{R}(2))$$

has image in $H^2_{\mathcal{D}}(C_{\mathbb{R}}, \mathbb{R}(2)) := H^2_{\mathcal{D}}(C_{\mathbb{C}}, \mathbb{R}(2))^+$ by (c), as desired.