Lecture 1: Symplectic resolutions and singularities.

General information:

Lecturer: Travis Schedler, Imperial

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Course Meetings: 8 2-hour sessions, Thursdays 4-6PM beginning 17 Jan 2019

Office Hours: 30 minutes after lecture; let me know if you want to skype in.

Course webpage + Online forum: www.imperial.ac.uk/people/t.schedler/page/courses.html

Coursework: Assigned weekly. Starred exercises will be collected 8 Feb + 8 Mar.

Assessment: Let me know if you require it. It will be based on coursework.

Prerequisites: Familiarity with algebraic geometry (affine, projective, schemes, sheaves), category theory (definition of abelian category, functor, natural transformation...), homological algebra (Ext + Tor); {derived categories helpful.}
{D-modules helpful (concurrent course offered)
Representation theory of semisimple Lie algebras helpful} Some background will be given in course.
Representation theory of finite groups (Maschke's Theorem)

Some references (more to be given later):

- Hotta, Takeuchi, and Tanisaki "D-modules...", Chapters 1-3, 9-11
- Gaitsgory, Geometric Rep. Theory notes (google it), sections 5+6

Fu, A survey on symplectic singularities and resolutions

Kaledin, Geometry and topology of symplectic resolutions

Braden, Licata, Proudfoot, Webster, Quantizations of conical symplectic...

Schedler, Deformations of algebras in noncommutative geometry

Background: Hartshorne, Fulton+Harris, Humphreys (Lie Algebras and Linear Algebraic Groups), Bellamy's notes on Lie algebras, MacLane (category theory), Weibel (homological algebra).

Rough outline of subject:

Algebraic geometry is about the interplay (or duality) between geometry and commutative algebra.

Geometric representation theory is about the application of geometric constructions to representation theory (of groups, Lie algebras, and noncommutative algebras).

Geometry $\xrightarrow{\text{quantization}}$	Algebra
T^*X	$\mathcal{D}_X$ (-modules)
$T^*(G/B)$ flag variety ($G = SL_n$: ordinary flag variety)	Reps of $\mathfrak{g}$ = semisimple Lie algebra ($\mathfrak{g} = \mathrm{Lie}\,G$)
$\cup$ Kostant–Slodowy slices	finite W-algebras
$\mathbb{C}^{2n}/\Gamma$, $\Gamma < Sp_{2n}$ finite	(Spherical) symplectic reflection algebras [or Cherednik algebras]
Higgs branch $\cup$ Quiver varieties $\subseteq$ $\cup$ hypertoric varieties; Hamiltonian (symplectic) reductions $T^*\mathbb{C}^n /\!/\!/ G$ (or $T^*X /\!/\!/ G$)	G-equivariant $\mathcal{D}_X$-modules, quantized quiver + hypertoric varieties

First main theorem (explained today):

Beilinson–Bernstein localization thm (BB): Have an equivalence

$$\mathcal{D}^{\lambda}_{G/B}\text{-mod} \xrightarrow[\Gamma \text{ (global sections)}]{\sim} \mathfrak{g}\text{-mods with appropriate action of } \underset{\text{center alg}}{Z}(U\mathfrak{g}),$$

for suitable $\lambda \in \mathfrak{h}^*$ ($\mathfrak{h}$ = Cartan subalg of $\mathfrak{g}$).

$\Rightarrow$ can understand rep theory of $\mathfrak{g}$ *locally* along the flag variety $G/\underset{\text{Borel subgroup.}}{B}$. (for $G = SL_n$: $G/B = \{0 \subseteq V_1 \subseteq \dots \subseteq V_n = \mathbb{C}^n : \dim V_i = i\}$)

Recall: Defn A representation ("rep") V of a (Lie or algebraic) group G is a (suitable) homomorphism $\rho: G \to GL(V)$.

Defn A representation V of a Lie algebra $\mathfrak{g}$ is a homomorphism $\rho: \mathfrak{g} \to \mathfrak{gl}(V) = End(V)$

Defn A rep V of an associative algebra A is a homomorphism $\rho: A \to End(V)$.

The class of representations of G, $\mathfrak{g}$, or A form an abelian category (= category with $\oplus$, sum of morphisms, and kernels + cokernels, satisfying axioms.)

Defn: Given $\mathfrak{g}$, $U\mathfrak{g} := T\mathfrak{g} / (x \otimes y - y \otimes x - [x,y])$ "universal enveloping algebra"

$T\mathfrak{g} = \bigoplus_{m \geq 0} \mathfrak{g}^{\otimes m}$ "tensor algebra"

$\Rightarrow$ Get a functor U: Lie algebras $\to$ Associative algebras

Rmk: Left adjoint to Associative algs $\to$ Lie, $A \mapsto (A, [\,,\,])$.

Induces a functor $\mathrm{Rep}\, \mathfrak{g} \longrightarrow \mathrm{Rep}\, U\mathfrak{g}$

$$(V, \rho) \longmapsto (V, \tilde{\rho}),$$

$\tilde{\rho} : U\mathfrak{g} \to \mathrm{End}\, V$ is the unique extension of $\rho : \mathfrak{g} \to \mathrm{End}\, V$ to an algebra homomorphism.

<u>Exercise</u>: This is an equivalence, with inverse

$$\rho \longmapsto \rho|_{\mathfrak{g}} .$$

$\therefore$ We can embed Lie algebra reps into associative algebra reps.

<u>Analogue for finite groups</u>: Recall: $\mathrm{Rep}\, G \longrightarrow \mathrm{Rep}\, \mathbb{C}[G]$ is an equivalence $\Rightarrow$ reps of finite groups $\hookrightarrow$ reps of assoc. algebras.

For Lie/algebraic groups, the situation is more complicated.

Representation theory of semisimple Lie algebras.

References include: Humphreys, Fulton + Harris.

I will just recall the basic outline.

<u>Example</u>: $\mathfrak{sl}_2 := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} \supseteq \mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right\}$ "a Borel subalgebra"

$$\supseteq \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}] = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\}.$$

$\mathfrak{h} := \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\}$, a "Cartan subalgebra".

<u>Theorem</u> $\exists!$ irrep of $\mathfrak{sl}_2$ up to $\cong$ of each dimension $n \geq 1$, V_n, characterized by: $V_n \underset{\mathfrak{h}}{=} \mathbb{C}_{1-n} \oplus \mathbb{C}_{3-n} \oplus \cdots \oplus \mathbb{C}_{n-3} \oplus \mathbb{C}_{n-1}$

where $\mathbb{C}_k$ is the 1-dim rep of $\mathfrak{h}$ with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acting by $k \cdot \mathrm{Id} = (k)$.

Geometric description: $V_n = \mathbb{C}[x,y]_{n-1}$, the space of homogeneous polynomials of degree $n-1$ in x, y;
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}_2$ acts by $ax\partial_x + bx\partial_y + cy\partial_x + dy\partial_y$
(usual action of $\mathfrak{gl}_2$ on $\mathbb{C}^2 = \langle x, y\rangle$, extended to be a derivation.)

Relation to $\mathbb{P}^1$: $V_n = \Gamma(\mathbb{P}^1, \mathcal{O}(n))$, action of $GL_2 \supseteq SL_2$ given by $GL_2 \longrightarrow \operatorname{Aut} \mathbb{P}^1 \cong PGL_2$. (More precisely, this gives V_n an action of GL_2 since $\mathcal{O}(n)$ is GL_2-equivariant.)

General case: $\mathfrak{g}$ = semisimple Lie algebra / $\mathbb{C}$
(a direct sum of simple (nonabelian) Lie algebras)
$\mathfrak{b}$ = a maximal solvable subalgebra (="Borel subalgebra")
(solvable $:= \exists\, m \geq 1$, $\mathfrak{b}^{(m)} = 0$, where $\mathfrak{b}^{(k+1)} = [\mathfrak{b}^{(k)}, \mathfrak{b}^{(k)}]$, $\mathfrak{b}^{(1)} := \mathfrak{b}$)
$\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{h} \subseteq \mathfrak{b}$ "Cartan subalgebra",
can take $\mathfrak{h}$ = any abelian subalgebra such that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

Example: $\mathfrak{g} = \mathfrak{sl}_n = \{\text{trace zero } n\times n \text{ matrices}\} \supseteq \mathfrak{b} = \{\text{upper triangular trace zero matrices}\}$
$\supseteq \mathfrak{h} = \{\text{diagonal trace zero matrices}\}$.

Theorem 1) $\mathfrak{h}$ acts semisimply on any finite-dimensional $\mathfrak{g}$-representation (V, ρ), i.e., $(V,\rho)|_{\mathfrak{h}} \cong \bigoplus_{\lambda \in I_V \subseteq \mathfrak{h}^*} (\mathbb{C}_\lambda^{r_\lambda}, \rho_\lambda)$ for
$I_V \subseteq \mathfrak{h}^*$ a finite subset, $r_\lambda \geq 1$ for $\lambda \in I_V$,
and $\mathbb{C}_\lambda$ = the one-dimensional representation where
$\rho_\lambda(x) = (\lambda(x)) \in \operatorname{End}(\mathbb{C}_\lambda)$ for $x \in \mathfrak{h}$.

(2) If V is irreducible, then $\exists! \lambda \in I_V$ s.t. V is generated, over $\mathfrak{b}$, by some v_λ, $\rho(x)(v_\lambda) = \lambda(x) v_\lambda$ for $x \in \mathfrak{h}$. Then $v_\lambda = 1$. λ is called the "highest weight" of V. This produces an injection

$$\left\{ \begin{array}{c} \text{irreducible finite-dim reps} \\ \text{of } \mathfrak{g} \end{array} \right\} \Big/ \sim \hookrightarrow \mathfrak{h}^*.$$

Image = "dominant integral weights."

(3) All finite-dimensional representations are direct sums of irreducible representations (uniquely up to isomorphism).

Note: there is a construction, given $\lambda \in \mathfrak{h}^*$, of an irreducible $\mathfrak{g}$-representation of highest weight λ. It is unique up to isomorphism. Call it V_λ.

Geometric construction: G = semisimple algebraic (or $\mathbb{C}$ Lie) group, $B < G$ Borel (maximal solvable subgroup), $T < B$ maximal torus (subgroup isomorphic to $(\mathbb{C}^\times)^m$, $m \geq 0$). Then $\mathfrak{h} := \operatorname{Lie} T < \mathfrak{b} := \operatorname{Lie} B < \mathfrak{g} = \operatorname{Lie} G$ are Cartan and Borel subalgebras. Moreover $T \hookrightarrow B \twoheadrightarrow B/[B,B]$ is an isomorphism.

Let $\lambda \in \operatorname{Hom}_{\text{group}}(T, \mathbb{C}^\times) \subseteq \mathfrak{h}^* \Rightarrow \mathbb{C}_\lambda$ is a rep of T.

Define the line bundle $\mathcal{O}(\lambda) = G \times_B \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is viewed as a B-representation with trivial $[B,B]$ action (as $B/[B,B] \cong T$).

Thm (Borel–Weil): $\Gamma(G/B, \mathcal{O}(\lambda))$ is either 0 or an irreducible G-representation of highest weight λ (V_λ) (the latter if and only if λ is dominant integral).

$\Rightarrow$ ALL finite-dimensional reps of G are realized by vector bundles on G/B!

Remark: Borel-Weil-Bott theorem extends this by computing $H^i(G/B, \mathcal{O}(\lambda))$ $\forall \lambda \in \mathrm{Hom}(T, \mathbb{C}^\times)$.
(it turns out to be concentrated in a single degree, where it is irreducible.)

This establishes a close link between the geometry of the flag variety G/B and the representation theory of $\mathfrak{g}$. But this only concerns finite-dimensional reps for now.

To go deeper, observe that $\mathfrak{g} \subseteq \mathrm{Vect}(G/B) = \Gamma(G/B, T_{G/B})$.

So if we had a sheaf on G/B on which vector fields act, we'd get a representation of $\mathfrak{g}$.

But vector fields are first-order differential operators. So a module over the sheaf $\mathcal{D}_{G/B}$ of differential operators on G/B is a $\mathfrak{g}$-module.

The following theorem makes this precise:

Let $\mathcal{D}^\lambda_{G/B} := \mathrm{Diff}(\mathcal{O}(\lambda))$, differential operators $\mathcal{O}(\lambda) \to \mathcal{O}(\lambda)$,
so $\mathcal{D}^0_{G/B} = \mathcal{D}_{G/B}$, and $\mathcal{D}^\lambda_{G/B}$ acts on $\mathcal{O}(\lambda)$.

(This can be extended to define $\mathcal{D}^\lambda_{G/B}$, $\forall \lambda \in \mathfrak{h}^*$ even though $\mathcal{O}(\lambda)$ itself only makes sense for $\lambda \in \underbrace{\mathrm{Hom}(T, \mathbb{C}^\times)}_{\text{a lattice}} \subset \mathfrak{h}^*$.)

"Recall" also there is an isomorphism (Harish-Chandra):
$\sigma : Z(U\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$, so $\lambda \in \mathfrak{h}^*$ also defines a
W = Weyl group $\quad \mathbb{C}[\mathfrak{h}^*]^W \subset \mathbb{C}[\mathfrak{h}^*]$
character $\sigma^*(\lambda) : Z(U\mathfrak{g}) \longrightarrow \mathbb{C}$.

<u>Theorem</u> (Beilinson—Bernstein):

1) $\Gamma(G/B, \mathcal{D}^{\lambda}_{G/B}) \cong U\mathfrak{g} / (\ker \sigma^*(\lambda)) =: (U\mathfrak{g})_\lambda$

(Modules over $(U\mathfrak{g})_\lambda$ = $\mathfrak{g}$-modules for which $z \in Z(U\mathfrak{g})$ acts by $\sigma^*(\lambda)(z) \cdot \mathrm{Id}$.)

2) The global sections functor

$$\Gamma : \mathcal{D}^{\lambda}_{G/B}\text{-mod} \longrightarrow \mathfrak{g}\text{-mod with central character } \sigma^*(\lambda)$$

is an <u>equivalence</u> for suitable λ (including λ <u>dominant</u>).

For λ dominant integral, this has the property

$$\mathcal{O}(\lambda) \longmapsto V_\lambda .$$

(So this recovers the Borel–Weil theorem.)

Connection with symplectic geometry:

$\mathcal{D}^{\lambda}_{G/B}$ is filtered by <u>order</u> of differential operators.

A key fact is: $\operatorname{gr} \mathcal{D}^{\lambda}_{G/B} \cong \mathcal{O}_{T^*G/B}$

$$\begin{array}{ccc} \operatorname{gr} \mathcal{D}^{\lambda}_{G/B} & \cong & \mathcal{O}_{T^*G/B} \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathrm{Vect}(G/B) & = & \mathrm{Vect}(G/B) \end{array}$$

<u>Theorem</u> $\Gamma(G/B, \mathcal{O}_{T^*G/B}) \cong \mathcal{O}(\mathcal{N})$,

$\mathcal{N} \subseteq \mathfrak{g}$ is the cone of ~~ad~~-nilpotent elements in $\mathfrak{g}$.

The resulting map (the affinization map) $T^*G/B \longrightarrow \mathcal{N}$ is a <u>resolution of singularities.</u> (By a symplectic variety!)

Such a resolution is called a <u>symplectic resolution</u>.