TCC course: symplectic resolutions + sings. (1/)

Lecture 2: Examples of quantisation

First comment: in case you didn't read slack, I verbally said something not right: $\mathcal{O}(\lambda)$ does not have action of $T_{G/B}$.

But $\exists$ SES $0 \to \mathcal{O}_{G/B} \to \mathrm{Diff}_{\leq 1}(\mathcal{O}(\lambda)) \twoheadrightarrow T_{G/B} \to 0$.

~~Well, say~~ let's explain this:

<u>Defn</u> Given two B-mods, M, N, a <u>deriv</u>
B = comm ring
$\phi : M \to N$ is a linear map s.t.
$\forall b \in B$, $m \mapsto \phi(bm) - b\phi(m)$ is <u>B-linear</u>.

<u>Defn</u> (inductive) $\mathrm{Diff}_{\leq n}(M, N) = \{\phi \in \mathrm{Hom}_{\mathbb{C}}(M, N) \mid (m \mapsto \phi(bm) - b\phi(m)) \in \mathrm{Diff}_{\leq n-1}(M, N), \forall b \in B\}$.

"Grothendieck diff. ops"

When $M = N$, this forms a <u>filtered algebra</u>, and when $M = N = B$, called "Grothendieck diff ops on B / spec B".

<u>Globalisation</u> Given X = variety (or scheme), $\mathcal{F}, \mathcal{G}$ = sheaves of $\mathcal{O}_X$-mods, we can define $\mathrm{Diff}_{\leq n}(\mathcal{F}, \mathcal{G})$, a sheaf of vector spaces;
$\mathcal{F} = \mathcal{G} \Rightarrow \mathrm{Diff}(\mathcal{F}, \mathcal{F})$ is a filtered algebra.

If $\mathcal{F} = \mathcal{G}$ is <u>locally free</u> of rank 1, then $\mathrm{Diff}(\mathcal{F}, \mathcal{F})$ is locally isomorphic to $\mathcal{D}_X := \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X)$. (i.e. if $\mathcal{F}$ = line bundle)

Lemma If $\mathcal{F} = \mathcal{L}$ is a line bundle, then
$$\mathrm{gr}\, \mathrm{Diff}(\mathcal{L},\mathcal{L}) \cong \mathrm{Sym}_{\mathcal{O}_X} T_X = \pi_* \mathcal{O}_{T^*X}.$$
($\pi : T^*X \longrightarrow X$ projection.)

[This does not depend on choice of $\mathcal{L}$!]

Proof: We have a SES
$$0 \to \mathcal{O}_X = \mathrm{Diff}_0(\mathcal{L},\mathcal{L}) \hookrightarrow \mathrm{Diff}_{\leq 1}(\mathcal{L},\mathcal{L}) \xrightarrow{\sigma} T_X \to 0$$
where $\sigma(\xi)(f) := \xi \circ f - f \circ \xi$, $\xi \in \mathrm{Diff}_{\leq 1}(\mathcal{L},\mathcal{L})$, $f \in \mathrm{Diff}_0(\mathcal{L},\mathcal{L})$.
(surjectivity of σ is checked locally). ∎

Defn A sheaf of twisted differential operators (TDOs) is a sheaf of filtered algebras $\mathcal{D}$ s.t. $\mathrm{gr}(\mathcal{D}) \cong \pi_* \mathcal{O}_{T^*X}$, as Poisson algebras.

What is a Poisson algebra?
Given $\phi \in \mathcal{D}_{\leq m}$, $\psi \in \mathcal{D}_{\leq n}$, then $\phi \circ \psi - \psi \circ \phi \in \mathcal{D}_{\leq m+n-1}$
(because $\mathrm{gr}\,\mathcal{D}$ is commutative).
$$\{\mathrm{gr}_m \phi, \mathrm{gr}_n \psi\} := \mathrm{gr}_{m+n-1}(\phi \circ \psi - \psi \circ \phi), \quad \mathrm{gr}_n : \mathcal{D}_{\leq n} \twoheadrightarrow \mathcal{D}_{\leq n}/\mathcal{D}_{\leq n-1} \text{ quotient}$$
This defines a Lie bracket on $\mathrm{gr}\,\mathcal{D}$ which is a derivation in each component.

Defn A Poisson algebra is a commutative algebra equipped with such a Lie bracket.

(The derivation condition says $\{ab, c\} = a\{b,c\} + b\{a,c\}$ "Leibniz identity".)

So a TDO is a $\mathcal{D}_X$ s.t. $\mathrm{gr}\,\mathcal{D} \cong \mathrm{gr}\,\mathcal{D}_X$ as Poisson algebras.
Clearly $\mathrm{Diff}(\mathcal{L},\mathcal{L})$ is a TDO since $\mathcal{L}$ is locally trivial (it is a "locally trivial TDO".

More general setup: A = filtered algebra, $\mathrm{gr}\, A =: B$ is commutative $\Rightarrow$ A called "filtered quantization" of $(B, \{-,-\})$.

Or: called a quantisation of the scheme Spec B. (3/)

So TDOs are the natural notion of quantisation of $\pi_* \mathcal{O}_{T^*X} = \mathrm{Sym}_{\mathcal{O}_X} T_X$, equipped with its Poisson bracket (from $\mathcal{D}_X$).

<u>Remark</u>: Another way to view this bracket: The unique extension of the Lie bracket $\{\xi, \eta\} = \xi\circ\eta - \eta\circ\xi = [\xi, \eta]$ on T_X to a Poisson bracket on $\mathrm{Sym}_{\mathcal{O}_X} T_X$.

$\rightsquigarrow$ Generally, given <u>any</u> "Lie algebroid" $\mathfrak{g}$ over $\mathcal{O}_X$, can define Poisson algebra $\mathrm{Sym}_{\mathcal{O}_X} \mathfrak{g}$.

Case $X = \mathrm{Spec}\,\mathbb{C}$: $\mathrm{Sym}_{\mathbb{C}} \mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$ "$\mathcal{O}(\mathfrak{g}^*)$", the standard Poisson bracket on $\mathfrak{g}^*$, making $\mathfrak{g}^*$ a <u>Poisson variety</u>:

Defn A Poisson variety (or scheme) X is a variety (scheme) where $\mathcal{O}_X$ is equipped with a Poisson bracket.

So the dual of a Lie algebra (or the total space of the dual of a Lie algebroid) is a Poisson variety.

Just as T^*X has a canonical quantisation $\mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X)$, so does $\mathfrak{g}^*$, $\mathfrak{g}$ = Lie alg:

$U\mathfrak{g}$ (= Universal enveloping algebra, $T\mathfrak{g}/(x\otimes y - y\otimes x - [x,y])$)

$\mathrm{gr}(U\mathfrak{g}) \overset{\boxed{\text{PBW theorem}}}{\cong} \mathrm{Sym}\,\mathfrak{g} = \mathcal{O}(\mathfrak{g}^*)$ with its Poisson bracket.

In general for any Lie algebroid $\mathfrak{g}$ (e.g. T_X), we get $U\mathfrak{g}$, a quantisation of $\mathfrak{g}^\vee$ ($UT_X = \mathcal{D}_X$) (relative version of PBW theorem).

Other examples of quantisation, coming from <u>symplectic singularities/resolutions</u>:

<u>Ex1</u>: $B = (\mathrm{Sym}\,V)^G = \mathcal{O}(V^*/G)$, $G \le Sp(V)$ finite (or reductive), V = symplectic vector space (e.g. $\mathbb{C}^{2n}$).

$A = \mathrm{Weyl}(V)^G$, $\mathrm{Weyl}(V) = \mathcal{D}(U)$, $U \le V$ Lagrangian subspace

Case $U = \mathbb{C}^n \le \mathbb{C}^{2n} = V$: $\mathrm{Weyl}(V) = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle / \left(\begin{array}{l} [\partial_i, x_j] = \delta_{ij} \\ [x_i, x_j] = 0 = [\partial_i, \partial_j] \end{array} \right)$.

Just as we considered TDOs, we can also define more general quantisations of $(\mathrm{Sym}\, V)^G$, i.e., of $V^*/\!/G$.

$\boxed{G \text{ finite:}}$

"Spherical symplectic reflection algebras" = universal deformation of $\mathrm{Weyl}(V)^G$, defined by Etingof + Ginzburg

Highly nontrivial result (Losev '16): these also give the universal quantisation of $(\mathrm{Sym}\, V)^G$. (Because V/G is a "symplectic singularity")

Difficulty: $\mathrm{Weyl}(V)^G$ is "homologically smooth" (it is Morita equivalent to $\mathrm{Weyl}(V \rtimes G)$), but $(\mathrm{Sym}\, V)^G = \mathcal{O}(V/G)$ is NOT (V/G is singular).

Example of the above: $V = \mathbb{C}^2$, $G = \{\pm I\} \cong \mathbb{Z}/2$

$\rightsquigarrow V/G \cong \mathrm{Nil}_{2\times 2} = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a^2 + bc = 0 \}$ nilp. matrices $\subseteq \mathfrak{sl}_2$

$\mathbb{C}^3 \supseteq \mathrm{Spec}\, \mathbb{C}[x^2, xy, y^2] = \mathrm{Spec}\, \mathbb{C}[a,b,c]/(a^2+bc)$

$(xy)^2 - (x^2)(y^2) = 0$

$(a^2 - bc = 0)$

Quantisations: $U\mathfrak{sl}_2 / (C - \lambda)$, $C = ef + fe + \frac{1}{2} h^2$

[here $\mathfrak{sl}_2$ has basis $e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; these are generators of $U\mathfrak{sl}_2$.]

In this case, these are precisely the spherical symplectic reflection algebras.

Exercise: Find the $\lambda \in \mathbb{C}$ s.t. $U\mathfrak{sl}_2 / C_\lambda \cong \mathrm{Weyl}(\mathbb{C}^2)^{\mathbb{Z}/2}$.

Ex 2: $\mathfrak{g}$ = semisimple Lie algebra, $\chi : Z(U\mathfrak{g}) \to \mathbb{C}$ algebra homomorphism

$A_\chi = U\mathfrak{g} / (\ker \chi)$ still filtered, $B = \mathrm{gr}\, A \cong \mathrm{Sym}\, \mathfrak{g} / \mathrm{gr}(\ker \chi)$.

By the Harish-Chandra isomorphism + a result of Kostant, $B \cong \mathbb{C}[\mathrm{Nil}\, \mathfrak{g}]$, $\mathrm{Nil}\, \mathfrak{g} := \{ x \in \mathfrak{g} \mid (\mathrm{ad}\, x)^N = 0, \text{ some } N \geq 1 \}$.

So we get: A_χ is a quantisation of $\underline{\mathrm{Nil}\,\mathfrak{g}} \subseteq \mathfrak{g}$, a Poisson cone.

Again, the result of Losev '16 implies that, as χ varies, A_χ produces the universal quantisation of $\mathrm{Nil}\,\mathfrak{g}$ (a symplectic singularity)

<u>Beilinson–Bernstein '81</u>: $A_\chi = \Gamma(G/B, \mathcal{D}^\lambda_{G/B})$, $\chi = \lambda \circ \sigma$

$$Z(U\mathfrak{g}) \xrightarrow[\text{"H-C isom."}]{\sigma} (\mathrm{Sym}\,\mathfrak{h})^W \xrightarrow{\lambda} \mathbb{C}$$

$$\lambda(x_1 \cdots x_n) = \lambda(x_1) \cdots \lambda(x_n), \quad x_i \in \mathfrak{h}$$

<u>Kostant '63 + Springer '69</u>: $B = \Gamma(G/B, T^*_{G/B})$

$$= \Gamma(T^*G/B, \mathcal{O}_{T^*G/B})$$

($T^*G/B \twoheadrightarrow \mathrm{Nil}\,\mathfrak{g}$ is a symplectic resolution of singularities)

That is, gr commutes with global sections, which relate G/B to $\mathfrak{g}$:

$$\begin{tikzcd}
\text{quantised resolution} & \mathcal{D}^\lambda_{G/B} \arrow[r, mapsto, "\Gamma", "B-B"'] \arrow[d, "gr"] & A_\chi \arrow[d, "gr"] & \text{quantisation} \\
\text{resolution} & T^*_{G/B} \arrow[r, mapsto, "\Gamma", "\text{Kostant+Springer}"'] & B = \mathbb{C}[\mathrm{Nil}\,\mathfrak{g}] & \text{singularity}
\end{tikzcd}$$

Ex 3: Hamiltonian / symplectic reduction.

<u>Motivation</u> Suppose that G "alg group" acts on X. $\rightsquigarrow X/\!\!/G$, the categorical quotient, when it exists.

Ex: $X = \mathrm{Spec}\, B \Rightarrow X/\!\!/G = \mathrm{Spec}\, B^G$.

Would like a construction on <u>cotangent bundles / Diff ops</u>:

$$T^*X \rightsquigarrow T^*(X/G)$$

$$\mathcal{D}_X \rightsquigarrow \mathcal{D}_{X/G}$$

But we expect $\dim (T^*X)/\!/G = \dim T^*X - \dim G$
whereas $\dim T^*(X/\!/G) = \underset{\underset{2\dim X}{\shortparallel}}{\dim T^*X} - 2\dim G$.

Fix: Symplectic / Hamiltonian reduction.

<u>General setup</u>: $X =$ variety

$\mathfrak{g} =$ Lie algebra

$\mu^* : \mathfrak{g} \longrightarrow \mathrm{Vect}(X) = \Gamma(X, T_X)$ Lie alg. homomorphism

$\Big[$ <u>extends</u> to: $\mu^* : \mathrm{Sym}\,\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*] \longrightarrow \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X} T_X) = \Gamma(T^*X, \mathcal{O}_{T^*X})$
i.e. $\mu : T^*X \longrightarrow \mathfrak{g}^*$. $\Big]$

$\rightsquigarrow$ consider symplectic / Hamiltonian reduction,

$$T^*X /\!/\!/ \mathfrak{g} := \mu^{-1}(0) /\!/ \mathfrak{g} \overset{\text{if } X \text{ affine}}{=\!=} \mathbb{C}[\mu^{-1}(0)]^{\mathfrak{g}}.$$

If $\mathfrak{g} = \mathrm{Lie}\, G$ and G acts on X, we define
$T^*X /\!/\!/ G := \mu^{-1}(0) /\!/ G$, similarly.

<u>Quantum version:</u> $\mathrm{Vect}(X) \subseteq \Gamma(X, \mathcal{D}_X)$

So define:
$$\left(\mathcal{D}_X / \mu^*(\mathfrak{g}) \cdot \mathcal{D}_X \right)^{\mathfrak{g}} =: \text{Quantum Hamiltonian reduction of } \mathcal{D}_X \text{ by } \mathfrak{g}.$$
(or $\mathfrak{g} \longmapsto G$ if G as before) "$\mathcal{D}_X /\!/\!/ \mathfrak{g}$ or G"

More general Poisson version: $X =$ Poisson variety

$\mu^* : \mathfrak{g} \longrightarrow \mathcal{O}(X)$ Lie homomorphism

$\rightsquigarrow \mu^* : \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathcal{O}(X)$ Poisson homomorphism, i.e. $X \to \mathfrak{g}^*$

$X /\!/\!/ \mathfrak{g} = \mu^{-1}(0) /\!/ \mathfrak{g}$; similarly with G.

If X affine, $A =$ quantisation of $(B = \mathcal{O}(X), \{-,-\})$, then $\left(A / \mu^*(\mathfrak{g}) A\right)^{\mathfrak{g}}$
$=:$ QHR.

Examples of this.

- Hypertoric varieties: $X = V = \mathbb{C}$-v.s.,
  $G = (\mathbb{C}^\times)^m =: \mathbb{T} = $ torus

Hamiltonian (symplectic) $(\mathbb{C}^\times)^{n-m} \curvearrowright T^*V /\!/\!/ \mathbb{T} =:$ hypertoric variety $\left(\text{symplectic analogue of } \underline{\text{toric variety}}\ V/\!/\mathbb{T} \curvearrowleft (\mathbb{C}^\times)^{n-m}\right)$ (open orbit)

$\dim V = n$.

- Quiver variety: $G = \prod_{i=1}^m GL_{n_i}$,

$$V = \bigoplus_{i,j=1}^m \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})^{r_{ij}}$$

(Quiver: directed graph with vertex set $\{1, \dots, m\}$, r_{ij} arrows from i to j $\forall i,j$.)

$T^*V /\!/\!/_G =:$ quiver variety

Variants / generalisations:

- Deformation: replace $\mu^{-1}(0)$ by $\mu^{-1}(\lambda)$
  $\lambda \in \mathfrak{g}^*$ a $\underline{\text{character}}$, i.e. $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = 0$

"$X/\!/\!/_\lambda G$"

$\Rightarrow G \cdot \{\lambda\} = \{\lambda\}$, $\mathfrak{g} = \mathrm{Lie}\, G$.

- partial resolution / Geometric Invariant Theory (GIT):

"$X/\!/\!/^\theta_G$" Replace $\mu^{-1}(0)$ (or $\mu^{-1}(\lambda)$) by $\mu^{-1}(0)^{\theta\text{-ss}}$ (or $\mu^{-1}(\lambda)^{\theta\text{-ss}}$),

$\theta : G \to \mathbb{C}^\times$ is a character, θ-ss = "θ-semistable"

$\mu^{-1}(0)^{\theta\text{-ss}} \subseteq \mu^{-1}(0)$ is the open locus of X where

in case X $\underline{\text{affine}}$: $\overline{G\cdot(x,1)} \cap (X \times \{0\}) = \emptyset$, where G acts on $X \times \mathbb{C}_\theta$ by $g\cdot(x,c) = (g\cdot x, \theta(g)^{-1} c)$.

$(x,1) \in X \times \mathbb{C}_\theta$

(i.e., $\exists f \in \mathbb{C}[X]^{m\theta}$ s.t. $f(x) \neq 0$, $m \geq 1$ ($m\theta$-isotypic functions).)

Now $\mu^{-1}(0)/\!/G$ is a cone (since $\mu^{-1}(0)$ is and G acts linearly) but in favorable cases, $\mu^{-1}(\lambda)/\!/G$ is a smoothing and $\mu^{-1}(0)^{\theta ss}/\!/G$ is a resolution of singularities.

Example: $T^*\mathbb{C}^2/\!/\!/^{\theta}\mathbb{C}^\times \cong T^*\mathbb{P}^1$, $\theta \neq 0$ [indeed $\mathbb{C}^2/\!/^{\theta}_{\mathbb{C}^\times} = \mathbb{P}^1$ for appropriate θ]
hypertoric (+quiver, quiver $= 1 \rightrightarrows 1$)

Use: Exercise: $X/\!/^{\theta}G := X^{\theta\text{-}ss}/\!/G \cong \mathrm{Proj}\left(\bigoplus_{m\geq 0}\mathbb{C}[X]^{m\theta}\right)$, for X affine, $\theta : G \to \mathbb{C}^\times$.

Rmk: If X not affine, do same thing to define $X/\!/\!/^{\theta}G$, $X/\!/^{\theta}G$, except replace $X \times \mathbb{C}_\theta$ by the total space of a G-linearised (ample) line bundle.

Then $T^*\mathbb{C}^2/\!/\!/\mathbb{C}^\times = Nil_{2\times 2} = Nil(\mathfrak{sl}_2) \cong \mathbb{C}^2/\mathbb{Z}_2$

Again we recover $T^*\mathbb{P}^1 \xrightarrow{\text{symplectic resolution}} Nil(\mathfrak{sl}_2)$.

Symplectic resolution := resolution of singularities by a symplectic variety.

Quantisation:

$$\begin{array}{ccc} \left(\mathcal{D}_{\mathbb{C}^2\setminus 0}/Eu\cdot\mathcal{D}_{\mathbb{C}^2\setminus 0}\right)^{\mathbb{C}^\times} & \cong & \mathcal{D}_{\mathbb{P}^1} \\ & & \downarrow \pi \\ \left(\mathcal{D}_{\mathbb{C}^2}/Eu\cdot\mathcal{D}_{\mathbb{C}^2}\right)^{\mathbb{C}^\times} & \cong & U\mathfrak{sl}_2/(\ker \chi) \end{array}$$

As before: $\chi : Z(U\mathfrak{sl}_2) = \mathbb{C}[C] \twoheadrightarrow \mathbb{C}$

$\chi(C) = \chi(ef + fe + \frac{1}{2}h^2) = 0$.

Upshot: 3 ways to realise $\subseteq \mathbb{C}^3$: • $\mathbb{C}^2/(\mathbb{Z}/2)$ (cone of t) • $T^*\mathbb{C}^2/\!/\!/\mathbb{C}^\times$ (hypertoric/quiver) • $Nil(\mathfrak{sl}_2)$ (nilcone).
Each gives quantisation. Second two give resolution/smoothing.

<u>Note</u>: Even for $T^*V/\!/\!/G$ with V = vector space, $G \le GL(V)$ reductive, it is not necessarily true that $T^*V/\!/\!/G$ is a <u>symplectic singularity</u> in general, although it is true in nice cases. (So it doesn't follow from Losev that one can necessarily classify <u>all</u> quantisations.)

<u>More nontrivial example</u>: G = reductive (e.g. semisimple)

G acts on $X := \mathfrak{g}$ by <u>adjoint action</u>

What is $T^*\mathfrak{g}/\!/\!/G$? (or $T^*G/\!/\!/G$, G acts on G by conjugation)

Joseph: underlying reduced variety is $T^*\mathfrak{h}/W = \operatorname{Spec} \mathbb{C}[T^*\mathfrak{h}]^W$.

<u>Conjecture</u>: Already reduced ($\mathbb{C}[\mu^{-1}(0)]^G$ has no nilpotents)

↳ proved for $G = GL_n$ (or PGL_n, SL_n) by Gan–Ginzburg

open in general.

Also, is $\mu^{-1}(0)$ itself reduced?

$G = GL_n$: $\mu^{-1}(0) \subseteq \mathfrak{g} \times \mathfrak{g}^* \cong$ Commuting scheme

$$:= \{(X, Y) \mid XY = YX\} \subseteq \mathfrak{gl}_n \times \mathfrak{gl}_n.$$

Open Q: Is it reduced?

Quantum version <u>easier</u>:

Thm (Levasseur–Stafford): $$\text{"}\mathcal{D}_\mathfrak{g}/\!/\!/G\text{"} := \left(\mathcal{D}(\mathfrak{g}) / \mu^*(\mathfrak{g}) \cdot \mathcal{D}(\mathfrak{g})\right)^G \cong \mathcal{D}_\mathfrak{h}^W.$$