Lecture 4, symplectic singularities.

Symp sing: · normal
X · $\exists\ \omega_{reg}$ on X_{reg}

· For $\rho: \tilde{X} \twoheadrightarrow X$ resol.,
$\rho^* \omega_{reg}$ extends to $\tilde{\omega} \in \Gamma(\tilde{X}, \Omega^2_{\tilde{X}})$.

<u>S.R.</u>: Such with $\tilde{\omega}$ <u>nondegen</u>.

<u>Rmk</u>: If only given $\rho: \tilde{X} \twoheadrightarrow X$ resol., with $\tilde{X}$ symplectic, $\Rightarrow \exists\ \omega_{reg}$ on X_{reg}.
($\Rightarrow X$ symp sing.)

<u>Rmk</u>: Symplectic sing $\Rightarrow$ Poisson.

[Smooth: symplectic = nondegen. Poisson.

$$T_X \xrightarrow{\sim} T_X^*$$
$$\xi \longmapsto i_\xi \omega$$

Poisson: $T_X^* \longrightarrow T_X$
$$df \longmapsto \{f, -\}$$
nondegen: $\xrightarrow{\sim}$

X_{reg} sympl $\Rightarrow$ X_{reg} Poisson.

X normal $\Rightarrow$ define $\{f, g\} = \{f|_{X_{reg}}, g|_{X_{reg}}\}$

$\Rightarrow$ $X \setminus X_{reg}$ has codim ≥ 2. extends uniquely to X.

Rmk: Symplectic resolutions are very restrictive, if exist

$\tilde{X} \twoheadrightarrow X$ S.R., $f: Y \twoheadrightarrow \tilde{X}$ blow-up

$f^* \omega_{\tilde{X}}$ degen.

BUT not unique.

Thm (Kaledin): $\tilde{X}$ S.R $\Rightarrow$ semismall

$\dim \tilde{X} \times_X \tilde{X} = \dim X$.

Thm (Namikawa): X conical $\Rightarrow$ $\exists$ only finitely many S.R.'s of X up to $\cong$.

Thm (Kaledin): Two S.R.'s are derived equivalent, étale-locally on base.

Rmk If $\tilde{X}$ symplectic $\Rightarrow$ $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$ Poisson

$\rho: \tilde{X} \twoheadrightarrow X$ resol $\Rightarrow$ X Poisson. (unique s.t. ρ Poisson)

$\Rightarrow$ X_{reg} Poisson, we can deduce it is symplectic.

3 equivalent notions of SR:

$$\left\{\begin{array}{c}\text{Poisson S.R.}\\ \text{of Poisson var.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Symp R.}\\ \text{of s. sing}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{resolution with}\\ \text{Symp. source}\end{array}\right\}$$

$\exists!$ structure on X.

Prop (Beauville) A s.s. is rational Gorenstein + canonical.

Proof: $\omega_{reg}^{\dim X/2}$ nonvan. gen of $K_{X_{reg}}$ extends to <u>holomorphic</u> section of $K_{\tilde{X}}$ $\Leftrightarrow$ "canonical of index 1" $\Longleftrightarrow$ "rational Gorenstein"

Rational: $R^i p_* \mathcal{O}_{\tilde{X}} = 0,\ i > 0.$

Prop (Beauville): A s.s. which is lci $\Rightarrow$ sing locus has codim ≤ 3.

Eg. $T^*\mathbb{P}^n \longrightarrow \min(\mathfrak{sl}_n)$ isolated sing iso!, $\Rightarrow$ NOT lci $n \geq 2$.

Rmk: Beauville: min normal nilp. orbit $\Leftarrow$ Symplectic sing with smooth proj. tangent cone.

Prop (Beauville) V s.s., $G < \mathrm{SpAut}(V)$ finite

$\Rightarrow V/G$ s.s.

Proof (sketch): V_{free}/G smooth, $V \setminus V_{free}$ has codim ≥ 2.

$(V/G)_{reg} = V_{free}/G \rightsquigarrow \omega_{(V/G)_{free}}$

Pull back: $f^* \omega_{(V/G)_{reg}}$ extends to X

$\Rightarrow$ extends to Y.

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, dotted] & V \arrow[d] \\ Y \arrow[r, "f"] & V/G \end{tikzcd}$$

Prop X symp sing, then $p : \widetilde{X} \twoheadrightarrow X$ is SR $\iff$ it is **crepant**.

(crepancy $\iff$ extendability of $p^* \omega_{reg}$ to $\widetilde{\omega}$ **nondegen.**)

Proof $\omega_{reg}^{\dim X/2}$ trivialises K_X, it pulls back to a triv of $K_{\widetilde{X}} \iff \widetilde{\omega}$ is nondegen.

Thm (Namikawa): X normal, $\exists\, \omega_{reg}$ symp on X_{reg} then s.sing $\iff X$ rat. Gorenstein.

Thm (Namikawa) X s.s. is **terminal**

$\iff$ $(X \setminus X_{reg})$ has codim ≥ 4.

Terminal: $\rho : \tilde{X} \longrightarrow X$ resolution,

$K_{\tilde{X}} = \rho^* K_X + \sum a_i E_i$, E_i = exceptional divisors

$\underline{a_i > 0 \ \forall i}$.

(canonical: $a_i \geq 0 \ \forall i$)

Proof follows from: $X \setminus X_{reg}$ has no 3-dim irreducible components.

This follows from:

Thm (Kaledin) A symp sing has finitely many symplectic leaves (finite alg. stratification)

· Recall: A (alg) symplectic leaf is a loc cl'sd conn subvariety of even dimension (symplectic)

· The open symplectic leaves = conn components of X_{reg}.

Recall: Thm (Namikawa) $\exists$ homogeneous CI s.s. $= \mathrm{Nil}(\mathfrak{g})$, $\mathfrak{g}$ s.s. fd Lie

2) X conical s.s., then:
$\shortparallel$
affine, $\mathcal{O}(X)$ nonneg graded
$\mathcal{O}(X)_0 = \mathbb{C}$

$\mathcal{O}(X)$ gen in deg 1 $\iff X \cong \overline{G \cdot e}$, $e \in \mathrm{Nil}\,\mathfrak{g}$

NORMAL.

Thm (Beauville/Panyushev) The normalisation $\widetilde{\overline{G \cdot e}}$ is always s.s., $e \in \mathrm{Nil}\,\mathfrak{g}$.

Sketch of proof: Jacobson-Morozov theorem:

$\mathfrak{g}$ = fd ss Lie, $e \in Nil(\mathfrak{g}) \Rightarrow \exists\ (e,h,f)$, satisfying relns of sl_2. $(e,h,f \in \mathfrak{g})$

$\Rightarrow \mathfrak{g} = \bigoplus \mathfrak{g}_i$, $\mathfrak{g}_i = \ker(h-i)$.

$\mathfrak{n} := \bigoplus_{i \geq 2} \mathfrak{g}_i \subseteq \mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}_i$, $P \leq G$ subgroup, $\mathrm{Lie}\,P = \mathfrak{p}$.

$\rho : G \times_P \mathfrak{n} \longrightarrow \mathfrak{g}$. $\quad P\cdot e$ dense in $\mathfrak{n}$

$\Rightarrow \omega_{G\cdot e}$ extends. $\quad \Rightarrow \overline{G\cdot e} = \mathrm{im}(\rho)$.

Situation for normality of $\overline{G\cdot e}$: (slack)

- $\mathfrak{sl}_n$: All normal
- $\mathfrak{so}_n, \mathfrak{sp}_{2n}$: Some are not: $\exists\, G\cdot e' \subseteq \partial G\cdot e$

Kraft–Procesi $\quad \nu : \widetilde{\overline{G\cdot e}} \twoheadrightarrow \overline{G\cdot e}$ 2-branched over $G\cdot e'$

Springer theory ($\{$irreps of Weyl group$\} \hookrightarrow \{$pairs $(G\cdot e, L)$, L = local system on $G\cdot e\}$)

$\rightsquigarrow$ formula for # branches

$\Rightarrow$ can detect all branched (non-normal) sings.

$\mathfrak{g}_2$: $\exists!$ nonnormal $\overline{G\cdot e}$, 8-dim.

$\mathfrak{so}(8)$

$$\mathfrak{g}_2 = \mathfrak{so}(8)^{S_3} \hookrightarrow \mathfrak{so}(8)^{C_2} = \mathfrak{so}(7) \hookrightarrow \mathfrak{so}(8)$$

$e \in \mathfrak{g}_2$ $\dim G\cdot e = 8$:

$$\widetilde{\overline{G\cdot e}} \cong \overline{\min(\mathfrak{so}_7)} \xrightarrow{\text{homeom}} \overline{G\cdot e}$$

$\Rightarrow \overline{G\cdot e}$ "unibranch"

$\mathfrak{e}_6$: Sommers classified (2003)

$\mathfrak{e}_7, \mathfrak{e}_8$: Not done, expected answer (Broer 1998)

(very even $\overline{G\cdot e} \subseteq \mathfrak{so}_{2n}$: normal (Sommers))

<u>Namikawa</u>: $\overline{G\cdot e}$ normal $\Longleftrightarrow$ $\mathcal{O}(\widetilde{\overline{G\cdot e}})$ gen. in deg 1.

Beyond degree 1:

<u>Thm (Namikawa)</u>: For each dim $2n$, each $N \geq 1$, $\exists$ only finitely many conical s.s. $/\cong$, $\dim = 2n$, $\mathcal{O}(X)$ gen in degs $\leq N$.

$\underset{\text{finite}}{\Gamma} \leq SL_2\mathbb{C} = Sp_2\mathbb{C} \rightsquigarrow$ McKay graph D

vertices = nontrivial irreps

edges $\rho_i - \rho_j$ if $\mathrm{Hom}_\Gamma(\rho_i, \mathbb{C}^2 \otimes \rho_j) \neq 0$.

Dynkin, $\mathfrak{g}_D$

Contrast to finite quotients:

$V/\!\!/\!\!/G$, $G < Sp(V)$ need <u>not</u> be symp sing.

$(\mathbb{C}^{2n})^{\oplus 2m}/\!\!/\!\!/Sp_{2n}$: <u>Thm (Becker)</u>: · <u>reduced</u> if $m \geq 2n$
· <u>normal</u> if $m \geq 2n+1$.

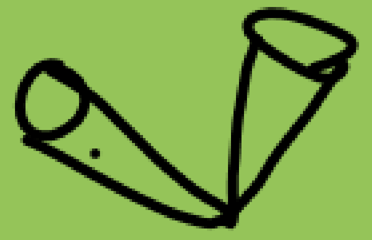
reduced = generically smooth

normal $\Rightarrow$ smooth outside codim 2

$(\mathbb{C}^2)^2/\!\!/\!\!/SL_2$: NOT reduced: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathbb{C}^2)^2$

$(ad-bc)^2 \in \mu^{-1}(0)$, not $(ad-bc)$.

$(\mathbb{C}^{2n})^{2n} /\!/ Sp_{2n} \cong$ adjoint for O_{2n}, not SO_{2n}



not normal. $\subseteq SO_{2n}$.

<u>Rmk:</u> Herbig–Seaton–Schwarz

<u>Thm:</u> If V is "3-large" (genericity)

$G \leq GL(V)$ reductive

$\Rightarrow T^*V /\!/\!/ G$ is symp sing.

<u>They believe:</u> Use real quotients, $\sqrt[\mathbb{R}]{I_{\mu_{\mathbb{R}}^{-1}(0)}}$, always give s.s.

Thm (Losev) $G \leq Sp(V)$, V symplectic then $V /\!/\!/ G$ has fin. many symplectic leaves.

Idea: leaves labelled by $\mathrm{Stab}_G(V)$/conjugation

Luna slice thm $\Rightarrow$ finitely many of these.

Rmk: Generally, there may not be many/any symp. leaves in a Poisson variety, only analytically locally. But if $\exists$ fin many leaves anal. locally $\Rightarrow$ algebraic.

Then they are components of

$$X_r := \{ x \in X \mid \operatorname{span}(\xi_f|_x) \text{ has dim} = r \}$$

and loc. finitely many leaves $\Leftrightarrow \dim X_r = r$ (or $X_r = \emptyset$).

General picture:

gen. symp $\Leftarrow$ finitely many symp leaves $\xleftarrow{\text{Kaledin}}$ symp sing $\Leftarrow \exists$ S.R.

$\uparrow$ Losev

$V /\!/\!/ G$

$\mathbb{C}^{2n}/\Gamma \leftarrow$ fin.

$\widetilde{G \cdot e}$

Nilcones

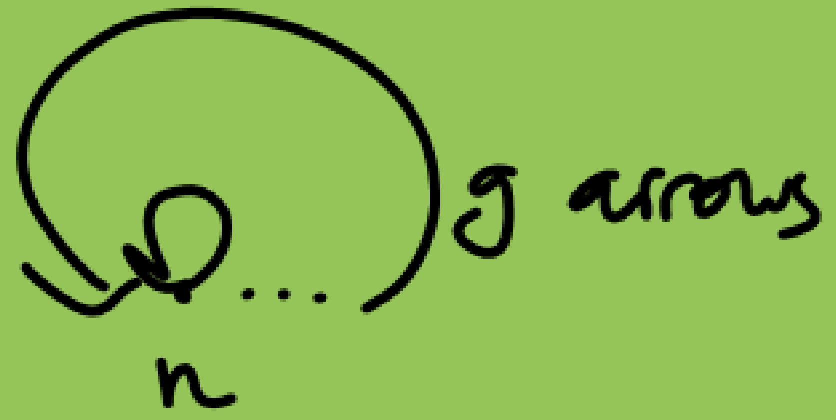
$\operatorname{Sym}^n X$

X = symp surface

Nak q. vars, hypertoric or du Val sing.

<u>Thm (Bellamy–S.)</u>: Nakajima q. vars are symp sings. Classified $\exists$ SR's.

Prototypical example:



<u>means</u>: $X = T^*(\mathfrak{gl}_n)^g /\!/\!/ GL_n$ (or PGL_n)

<u>Cases</u>: $g=1$: $T^*\mathfrak{gl}_n /\!/\!/ GL_n \underset{\text{Gan–Gin.}\ \text{Joseph}}{\cong} \mathbb{C}^{2n}/S_n \twoheadleftarrow \mathrm{Hilb}^n \mathbb{C}^2$.

$n=1$: $\mathbb{C}^{2g}$.

$(g,n)=(2,2)$: O'Grady (K3 surfaces)

$\tilde{X} \twoheadrightarrow X$ S. R., blow up set-theoretic sing locus (once)

$(g,n)\neq(2,2)$, $g,n\geq 2$: $\nexists$ S. R.

(Kaledin–Lehn–Sorger)

Idea: They show that these are factorial + terminal.

$\mathbb{Q}$-factorial: Some multiple of every Weil divisor is Cartier.

factorial: local rings are UFD

Every Weil divisor is Cartier

terminal $\Uparrow$ sing locus has codim ≥ 4

<u>Terminal</u>: $\rho : \tilde{X} \twoheadrightarrow X$ resol. of sings,

$E_i :=$ exceptional irreducible divisors

$$K_{\tilde{X}} = \rho^* K_X + \sum a_i E_i$$

Terminal $\iff a_i > 0$.

<u>Van de Waerden purity</u>: $\mathbb{Q}$-factorial $\Rightarrow$ exc. locus is divisor (pure codim 1)

Together. $\tilde{X} \twoheadrightarrow X$ not isom $\Rightarrow$ not crepant ($K_{\tilde{X}} \neq \rho^* K_X$).

$\Rightarrow \nexists$ crepant resol.

<u>BCHM</u>: Corollary 1.4.3 : existence of relative minimal models

$\Rightarrow$ Crepant $\rho: \tilde{X} \longrightarrow X$, "$\mathbb{Q}$-fact terminalization"

$\tilde{X}$ is $\mathbb{Q}$-factorial + terminal.

[$\Rightarrow \tilde{X}$ has S.R. $\Leftrightarrow$ smooth]

<u>Thm (Namikawa)</u>: If X conical, then one "$\mathbb{Q}$-fact. term" is smooth $\Leftrightarrow$ they <u>all</u> are.

$\therefore X$ has S.R. $\Leftrightarrow \tilde{X}$ smooth.