What makes $T^*G/B \twoheadrightarrow Nil\,\mathfrak{g}$ so nice?

B–B: "D-affinity" of G/B:

Defn A variety X is D-affine if the global sections functor is an equivalence:

$$D_X\text{-mod} \xrightarrow{\Gamma} \Gamma(X, D_X)\text{-mod}.$$

Note: X affine $\Rightarrow$ X D-affine.

$\not\Leftarrow$: by B-B, G/B (projective) is D-affine e.g. $\mathbb{P}^1$.

Taking associated graded, we have an analogous map,

$$\rho : T^*X \twoheadrightarrow Aff(T^*X) := Spec(\Gamma(T^*X, \mathcal{O}_{T^*X}))$$

$$\Gamma(T^*X, \mathcal{O}_{T^*X}) = \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X} T_X).$$

Observe: if X is not affine, this is NOT an isomorphism. But in order for X to be D-affine we can expect ρ to have some nice properties.

$X = G/B$: Thm (Springer): ρ is projective + birational

$$(Aff(T^*G/B) = Nil(\mathfrak{g}) \subseteq \mathfrak{g}).$$

Explicit description of ρ: $T^*G/B \cong G \times_B \mathfrak{n} \xrightarrow{\rho} \mathfrak{g}$

$$\rho(g, x) = (Ad\, g)(x)$$

Image $= Nil(\mathfrak{g})$. See CW (shows for $G = SL_n$).

Conjecture (Demailly, Campana, Peternell):

If $\rho: T^*Z \to Y$ is a resolution of singularities (= proper (or projective) birational map with smooth source), Z is projective, and Y is affine,

then $Z \cong G/P$, G = semisimple $\mathbb{C}$ alg group, P = parabolic subgroup.

<u>Recall</u>: P is parabolic $\iff$ G/P projective
$\iff P \supseteq B$, $B =$ a Borel subgroup.

G/P is called a "(generalised) partial flag variety".

Case $G = SL_n$ (or GL_n): Up to conjugation,

$$P = \begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix} \quad \text{block-upper triangular}$$
blocks of size $n_1, n_2, \dots, n_k$, $\sum n_i = n$

$$\| \quad \mathrm{Stab}_{SL_n \text{ (or } GL_n)} \left(0 \subseteq \mathbb{C}^{n_1} \subseteq \mathbb{C}^{n_1+n_2} \subseteq \dots \subseteq \mathbb{C}^{n_1+\dots+n_k} = \mathbb{C}^n \right)$$

$$\implies G/P \cong \{ \text{Partial flags } 0 \subseteq V_1 \subseteq \dots \subseteq V_k \subseteq \mathbb{C}^n \mid \dim V_i/V_{i-1} = n_i \}$$

<u>Theorem (Beilinson–Bernstein)</u>: G/P is D-affine.

(Also, $D^\lambda_{G/P}\text{-mod} \xrightarrow{\sim} \Gamma(G/P, D^\lambda_{G/P})\text{-mod}$ for λ dominant.)

<u>Conjecture</u>: if Z is projective and D-affine, then
$Z \cong G/P$, some G, P.

<u>Question</u>: What is $\Gamma(T^*G/P, \mathcal{O}_{T^*G/P}) = \mathrm{Aff}(T^*G/P)$?

<u>Answer</u>: Have $T^*G/P \cong G \times_P \mathfrak{u}$, $\mathfrak{u} \subseteq \mathfrak{p} := \mathrm{Lie}\, P$ is the nilpotent radical ($\mathfrak{u} = \mathrm{Lie}\, U$, $U \subseteq P$ unipotent radical).

So get (partial) Springer map,

$$\begin{aligned} \rho_{\mathrm{Spr}_P} : G \times_P \mathfrak{u} &\longrightarrow \mathfrak{g} \\ (g, x) &\longmapsto \mathrm{Ad}(g)(x) \end{aligned}$$

Image $= (\mathrm{Ad}\, G)(\mathfrak{u})$.

<u>Thm (Richardson)</u>: $\exists\, x \in \mathfrak{u}$ s.t. $\overline{\mathrm{Ad}\, P(x)} = \mathfrak{u}$.

$\therefore \mathrm{Im}(\rho_{\mathrm{Spr}}) = \overline{G \cdot x}$, $x =$ <u>Richardson element</u>.

"(nilpotent) orbit closure" — note $x \in \mathfrak{u} \subseteq \mathrm{Nil}(\mathfrak{g})$ so x nilpotent.

<u>Thm</u> ρ_{Spr_P} is projective. But <u>NOT</u> always birational! (True for $\mathfrak{g} = \mathfrak{sl}_n$.)

Actually get resol. of sings, $T^*G/P \twoheadrightarrow \mathrm{Aff}(T^*G/P)$
$$\downarrow \text{finite-to-1}.$$
$$\overline{G\cdot X}$$

Two issues:
- $\mathrm{Aff}(T^*G/P) \twoheadrightarrow \overline{G\cdot X}$ not always 1-1
- $\overline{G\cdot X}$ not always normal.

$\}$ Both true for $\mathfrak{g} = \mathfrak{sl}_n$ (and many cases). Non-normal is typically also.

The entire Nil cone, $\mathrm{Nil}(\mathfrak{g})$, is normal, thanks to:

Thm (Kostant) $\mathrm{Nil}(\mathfrak{g}) \underset{\text{scheme-theoretically}}{=} Z(C_1, \dots, C_r)$,

C_i = generators of $\mathcal{O}(\mathfrak{g})^G \underset{\text{Chevalley}}{\cong} \mathcal{O}(\mathfrak{h})^W \underset{\text{Chevalley}}{\cong} \mathbb{C}[x_1, \dots, x_r]$

$r = \operatorname{rank} \mathfrak{g} := \dim \mathfrak{h}$.

$\therefore$ $\mathrm{Nil}(\mathfrak{g})$ is a complete intersection and reduced
$$\Downarrow$$
S_2 (Serre condition)

Thm II $\mathrm{Nil}(\mathfrak{g})$ is smooth outside a codimension-2 subset
"R_1 (Serre condition)"

The two theorems together imply $\mathrm{Nil}(\mathfrak{g})$ is normal ($\underset{\text{Serre}}{=} S_2 + R_1$).

Stronger version of second theorem follows from some general considerations:

Meaning of $G\cdot x$ (orbit):

Let $\mathfrak{g}$ = any f.d. Lie alg, $\mathfrak{g} = \mathrm{Lie}\, G$, G connected. Let $X = \mathfrak{g}^*$, Poisson.

[Rmk: For $\mathfrak{g}$ = semisimple Lie alg, $\mathfrak{g} \cong \mathfrak{g}^*$ via Killing form (or trace pairing for $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$)]

$(\mathrm{Ad}\, G)\cdot \varphi \subseteq \mathfrak{g}^*$, for $\varphi \in \mathfrak{g}^*$, called coadjoint orbit.

Prop (Kostant–Kirillov–Souriau): The coadjoint orbits $G\cdot\varphi$ are precisely the symplectic leaves.

Defn A symplectic leaf is an orbit (integral) of the Hamiltonian foliation.

X Poisson:

Hamiltonian v.fds: $\xi_f := \{f, -\}$, $f \in \mathcal{O}_X$.
(local or global)

Recall if X is smooth, it is symplectic with $i_{\xi_f}(\omega) = df$ $\Leftrightarrow \{-,-\}$ NONDEGENERATE.

Defn $Z \subseteq X$ is a closed Poisson subvariety if $I_Z \subseteq \mathcal{O}_X$ (ideal sheaf) is a Poisson ideal.

Exercise: Equivalently, $\forall z \in Z$, $\forall f \in \mathcal{O}_X$ local section, $\xi_f|_z \in T_z Z$.

all Hamiltonians ξ_f are parallel to Z
$\Leftrightarrow \{f, I_Z\} \subseteq I_Z \; \forall f \in \mathcal{O}_X$.

Defn $U \subseteq X$ symplectic leaf is a maximal connected locally closed subvariety such that:

- $\overline{U}$ is a closed Poisson subvariety
- U is nondegenerate ($=$ symplectic) in $\overline{U}$.

$\Leftrightarrow \{\forall u \in U,\ T_u U = \mathrm{Span}(\xi_f|_u)\}$ f loc.

Ex X is smooth conn. symplectic $\Leftrightarrow X$ is a symplectic leaf.

$\triangle$ Although the Jacobi identity for $\{-,-\}$ implies the Hamiltonian foliation is integrable:
$$\{\xi_f, \xi_g\} = \xi_{\{f,g\}} \quad (\Leftrightarrow \text{Jacobi id.})$$
it need only integrate to immersed analytic (or local analytic) leaves, not to algebraic ones.

Ex: On $\mathbb{C}^\times \times \mathbb{C}^\times$, the vector field $\xi = x\partial_x - cy\partial_y$, $c \in \mathbb{C} \setminus \mathbb{Q}$ irrational, spans an abelian Lie algebra ($[\xi,\xi]=0$), but the local / immersed leaves are level sets of $x^c y$, NOT algebraic (or globally analytic)

Exer: Use the previous example to cook up an algebraic Poisson variety where the immersed analytic symplectic leaves are NOT algebraic (or globally analytic $=$ embedded).

Upshot: X need not be a union of (or even have any) algebraic symplectic leaves. But in many examples it is such a union.

G = conn alg gp, $\mathfrak{g} = \mathrm{Lie}\, G$.

Prop (Kirillov–Kostant–Souriau): $G\cdot\varphi$, $\varphi \in \mathfrak{g}^*$ are the (alg.) symplectic leaves of $\mathfrak{g}^*$ (in particular it is a union of them),

$\omega_{G\cdot\varphi}$ is given by: • $T_\varphi G\cdot\varphi$ is spanned by $\xi_x|_\varphi$, $x \in \mathfrak{g}$

• $\omega_{G,\varphi}(\xi_x, \xi_y) = \varphi(\xi_{[x,y]})$, $x, y \in \mathfrak{g}$.

[Or just say: restriction of $\{-,-\}$ on $\mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}\,\mathfrak{g}$.]

Case $\mathfrak{g}$ semisimple: $\mathfrak{g}^* \cong \mathfrak{g}$ under the Killing form

$$\mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto \mathrm{tr}(\mathrm{ad}\,x\, \mathrm{ad}\,y).$$

$\Rightarrow \mathrm{Nil}(\mathfrak{g}) \subseteq \mathfrak{g}$ is closed Poisson.

Thm (Richardson): $\mathrm{Nil}(\mathfrak{g})$ is a union of finitely many adjoint orbits (called nilpotent orbits).

$\mathfrak{g} \xrightarrow{c} \mathfrak{g}/\!/G \simeq \mathfrak{h}/W$
c is Poisson, $\mathfrak{g}/\!/G$ has trivial Poisson str.
$\Rightarrow c^{-1}(\bar{h}) \subseteq \mathfrak{g}$ closed Poisson $\forall \bar{h} \in \mathfrak{h}/W$
$c^{-1}(0) = \mathrm{Nil}(\mathfrak{g})$.
$\mathcal{O}(\mathfrak{g}/\!/G) = Z(\mathcal{O}(\mathfrak{g})) = \mathcal{O}(\mathfrak{g})^{\mathfrak{g}}$?

$\overset{\text{KKS}}{\Longrightarrow}$ $\mathrm{Nil}(\mathfrak{g})$ has finitely many symplectic leaves.

General statement: $\boxed{X \text{ a normal variety}}$

$\exists$ proper (or projective) symplectic resolution $\tilde{X} \to X$ of X

$\Downarrow$ by defn

X has "symplectic singularities"

$\Downarrow$ Thm (Kaledin)

X has finitely many symplectic leaves

So Springer resolution $\Rightarrow$ Richardson's thm (but usually Richardson's thm is used to show that the Springer map is birational - to see also problem sheet 1.

Last time we gave some examples of symplectic resolutions. More comments:

1) For any $\Gamma \leq SL_2\mathbb{C}$ finite $\rightsquigarrow$ $\mathbb{C}^2/\Gamma$ "du Val/Kleinian" sing.

The minimal resolution $\rho: \widetilde{\mathbb{C}^2/\Gamma} \twoheadrightarrow \mathbb{C}^2/\Gamma$ is symplectic.

<u>McKay correspondence</u>: The zero fibre $\rho^{-1}(0)$ is a union of $\mathbb{P}^1$'s (ex: $\Gamma = \mathbb{Z}/n$:  $n-1$ $\mathbb{P}^1$'s)

Intersection graph is ADE Dynkin $\cong$ the graph whose vertices are the nontrivial irreps of Γ,

$\rho_i \text{---} \rho_j \iff \mathrm{Hom}_\Gamma(\rho_i, \mathbb{C}^2 \otimes \rho_j) \neq 0.$

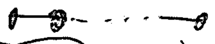
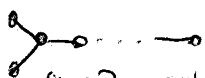
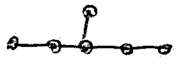
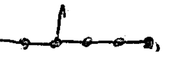
<u>Fact (Brieskorn, Slodowy)</u>: The resolution fits into Springer resolution of that ADE type:

$\mathbb{C}^2/\Gamma \cong$ a transverse slice to the unique codim 2 nilpotent orbit in $Nil(\mathfrak{g})$ (= symplectic leaf)

$\widetilde{\mathbb{C}^2/\Gamma} \cong$ its preimage under $\rho: T^*G/B \twoheadrightarrow Nil(\mathfrak{g})$.

<u>Rmk Quantisations of $\widetilde{\mathbb{C}^2/\Gamma}$ studied by M. Boyarchenko while a PhD student!</u>

Explicitly:

- $\Gamma = \mathbb{Z}/n$: graph =  $n-1$ vertices, Lie algebra: $\mathfrak{sl}_n$
- $\Gamma = \widetilde{D}_{2n}$: graph =  ($n+2$ vertices) Lie algebra $= \mathfrak{so}_{2(n+2)}$
- $\Gamma = \widetilde{A}_4$ (binary tetrahedral): graph =  Lie alg $= e_6$
- $\Gamma = \widehat{S}_4$:  e_7 • $\Gamma = \widetilde{A}_5$:  e_8

2) $\mathrm{Hilb}^n \mathbb{C}^2 \twoheadrightarrow \mathrm{Sym}^n \mathbb{C}^2 = \mathbb{C}^{2n}/S_n$

More generally, for X = symplectic surface,

$\mathrm{Hilb}^n X \twoheadrightarrow \mathrm{Sym}^n X = X^n/S_n$

Composing these constructions (1+2):

$\mathrm{Hilb}^n \widetilde{\mathbb{C}^2/\Gamma} \twoheadrightarrow \mathrm{Sym}^n \mathbb{C}^2/\Gamma \cong \mathbb{C}^{2n}/(\Gamma^n \rtimes S_n)$

"wreath product group"

Remark: these are also special cases of quiver varieties, using the McKay graph of Γ (extended + framed, with appropriate dimension vector)

3) Finite linear quotients ($\mathbb{C}^{2n}/G$, $G < Sp_{2n}\mathbb{C}$ finite):

Thm (many authors): The groups s.t. $\mathbb{C}^{2n}/G$ admits a symplectic resolution are products of:
- wreath product groups $\Gamma^n \rtimes S_n \circlearrowright \mathbb{C}^{2n}$
- Two exceptional cases in $Sp_4\mathbb{C}$
- At most finitely many cases to check (in dims ≤ 10), but we doubt there are any other examples.

This motivates symplectic singularities:

Defn (Beauville): X is a symplectic singularity if:

a) it is normal b) X_{reg} (= smooth locus) has a symplectic form ω_{reg}

c) For some ($\Leftrightarrow$ every) resolution of sings $\tilde{X} \twoheadrightarrow X$,

$p^*\omega_{reg}$ extends to a regular 2-form. [NOT necessarily nondegenerate!]

This is less restrictive. Ex: $\mathbb{C}^{2n}/G$ ALWAYS symp sing, $G < Sp_{2n}\mathbb{C}$ finite.

Same is unfortunately <u>NOT</u> in general <u>true for $V/\!\!/\!\!/G$</u>, $G < Sp_{2n}\mathbb{C}$ infinite (= positive-dimensional.)

Case of $\mathfrak{g}$ = f.d. ss, $X = Nil(\mathfrak{g}) \subseteq \mathfrak{g}$: Namikawa proved:

<u>Thm</u> A symplectic singularity is a homogeneous complete intersection in $\mathbb{A}^n \iff$ it is $\cong Nil(\mathfrak{g})$, $\mathfrak{g}$ f.d. ss!

$\rightsquigarrow$ Favorite definition of Allen Knutson of a f.d. ss Lie alg: <u>homogeneous complete intersection symplectic singularity.</u>

<u>Note</u>: $\Leftarrow$ is Kostant's theorem:

$Nil(\mathfrak{g})$ is <u>scheme-theoretically</u> cut out by $c_1, \dots, c_r$,

$(\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}} \overset{\text{Chevalley}}{\cong} \mathbb{C}[\mathfrak{h}^*]^W \cong \mathbb{C}[x_1, \dots, x_r]$, $r = \mathrm{rank}\,\mathfrak{g} := \dim \mathfrak{h}$

$c_1, \dots, c_r \longmapsto x_1, \dots, x_r$. W = Weyl group

$\Rightarrow$ it is normal and a complete intersection.

Thanks to the S.R. $T^*G/B \longrightarrow Nil\,\mathfrak{g} \Rightarrow Nil(\mathfrak{g})$ is a <u>symplectic singularity</u>.