Lecture 4: Symplectic singularities and resolutions.

Recap: <u>Defn</u> A <u>symplectic singularity</u> is a variety X s.t.

a) X is <u>normal</u>

b) $\exists\ \omega_{reg}$ = symp. str. on X_{reg} (= smooth locus)

c) For some ($\Leftrightarrow$ every) resolution $\rho: \tilde{X} \twoheadrightarrow X$, then $\rho^* \omega_{reg}$ extends to $\tilde{X}$. ($\tilde{\omega} \in \Gamma(\tilde{X}, \Omega^2_{\tilde{X}})$, unique)

<u>Defn</u> ρ as above is a <u>symplectic resolution</u> $\Leftrightarrow$ $\tilde{\omega}$ is <u>nondegenerate</u> ($\Rightarrow \tilde{X}$ symplectic).

<u>Rmk</u> Conversely, if $\rho: \tilde{X} \twoheadrightarrow X$ is resol^{n} of sings with $(\tilde{X}, \tilde{\omega})$ symplectic $\Longrightarrow$ X is a symp. sing, $\exists\ \omega_{reg}$ on X_{reg} s.t. $\tilde{\omega}$ extends $\rho^* \omega_{reg}$.

<u>Rmk</u> Symplectic singularity $\Rightarrow$ Poisson.

Indeed X_{reg} symplectic $\Rightarrow$ it is Poisson. But X normal, $\operatorname{codim}(X \setminus X_{reg}) \geq 2$ (implied by normality) $\Rightarrow$ every function on X_{reg} extends (uniquely) to X.

So $\{f, g\}$ = unique extn of $\{f|_{X_{reg}}, g|_{X_{reg}}\}$.

[Can do this locally on X as well.]

<u>Rmk</u> A symplectic resolution as in the first two definitions is automatically Poisson, since it is over X_{reg} (where ρ is an isomorphism).

$\therefore$ If X is Poisson and $\rho: \tilde{X} \twoheadrightarrow X$ is a symplectic resolution which is also Poisson, then X_{reg} is symplectic and we are in the situation of the defns.

<u>I.e.</u>: The following are equivalent, where all varieties are <u>normal</u>:

{(Poisson) Symplectic resolutions of Poisson varieties} $\longleftrightarrow$ {Symplectic resolutions of symplectic varieties} $\longleftrightarrow$ {resolutions of singularities with symplectic source}

[by existence of a unique symplectic form on the regular locus]

Some general properties / characterisations:

Proposition (Beauville): A symplectic singularity is rational Gorenstein, and canonical.

Proof: $\omega_{reg}^{\dim X/2}$ is a nonvanishing generator of $\Omega_{X_{reg}}^{\dim X_{reg}} = K_{X_{reg}}$, which extends to a holomorphic form on a resolution $\Leftrightarrow$ $\boxed{\text{canonical of index 1.}}$ By Reid [Canonical 3-folds] $\Leftrightarrow$ rat. Gorenstein. $\square$

Prop (Beauville) A symplectic singularity which is locally a complete intersection has singular locus in codimension ≤ 3

$\rightsquigarrow$ So isolated symplectic singularities in dim >2 are NOT lci

(e.g. $T^*\mathbb{P}^n \twoheadrightarrow$ min. nilpotent orbit in $\mathfrak{sl}_n$ $\cup \{0\}$ $\leftarrow$ NOT lci.)

Rmk min. nilp. orbit = isolated symplectic singularity with smooth projective tangent cone. (Beauville)

Prop (Beauville): If V = symplectic sing., G = finite group of symplectomorphisms, then V/G = symplectic sing.

Sketch of proof:

$$\exists \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[r, swap, "\text{resolution}"] \arrow[d] & V \arrow[d] \\ Y \arrow[r, "g"] \arrow[r, swap, "\text{any resolution}"] & V/G \end{tikzcd}$$

V/G — has $\omega_{(V/G)_{reg}}$ by extending from V_{free}/G ($V \setminus V_{free}$ has codim ≥ 2 as G acts symplectically)

$g^*\omega_{(V/G)_{reg}}$ extends to X (= no poles) $\Rightarrow$ extends to Y. (= no poles)

Prop: If X is a symplectic singularity, then a resolution of sings $\tilde{X} \twoheadrightarrow X$ is symplectic $\Leftrightarrow$ it is crepant.

Proof ($\Leftarrow$) $\omega_{reg}^{\dim X/2}$ trivialises K_X, and pulls back to a trivialisation of $K_{\tilde{X}} \Leftrightarrow \tilde{\omega}$ is nondegenerate. [Note: $\Rightarrow$ does not require symp sings]

Theorem (Namikawa): If X is a normal variety and $\exists\, \omega_{reg}$ a symplectic structure on X_{reg}, then X is a symplectic singularity $\Leftrightarrow$ X has rational Gorenstein singularities. [Already proved $\Rightarrow$.]

Theorem (Namikawa): A symplectic singularity is terminal $\Leftrightarrow$ its singular locus has codimension ≥ 4.

Extra

Added in lecture:

Thm (Namikawa): X conical (+normal) $\Rightarrow$ $\exists$ only finitely many S.R's up to $\cong$.

Thm (Kaledin): Two symplectic resolutions of any normal variety X are derived equivalent étale-locally on X.

The proof mostly follows from the theorem: if X is a symplectic singularity, then $\mathrm{Sing}(X) := X \setminus X_{reg}$ has no 3-dimensional irreducible components. This is now a consequence of the stronger statement (last time):

<u>Thm (Kaledin)</u> A symplectic singularity is the union of finitely many (algebraic) symplectic leaves.

Recall: [A symplectic leaf is locally closed of <u>even</u> dimension.]
[The open symplectic leaves are the connected components of X_{reg}.]
$\therefore \nexists$ codim 3 components of the singular locus.

Classification results: Last time:

<u>Thms (Namikawa)</u>: 1. If a symplectic singularity is a homogeneous complete intersection in $\mathbb{A}^n$, then it is $\cong \mathcal{N}il(\mathfrak{g})$ for some f.d. semisimple Lie alg $\mathfrak{g}$.

2. More generally, if a symp. sing. X is <u>conical</u> ($\Leftrightarrow$ $\mathcal{O}(X)$ is connected + ≥ 0-graded + X is affine) and $\mathcal{O}(X)$ is generated <u>in degree 1</u>, then $X \cong \overline{G \cdot e} \subseteq \mathcal{N}il\,\mathfrak{g}$, some $e \in \mathcal{N}il\,\mathfrak{g}$, some $\mathfrak{g}$ (f.d. s.s.).

Note: these are all symplectic singularities up to normalisation:

<u>Thm (Beauville / Panyushev)</u>: Each symplectic leaf closure $\overline{G \cdot e} \subseteq \mathcal{N}il\,\mathfrak{g}$ is a symplectic singularity $\boxed{\text{after normalisation.}}$

<u>Sketch of Proof</u>: By the <u>Jacobson-Morozov</u> theorem, $e \in \mathcal{N}il\,\mathfrak{g}$ can be extended to an $\mathfrak{sl}_2$-triple (e,h,f) (i.e. satisfying relations of $\mathfrak{sl}_2$), $e,h,f \in \mathfrak{g}$. Let $\mathfrak{n} := \bigoplus_{i \geq 2} \underbrace{\ker(h-i)}_{\text{"}\mathfrak{g}_i\text{"}} \subseteq \mathfrak{g}$, a nilpotent subalgebra, and $P := \exp\left(\bigoplus_{i \geq 0} \mathfrak{g}_i\right) \leq G$, a subgroup.

Now $\underline{P \cdot e}$ is dense in $\mathfrak{n}$ by a result in Carter's book (Prop 5.7.3) "Finite groups of Lie type"
$\Rightarrow \overline{G \cdot \mathfrak{n}} = \overline{G \cdot e}$. Now form $\rho : G \times_P \mathfrak{n} \to \mathfrak{g}$, $(g, x) \mapsto g \cdot x$. $\Rightarrow \mathrm{im}(\rho) = \overline{G \cdot e}$.

But $G \times_P \mathfrak{n}$ has a regular 2-form ω':

$$T_{(g,x)}(G \times_P \mathfrak{n}) = \mathfrak{g} \oplus \mathfrak{n} / \mathfrak{p}' = \{(-z, [x,z]) \mid z \in \mathfrak{p}\}$$

$$\rightsquigarrow \omega'((A_1, y_1), (A_2, y_2)) = \kappa(y_2, A_1) - \kappa(y_1, A_2) + \kappa(x, [A_1, A_2]),\ \kappa\text{-killing form.}$$

Computation $\Rightarrow \omega'$ extends $\rho^* \omega_{G \cdot e}$. $\square$

But <u>NOT ALL</u> symplectic leaf closures $\overline{G \cdot e}$ are normal. I wrote about this on slack: • $\mathfrak{sl}_n$: All are normal • $\mathfrak{so}_n$, $\mathfrak{sp}_n$: Some are "2-branched", i.e. a ~~resolution~~ normalisation $\nu : \tilde{X} \twoheadrightarrow X = \overline{G \cdot e}$ is 2-branched over some $G \cdot e' \subseteq \partial(G \cdot e)$.

$\mathfrak{g}_2, \mathfrak{e}_6$: Normal $\overline{G\cdot e}$ are classified. For $\mathfrak{g}_2$ all but one is normal, but the non-normal one is unibranch $\rightsquigarrow$ normalisation $\cong \underbrace{\min(\mathfrak{so}_7)}$, bijective (homeom.)

$\mathfrak{e}_7, \mathfrak{e}_8$: Not fully classified but there is an expected answer (in Broer 1998)

Note: Representation theory (the Springer correspondence $\{\text{irreps of Weyl group}\} \hookrightarrow \{G\cdot e, \text{Loc Syst.}\}$) implies a formula for the # of branches under normalisation $\Rightarrow$ we can detect all nonnormal orbits that are NOT unibranch.

So Namikawa's result says:
The normal nilpotent orbit closures $\overline{G\cdot e}$ = those s.t. $\mathcal{O}(\overline{G\cdot e})$ is generated in degree 1.

Going beyond degree 1:

Thm (Namikawa): For each dimension $2n$ and degree bound N, $\exists$ only finitely many conical symplectic singularities X of dimension $2n$ ~~gener~~ with $\mathcal{O}(X)$ generated in degrees $\leq N$, up to isomorphism.

Ex: dimension 2: All symplectic singularities are locally (formally or étale) du Val singularities, since they are rational Gorenstein (and these are classified).

$\mathbb{C}^2/\Gamma$ List of these: [Recall: $\Gamma \rightsquigarrow$ Dynkin diagram D, vertices = nontriv irreps; $\mathbb{C}^2/\Gamma$ = transverse slice to $\overline{G\cdot e} \subseteq Nil\, \mathfrak{g}_D$, codim $G\cdot e = 2$]

Γ	Dynkin diagram	Equation	Max degree
$\mathbb{Z}/_{n+1}$	A_n $(\mathfrak{sl}_{n+1})$	$xy = z^{n+1}$, $n\geq 1$	$n+1$ (or $\frac{n+1}{2}$, n odd)
$\widetilde{D}_{2(n-2)}$	D_n $(\mathfrak{so}_{2n})$	$x^2+zy^2+z^{n-1}=0, n\geq 3$ (4)	$n-1$ $(n-1, n-2, 2)$
$\widetilde{A}_4$	E_6 $(\mathfrak{e}_6)$	$x^2+y^3+z^4=0$	6 $(6,4,3)$
$\widetilde{S}_4$	E_7 $(\mathfrak{e}_7)$	$x^2+y^3+yz^3=0$	9 $(9,6,4)$
$\widetilde{A}_5$	E_8 $(\mathfrak{e}_8)$	$x^2+y^3+z^5=0$	15 $(15,10,6)$

$\rightsquigarrow$ only type A_1 is generated in degree 1 $(\mathbb{C}^2/\mathbb{Z}_2 \cong Nil(\mathfrak{sl}_2))$.

In contrast to the situation for finite quotients and nilpotent orbit closures, Hamiltonian reductions need not be symp. sings:

Example: $(\mathbb{C}^{2n})^{\oplus 2m} /\!/\!/ Sp_{2n}$: $\boxed{\text{Thm (Becker): } \bullet \text{ reduced if } m \geq 2n \quad \bullet \text{ normal if } m\geq 2n+1}$

[reduced = smooth in codim 0 (generically); normal $\Rightarrow$ smooth in codim 1 (outside codim 2)]

Simple calculation shows: $(\mathbb{C}^2)^2 /\!/\!/ Sp_2 = SL_2$ NOT reduced

[for $A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix} \in (\mathbb{C}^2)^2$, then $(\det A)^2 \in I_{\mu^{-1}(0)}$ but $\det A \notin I_{\mu^{-1}(0)}$]

Similarly, $(\mathbb{C}^2)^4 /\!/\!/ SL_2$ NOT normal.

In general, $(\mathbb{C}^{2n})^{2n} /\!/\!/ Sp_{2n} \cong$ an adjoint orbit for O_{2n} which has two components, the adjoint orbits for SO_{2n}.

> Herbig—Seaton—Schwarz: this can be fixed (conjecturally in all cases) by instead complexifying a real quotient where we first pass to the real radical of $\mu_{\mathbb{R}}^{-1}(0)$, i.e. the real ideal of all functions vanishing on the real locus.

Nonetheless Losev proves:

Prop (Losev): For $G \leq Sp(V)$ reductive, (V symplectic), $V /\!/\!/ G$ has finitely many symplectic leaves.

Rmk: Although in general a Poisson variety X is not a union of its algebraic symplectic leaves (or need not have any at all), if it has, locally, finitely many analytic symplectic leaves, then these leaves must be algebraic and X is the union of them. Indeed, then the symplectic leaves are the connected components of $X_r := \{x \in X \mid \dim(\mathrm{Span}(\xi_f|_x)) = r\} \underset{\text{loc clsd}}{\subseteq} X$. [Exer.]

General picture:

generically symplectic $\Longleftarrow$ finitely many symplectic leaves $\Longleftarrow$ symplectic singularities $\Longleftarrow$ exists a symplectic resolution

- finitely many symplectic leaves: linear reductive Ham. reductions
- symplectic singularities: Normalised Nilpotent orbit closures; finite linear quotients; Nakajima quiver varieties; Hypertoric varieties
- exists a symplectic resolution: Nilpotent cones, certain nilpotent orbits (normalised); $\mathrm{Sym}^n X$, X = symp surface OR du Val sing.; "Most" Nak.-quiver varieties, Most hypertoric varieties

Thm (Bellamy—S.): Nakajima quiver varieties are symplectic singularities. "Most" admit symplectic resolutions: we classify these.

~~Prototypical case: quiver = $\underset{n}{\bullet}$ g loops, which means~~
~~$X = T^*(\mathfrak{gl}_n)^g /\!/\!/ GL_n$ (or PGL_n):~~ ~~$n=1$: Get $\mathbb{C}^{2g}$ (trivial)~~
~~$g=1$: Get $T^*\mathfrak{gl}_n /\!/\!/ GL_n \cong \mathbb{C}^{2n}/S_n \leftarrow \mathrm{Hilb}^n \mathbb{C}^2$~~
~~$(g,n)=(2,2)$: O'Grady resolved by one blowup.~~ OTHERS: $\nexists$ S.R.

Prototypical case:  g arrows, meaning:

$X = T^*(\mathfrak{gl}_n)^g /\!/\!/ GL_n$. <u>Cases</u>: $n=1$: $X = \mathbb{C}^{2g}$ smooth

$g=1$: $X = T^*\mathfrak{gl}_n /\!/\!/ GL_n \underset{\text{Gan–Ginzburg}}{\cong} \mathbb{C}^{2n}/S_n \twoheadleftarrow \mathrm{Hilb}^n\mathbb{C}^2$ sympl.

$(g,n)=(2,2)$: O'Grady observed that a symplectic resolution Joseph $\tilde{X} \twoheadrightarrow X$ is given by blowing up the <u>set-theoretic</u> (ie reduced ideal of the) singular locus (one time).

OTHER cases: Thm (Kaledin–Lehn–Sorger): $\nexists$ symplectic resolution of singularities.

Why? They show that these are <u>factorial</u> + <u>terminal</u>

factorial $\Leftrightarrow$ local rings are UFDs $\Leftrightarrow$ Every Weil divisor is Cartier.

terminal $\Leftarrow$ symp sing, $\operatorname{codim}(\operatorname{Sing}(X)) \geq 4$

<u>More general</u>:
<u>Defn</u> $\mathbb{Q}$-factorial means some multiple of every Weil divisor is Cartier.

Van de Waerden purity: If X is factorial (or even $\mathbb{Q}$-factorial), and $\rho: \tilde{X} \twoheadrightarrow X$ is a resolution of singularities, then the exceptional locus (ρ is not injective) is a <u>divisor</u>.

But <u>terminal</u> means that:

$$K_{\tilde{X}} = \rho^* K_X + \sum a_i E_i, \quad E_i \text{ are the exceptional divisors, } \underline{\underline{a_i > 0}}\ \forall i.$$

So terminal + ($\mathbb{Q}$-)factorial $\Rightarrow K_{\tilde{X}} \neq \rho^* K_X$, ie., ρ <u>NOT</u> <u>crepant</u>. So $\nexists$ crepant resolution $\Rightarrow \nexists$ S.R.

<u>BCHM, Corollary 1.4.3</u> (minimal model programme) implies that every symplectic singularity admits a <u>relative</u> minimal model which is $\mathbb{Q}$-factorial.

Here it implies: $\exists$ <u>crepant</u> $\rho: \tilde{X} \twoheadrightarrow X$, $\tilde{X}$ <u>terminal + $\mathbb{Q}$-factorial</u>.
X = symp sing

"<u>$\mathbb{Q}$-factorial terminalisation</u>" $\Rightarrow$ $\boxed{\tilde{X} \text{ has NO S.R, if sing.}}$

<u>Thm (Namikawa)</u>: If X is conical, then one $\mathbb{Q}$-factorial terminalisation is smooth $\Longleftrightarrow$ they <u>all</u> are. So $\tilde{X}$ is smooth $\Leftrightarrow$ X admits a S.R.