Symplectic Representation Theory: Sheet 1

Travis Schedler

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The starred exercises will be collected on Thursday, 14 Feb, 2019. If you are taking the course for credit, please attempt these and hand them on on that date.

1. (a) Let G be an algebraic group acting on a variety X. Let $a : G \times X \to X$ be the action and $p : G \times X \to X$ be the second projection. Recall that a sheaf $\mathcal{F}$ on X is G-equivariant if we are given an isomorphism $a^*\mathcal{F} \simeq p^*\mathcal{F}$. Prove that, if $\mathcal{F}$ is G-equivariant, then $\Gamma(X, \mathcal{F})$ is a G-representation.

   (b) Let $X = G \times_H V := (G \times V)/H$ where $H < G$ is a subgroup and V is a variety with an action of H. Prove that G-equivariant sheaves on X are the same as H-equivariant sheaves on V. More precisely, the pullbacks under $G \times V \to X$ and $G \times V \to V$ induce equivalences of categories between each of these and $(G \times H)$-equivariant sheaves on $G \times V$. (Note: in terms of stacks, G-equivariant sheaves on X are, by definition, sheaves on the stack quotient X/G, so part (b) is just saying that $G \setminus (G \times V)/H = V/H$.)

   (c) Show that the cotangent bundle T^*G/B is isomorphic to $G \times_B \mathfrak{b}^\perp$ where $\mathfrak{b}^\perp \subseteq \mathfrak{g}^*$ is the annihilator of $\mathfrak{b}$. Under the Killing form (for G semisimple) this identifies with $G \times_B \mathfrak{n}$ for $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

   (d) Using (b) and (c), show that every line bundle $\mathcal{O}(\lambda)$ is indeed G-equivariant and when λ is dominant, compute its global sections.

2. Let $G = \mathrm{SL}_n(\mathbf{C})$ and $\mathfrak{g} = \mathfrak{sl}_n$. (For general complex semisimple Lie groups, see, e.g., Chriss–Ginzburg, Section 3.)

   (a) $\star$ Define the Springer map $\rho : T^*G/B \cong G \times_B \mathfrak{b}^\perp \to \mathfrak{g}^*$, by $\rho(g, x) = \mathrm{Ad}(g)(x)$. First, explain why we can instead define this as

   $$\rho : G \times_B \mathfrak{n} \to \mathfrak{g}, \quad \rho(g, x) = \mathrm{Ad}(g)(x).$$

   (More precisely, using the trace pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbf{C}, (x, y) \mapsto \mathrm{Tr}(x, y)$, or the Killing form, identify $\mathfrak{b}^\perp \cong \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{g} \cong \mathfrak{g}^*$, compatibly with the G action). Next, with this done, show that the image in $\mathfrak{g}$ is indeed the subset of ad-nilpotent elements, called $\mathrm{Nil}(\mathfrak{g})$. We get a map, called the Springer resolution, $\rho : T^*G/B \to \mathrm{Nil}(\mathfrak{g})$.

   (b) $\star$ Prove that the Springer resolution can also be described as

   $$\{(\mathfrak{b}, x) \mid x \in [\mathfrak{b}, \mathfrak{b}], \mathfrak{b} \subseteq \mathfrak{g} \text{ a Borel}\} \to \mathrm{Nil}(\mathfrak{g}), \quad (\mathfrak{b}, x) \mapsto x.$$

You may use the fact, from Lie theory, that all Borel subalgebras of a semisimple Lie algebra are conjugate and that the normalizer $N_G(\mathfrak{b})$ is B (although this is easier to prove for $\mathfrak{sl}_n$ than in the general case).

(c) $\star$ Using (b), show that the Springer resolution is projective and birational.

(d) $\star$ The Grothendieck–Springer resolution can be described by $\rho : G \times_B \mathfrak{b} \to \mathfrak{g}$, $(g, x) \mapsto \mathrm{Ad}(g)(x)$. Show that this is surjective. Find the size of the fiber over regular nilpotent elements and over regular semisimple elements (in both cases, "regular" means that the centraliser has the minimum dimension, equal to the dimension of a Cartan subalgebra, called the *rank* of $\mathfrak{g}$; for regular semisimple elements, this is the same as saying that the image in the adjoint quotient $\mathfrak{g}//G$ followed by the Chevalley isomorphism to $\mathfrak{h}/W$ (the characteristic polynomial map) is a free $W = S_n$-orbit, i.e., one not touching a root hyperplane).

Hint 1: A regular semisimple element here is just a diagonalisable matrix with distinct eigenvalues. Show that a Borel subalgebra contains such a diagonal matrix if and only if it contains all diagonal matrices and prove that corresponding flags are invariant under invertible diagonal matrices (the standard torus).

Hint 2: Show that if you have a regular nilpotent element, then multiplication by this element determines a complete flag, and that this corresponds to a Borel containing the element.

(e) $\star$ Similarly to before, prove that the Grothendieck–Springer resolution can also be described as

$$\{(\mathfrak{b}, x) \mid x \in \mathfrak{b}, \mathfrak{b} \subseteq \mathfrak{g} \text{ a Borel}\} \to \mathfrak{g}, \quad (\mathfrak{b}, x) \mapsto x.$$

Deduce that it is also projective. Is it birational?

(f) Consider the commutative diagram,

$$\begin{tikzcd} G \times_B \mathfrak{b} \arrow[r, "\rho"] \arrow[d, "\widetilde{q}"] & \mathfrak{g} \arrow[d, "q"] \\ \mathfrak{h} \arrow[r] & \mathfrak{h}/W, \end{tikzcd}$$

where $\widetilde{q}(g, x)$ is the image of x in $\mathfrak{h} \cong \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$, and q is the adjoint quotient map $\mathfrak{g} \to \mathfrak{g}//G$ composed with the Chevalley isomorphism (characteristic polynomial map) for $\mathfrak{h}/W$. Show that the left vertical map $\widetilde{q}$ is smooth (in particular having smooth fibers), and that over the regular locus $\mathfrak{h}_{\mathrm{reg}} = \{h \in \mathfrak{h} \mid \alpha(h) \neq 0, \forall \alpha \in \Phi\}$ (with Φ the set of rots), the horizontal map yields an isomorphism $\widetilde{q}^{-1}(h) \cong q^{-1}(W \cdot h)$, i.e., the diagram restricts to a Cartesian one over the regular locus.

3. (a) $\star$ As discussed in detail in Exercise 5 and referenced exercises, one way to define the category of $\mathcal{D}^{\lambda}_{\mathbb{P}^1}$-modules is as the category of graded $\mathcal{D}(\mathbf{A}^2)$-modules modulo torsion modules supported at the origin, such that the Euler operator $\mathrm{Eu} := x\partial_x + y\partial_y$ acts in degree m as multiplication by $m + \lambda$. Prove the Beilinson–Bernstein theorem, that the

functor Γ (of "global sections on $\mathbf{P}^1$") which takes a module to its degree zero component, is an equivalence of abelian categories $\mathcal{D}^\lambda_{\mathbb{P}^1}$-mod $\to \Gamma(\hat{\mathcal{D}}^\lambda_{\mathbb{P}^1}) - mod$, provided that λ is not a negative integer. Here $\hat{\mathcal{D}}^\lambda_{\mathbb{P}^2}$ denotes the $\mathcal{D}^\lambda_{\mathbb{P}^1}$-module $\mathcal{D}(\mathbf{A}^2)/\mathcal{D}(\mathbf{A}^2)(\mathrm{Eu} - \lambda)$, which has the property that $\mathrm{Hom}(\hat{\mathcal{D}}_{\mathbb{P}^1}, M) = \Gamma(M)$ (verify it!): this implies that $\Gamma(M)$ is a module over $\Gamma(\hat{\mathcal{D}}_{\mathbb{P}^1})$.

(b) **Not to hand in**: Show that the last algebra is isomorphic to $U\mathfrak{sl}_2/(C - \frac{1}{2}\lambda^2 - \lambda)$. Show that the above generalises from $\mathbb{P}^1$ to $\mathbb{P}^n$, putting $\mathrm{Eu} = \sum_{i=1}^{n+1} x_i \partial_{x_i}$ and replacing $\mathbf{C}^2$ by $\mathbf{C}^n$.

For the next few exercises, one needs the notion of (weakly) equivariant D-modules: see Exercise 6, or a suitable reference on D-modules (e.g., Section 11.5 of Hotta et al), for details.

4. Let $G = \mathbf{G}_m$ be the multiplicative group ($\mathbf{C}^\times$), and suppose that it acts on an affine variety X. Prove the following: (a) the structure of the action is the same as a $\mathbf{Z}$-grading on X; (b) a G-equivariant sheaf on X is the same thing as a $\mathbf{Z}$-graded $\mathcal{O}(X)$-module; (c) supposing now X is smooth: a weakly G-equivariant $\mathcal{D}_X$-module is the same thing as a $\mathbf{Z}$-graded $\mathcal{D}(X)$-module (included in this is the fact that $\mathcal{D}(X)$ is $\mathbf{Z}$-graded); (d) still supposing X is smooth, a (strongly) G-equivariant $\mathcal{D}_X$-module is the same thing as a $\mathbf{Z}$-graded $\mathcal{D}(X)$ module V such that the Euler vector field Eu acts by $\mathrm{Eu}(v) = |v|v$ for all homogeneous elements $v \in V$; (e) In the case that $X = \mathbf{A}^n$ with the usual scaling action, show that the Euler vector field is $x_1\partial_1 + \cdots + x_n\partial_n$. Note that the grading in this case is by $|x_i| = 1, |\partial_i| = -1$.

   Recollection: The Euler vector field is defined as the image of $z\partial_z \in \mathrm{Lie}G$ in $\mathcal{D}(X)$ under the action, i.e., the vector field Eu such that $\mathrm{Eu}(f) = |f|f$ for all homogeneous functions $f \in \mathcal{O}(X)$.

5. If X is a projective variety, then by descent, $\mathcal{D}_X$-modules are equivalent to (strongly) $\mathbf{G}_m$-equivariant $\mathcal{D}_{Y\setminus\{0\}}$-modules, where Y is the affine cone over X. Show that this category is equivalent to the category of $\mathbf{Z}$-graded $\mathcal{D}(Y)$-modules, such that the Euler operator acts by multiplication by degree, modulo torsion modules supported at the origin.

6. Let G be an algebraic group acting on a smooth variety X. Let $\alpha : G \times X \to X$ be the action map and $p : G \times X \to X$ the second projection. One defines a *G-equivariant left $\mathcal{D}$-module*, $\mathcal{F}$, on X to be a $\mathcal{D}_X$-module $\mathcal{F}$ equipped with an isomorphism of left $\mathcal{D}_{G\times X}$-modules, $a^*\mathcal{F} \cong p^*\mathcal{F}$. Similarly, one defines a *weakly G-equivariant left $\mathcal{D}$-module* to be the same except that we only require the isomorphism to be one of $(\mathcal{O}_G \boxtimes \mathcal{D}_X)$-modules. Prove the following: (a) if X is affine and G is a linear algebraic group, then a weakly G-equivariant $\mathcal{D}$-module $\mathcal{F}$ is the same as a rational G-module $V = \Gamma(X, \mathcal{F})$ together with an action of $\mathcal{D}(X)$ satisfying, for all $g \in G, \Phi \in \mathcal{D}(X)$, and $v \in V$, the identity $g \cdot \Phi(v) = (\alpha(g)^*(\Phi))(g \cdot v)$.

(b) In general, a G-equivariant $\mathcal{D}$-module $\mathcal{F}$ is a weakly G-equivariant $\mathcal{D}$-module $\mathcal{F}$ such that, for every $\xi \in \mathfrak{g}$, the two natural actions of $\mathfrak{g}$, via the homomorphism $d\alpha : \mathfrak{g} \to \mathcal{D}(X)$, and via the equivariant structure, $\mathfrak{g} \to \mathrm{Der}(\mathcal{F})$, agree. In terms of part (a), in the affine setting, this says that, for $\xi \in \mathfrak{g}$, $d\alpha(\xi)(v) = \xi \cdot v$ for all $\xi \in \mathfrak{g}, v \in V$, interpreting the left-hand side as the action via $d\alpha(\xi) \in \mathcal{D}(X)$ and the right-hand side as the action via the equivariant structure.

Recollections for the above problem: See, e.g., Hotta et al, Section 1.3, for the definition of the pullback of a $\mathcal{D}$-module: this is the same as the $\mathcal{O}$-module pullback but one must define the $\mathcal{D}$-module structure. Briefly, $f^*\mathcal{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$, with the $\mathcal{D}_X$-module structure defined by allowing vector fields $\xi \in T_X$ to act locally by $\xi(\psi \otimes s) = \xi(\psi) \otimes s + \psi \otimes f_*\xi(s)$. Here, $f_*\xi(s)$ makes sense since, if s is a section over an open subset $U \subseteq X$, then it corresponds to a section $\tilde{s}$ over $f(U)$ (more precisely, over open subsets containing $f(U)$), and we set $f_*\xi(s)(x) = (f_*\xi|_x)(\tilde{s})$. Recall in part (a) that a rational G-module is a comodule over the bialgebra $\mathcal{O}(G)$, or equivalently, a filtered union of finite-dimensional vector spaces W equipped with algebraic maps $G \to \mathrm{GL}(W)$; also, $\alpha(g)^*(\Phi) \in \mathcal{D}(X)$ is defined by $\alpha(g)^*(\Phi)(\psi) = \alpha(g)^*(\Phi(\alpha(g^{-1})^*(\psi)))$.