Symplectic Resolutions and Singularities.
Lectures 5, 6

(1/11)

Today: Quiver varieties.

Given an oriented graph, with multiple edges and loops allowed,

$$Q = \text{"quiver"} = (Q_0, Q_1), \quad \text{e.g.}$$

(coined by Gabriel) "carquois"

lit.: collection of arrows

$Q_0 = \{\text{vertices}\}$, $Q_1 = \{\text{arrows}\}$

→ For simplicity, let our graphs be connected.

Defn A representation of Q is an assignment, to each vertex, of a vector space (over $\mathbb{F}$), and to each arrow, of a linear map.

$$(V_i, \rho_a)_{i \in Q_0, a \in Q_1}.$$

Defn $\dim (V_i, \rho_a) = (\dim V_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$.

Defn A homomorphism of reps $T: (V_i, \rho_a) \to (W_i, \tau_a)$ is $T_i : V_i \to W_i$ $\forall i \in Q_0$, such that:

$\forall a: i \to j$,

$$\begin{tikzcd} V_i \arrow[r, "T_i"] \arrow[d, "\rho_a"'] & W_i \arrow[d, "\tau_a"] \\ V_j \arrow[r, "T_j"] & W_j \end{tikzcd}$$

commutes.

Isomorphism = one s.t. T_i is an isom $\forall i \in Q_0$.

Example. $Q = $ (one vertex with a loop) = "Jordan" quiver = "type $\tilde{A}_0$ quiver"

A rep is just a pair (V, ρ), $\rho: V \to V$ linear.

Exer: Up to isomorphism, this is given by $(\dim V, \mathcal{C})$, $\mathcal{C} \subseteq \mathrm{Mat}_{\dim V} \mathbb{F}$ conjugacy class.

[first write $V \cong \mathbb{F}^{\dim V}$, then $\rho \mapsto X \in \mathrm{Mat}_{\dim V} \mathbb{F}$, unique up to conjugacy.]

(2/12)

<u>Ex</u> $Q = \bullet \rightrightarrows \bullet$ = "Kronecker" quiver = "type $\tilde{A}_1$ quiver" (up to choice of orientation) e.g. $\rightleftarrows$ also type $\tilde{A}_1$.

Now reps are more complicated:
(V, W, ρ_1, ρ_2), $\rho_1, \rho_2 : V \longrightarrow W$ linear.

<u>Special case ρ_1 is an isomorphism</u>: then reps up to isom are given by $(V, \rho_2 \circ \rho_1^{-1})_{/\cong}$ $\rightsquigarrow$ just reps of the Jordan quiver up to isomorphism.
[Same is true for the special case ρ_2 is an isom.]

Ex. The preceding both result in <u>infinitely</u> <u>many</u> isomorphism classes of reps, even fixing $\dim V_i$ $\forall i$.

In contrast: $Q = \bullet \rightarrow \bullet$ = "type A_1 quiver" Dynkin

<u>Exercise</u> Every representation is isomorphic to a direct sum of:
a) $\underset{\mathbb{F}}{\bullet} \xrightarrow{0} \underset{0}{\bullet}$, b) $\underset{0}{\bullet} \xrightarrow{0} \underset{\mathbb{F}}{\bullet}$, c) $\underset{\mathbb{F}}{\bullet} \xrightarrow{I} \underset{\mathbb{F}}{\bullet}$

[Hint: it's just saying that up to change of basis on both sides an $m \times n$ matrix can be brought to $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & 0 & \end{pmatrix}$ or $\begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{pmatrix}$
i.e. $PAQ^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ ~~a~~ in some block form.]

So there are only 3 isoclasses of <u>indecomposable</u> reps (note: (a), (b) are irreducible, but not (c).)

Krull-Schmidt theorem $\Rightarrow$ every rep is, up to isom, uniquely a $\oplus$ of <u>indecomposable</u> ones.

<u>Observe</u>: The positive roots of $\bullet - \bullet$ (Dynkin diag. of $\mathfrak{sl}_3$) are: $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$.

Thm (Gabriel): There are finitely many isoclasses of indecomposable reps $\Longleftrightarrow$ the quiver Q is Dynkin of types A, D, or E (, , , , or), and the dim vectors of these are precisely the positive roots of the associated simple Lie alg (or root system) [one of each].

Thm (Kac): For a general quiver without loops, the dimension vectors of indecomposable reps are precisely the positive roots of the associated Kac-Moody Lie algebra.

(these are infinite-dimensional analogues of simple Lie algebras over $\mathbb{C}$)

Furthermore, given a positive root α, there is a unique isomorphism class of indecomposable reps of this dimension vector $\Longleftrightarrow$ the root is real. Otherwise (the root is imaginary), there are ∞ many isomorphism classes (over $F=\mathbb{C}$)

Ex: Back to the Kronecker quiver:

$\bullet \rightrightarrows \bullet$. Positive Real roots: $\underset{m+1}{\bullet} \rightrightarrows \underset{m+0}{\bullet}$ $\quad \underset{m+0}{\bullet} \rightrightarrows \underset{m+1}{\bullet}$

isom classes here $(m=0)$: one each ($(F,0)$ and $(0,F)$) maps are 0.

positive imaginary roots: $\underset{m}{\bullet} \rightrightarrows \underset{m}{\bullet}$ $\quad \forall m \geq 1$.

We saw that there are ∞ many isoclasses for $m=1$.

What are Kac-Moody Lie algebras?

Given a quiver $Q \rightsquigarrow$ adjacency matrix $A \in Mat_{Q_0}(\mathbb{N})$

$A + A^t$ = adjacency matrix of doubled quiver.

$C :=$ Cartan matrix $:= 2I - (A+A^t)$.

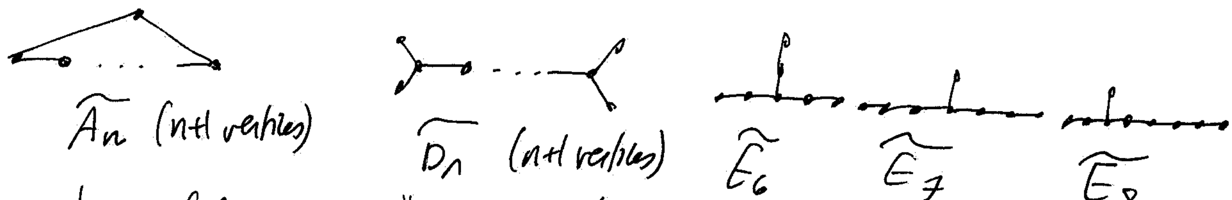
Let Q have no loops.
Then the Kac-Moody algebra associated to Q (or C) can be defined via generators E_i, F_i, H_i, $i \in Q_0$, and relations:

- $[H_i, H_j] = 0$
- $[H_i, E_j] = C_{ij} E_j$
- $[E_i, F_j] = \delta_{ij} H_i$
- $[H_i, F_j] = -C_{ij} F_j$
- For $i \neq j$, $\left.\begin{array}{l} \mathrm{ad}(E_i)^{1-C_{ij}}(E_j) = 0 \\ \mathrm{ad}(F_i)^{1-C_{ij}}(F_j) = 0 \end{array}\right\}$ "Serre relations"

[Usually one also adds in some additional elements to make up for the fact that C can be singular: this is in the non-Dynkin case. See, e.g., Wikipedia. These elements have relations just like the H_i.]

<u>Thm</u> For Q Dynkin, the above "Chevalley generators" & rels define the associated simple complex Lie algebra $\mathfrak{g}$.

<u>Thm</u> For Q "Extended Dynkin":



the above defines an "<u>affine Kac-Moody Lie algebra</u>" $\hat{\mathfrak{g}}$ (but one should add one extra generator) ("Central extension of $\mathfrak{g}[t, t^{-1}]$.")

<u>Defn</u> A <u>root</u> of a Kac-Moody Lie algebra $\mathfrak{g}$ is an element $\alpha \in \mathfrak{h}^*$, $\mathfrak{h} := \mathrm{Span}(H_i)$, such that $\exists X \in \mathfrak{g}$, $[H_i, X] = \alpha(H_i) X$ $\forall i$.

<u>Fact:</u> $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \text{ a root}} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha := \mathrm{Span}(X \text{ as above})$.

"Weight space decomposition" under $\mathfrak{h}$
$=:$ root space decomposition

Ex: $\boxed{\text{Kronecker quiver: } \bullet \rightrightarrows \bullet = \text{type } \widehat{A}_1}$ ($\bullet \rightarrow \bullet$ = type A_2)
($\bullet$ = type A_1)

$\bullet \rightrightarrows \bullet$ Corr to $\widehat{\mathfrak{sl}_2}$ ($\mathfrak{sl}_2$ corr to type A_1). $0 \to \mathbb{C} \hookrightarrow \widehat{\mathfrak{sl}_2} \twoheadrightarrow \mathfrak{sl}_2[t, t^{-1}] \to 0$

central extension.

→ root spaces are · $\mathbb{C}\cdot et^m$ (root $(m+1,m)$)

· $\mathbb{C}\cdot ft^m$ (root $(m,m+1)$)

· $\mathbb{C}\cdot ht^m$ (root (m,m)) [note $\mathbb{C}\cdot h \in \mathfrak{h}$], $m \neq 0$

· $\mathfrak{h} = \mathrm{Span}(h, c)$, $c =$ the central element (from extension).

(the zero weight — NOT a root.

General way to describe the roots combinatorially:

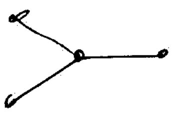
· real roots are given by applying reflections to elementary vectors e_i

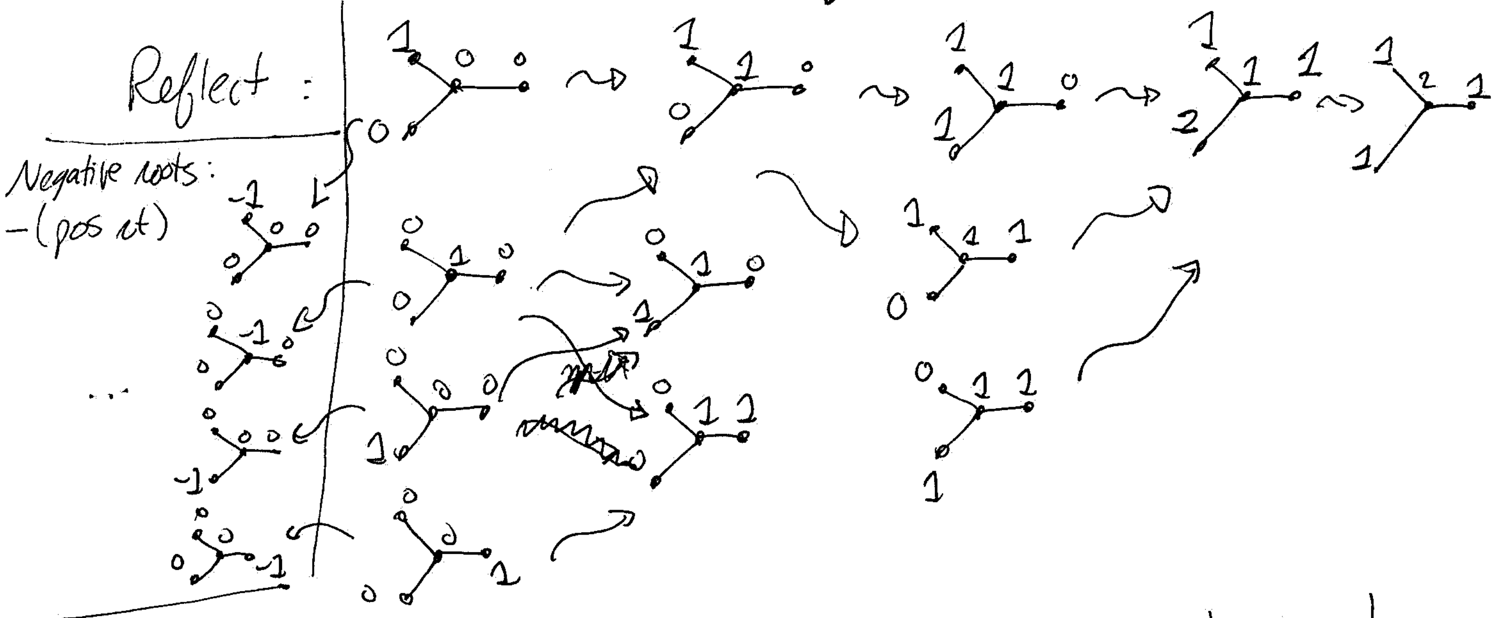
Reflection: $s_i\alpha := \alpha - (\alpha, e_i)e_i$

$(-,-) =$ Cartan pairing, $(e_i, e_j) = C_{ij}$ (Cartan matrix)

Concretely, if there are no loops,

$$(s_i\alpha)_j = \begin{cases} \alpha_j, & j \neq i \\ -\alpha_i + \sum\limits_{\substack{j \to i \\ \text{or } i \to j}} \alpha_j \end{cases}$$

Example: Q of type D_4 

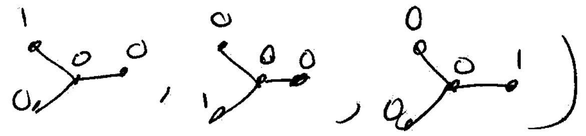


positive roots: 4 + 3 + 3 + 1 + 1 = $\boxed{12}$.

This says (since there are no imaginary roots in the Dynkin case) that there are $\underline{\underline{12}}$ isomorphism classes of irreps of Q.

E.g. "triple of subspaces problem": classify triples $V_1, V_2, V_3 \subseteq V$ of subspaces up to isom.

Solution: 9 parameters, corresponding to the indecomposable reps of  with injective arrows

(all dim vectors except ).

$\rightsquigarrow$ parameters are: $\dim V_i$, $\dim V_i \cap V_j$ ($i \neq j$), $\dim V_1 \cap V_2 \cap V_3$, $\dim V$,

params: 3, 3, 1, 1

$\dim V_1 + V_2 + V_3$.
1

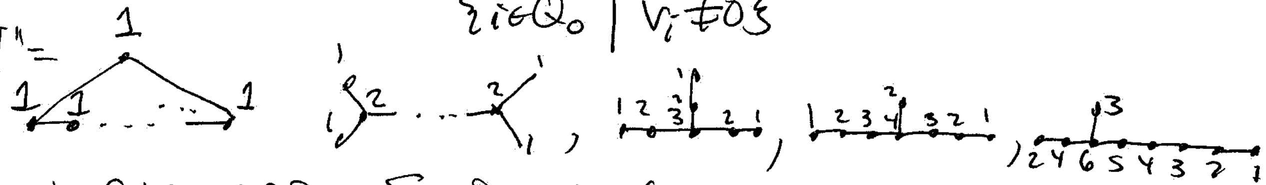
needed to distinguish, e.g., x-axis, y-axis $\to \mathbb{C}^3 \leftarrow$ z-axis from:

x-axis, y-axis $\to \mathbb{C}^3 \leftarrow$ $x=y$ line.

Imaginary roots := those vectors (in $\mathbb{Z}^{Q_0}$) obtained by reflections from an element of the "fundamental ~~domain~~ region" $\mathcal{F}$.

$\mathcal{F} := \{ v \in \mathbb{Z}^{Q_0} \mid (v, e_i) \leq 0\ \forall i,$ $\mathrm{supp}(v)$ is connected $\}$.

$\mathrm{supp}(v) = \{ i \in Q_0 \mid v_i \neq 0 \}$

Ex: "δ" = 

Here $(\delta, e_i) = 0\ \forall i \iff 2\delta_i = \sum_{j - i} \delta_j$, $\forall i$. ($\delta \in \ker C$; it spans the kernel.)

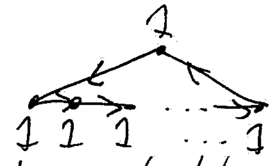
(7/11)

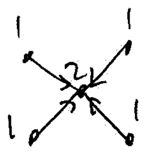
Rmk This is perhaps the easiest way to classify ADE Dynkin diagrams: the extended ~~ADE~~ Dynkin diags are the graphs with a vector δ satisfying $2\delta_i = \sum_{i-j} \delta_j \ \forall i$, and the Dynkin diags are obtained by chopping off a vertex i_0 "extending vertex" with $\delta_{i_0} = 1$. [Doesn't matter which one by symmetry!]

Exs of Reps for imaginary roots (actually for δ as above):

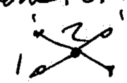
Remember: · For  $\tilde{A}_0$, $\exists\ \infty$-many isoclasses of reps of dim 1

· For  $\tilde{A}_1$, $\exists\ \infty$-many iso classes of reps of dim $(1,1)$

· Similarly for  dims indicated ∴ going all around the cycle gives a continuous invariant in $\mathbb{C}$.

· For  $\tilde{D}_4$, have continuous invariant: $\boxed{\text{quadruples of lines in } \mathbb{C}^2}$ generically, up to $GL_2\mathbb{C}$, can make the first three ~~subspaces~~ lines in $\mathbb{C}^2$ = x-axis, y-axis, x=y line: $\mathbb{C}\cdot(1,0)$ $\mathbb{C}\cdot(0,1)$ $\mathbb{C}\cdot(1,1)$

Then the fourth $= \mathbb{C}\cdot(1,\lambda)$, λ is an invariant (λ = the cross-ratio of the four points of $\mathbb{CP}^1$ corresponding to the four ~~subspaces of $\mathbb{C}^2$~~ lines in $\mathbb{C}^2$.)

[the quadruples of lines problem has: ∞-many up to isomorphism for each fixed dimension, if $\geq$ .]

Quiver varieties:

(8/12)
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The interesting geometric object we alluded to is:

For $\alpha \in \mathbb{N}^{Q_0}$, take $M_\alpha Q := \mathrm{Rep}_\alpha Q /\!/ GL_\alpha$

$$GL_\alpha := \prod_{i \in Q_0} GL_{\alpha_i}, \quad \mathrm{Rep}_\alpha Q := \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{a_t}}, \mathbb{C}^{\alpha_{a_h}})$$

$$a : \underset{\text{"tail"}}{a_t} \longrightarrow \underset{\text{"head"}}{a_h}$$

$M_\alpha Q = \underline{\overset{\text{coarse}}{\text{moduli}} \text{ space of reps of } Q \text{ of dim } \alpha}$. [infinite $\Leftrightarrow$ $\alpha \geq$ imag. rt.]

[Could also take moduli $\underline{\text{stack}}$, $\mathrm{Rep}_\alpha Q / GL_\alpha$].

$\underline{\text{Symplectic version}}$: "$T^*(M_\alpha Q)$" (for real if we use the stack):

$$X := \mathfrak{M}_{\lambda,\theta}(Q,\alpha) := \mu^{-1}(\lambda)^{\theta\text{-ss}} /\!/ GL_\alpha \text{ (or } PGL_\alpha)$$

$$\mu : T^* \underset{\underset{\mathrm{Rep}_\alpha \bar{Q},\ \bar{Q} = \text{doubled } Q.}{\shortparallel}}{\mathrm{Rep}_\alpha Q} \longrightarrow \mathfrak{gl}_\alpha^* \cong \mathfrak{gl}_\alpha := \mathrm{Lie}\, GL_\alpha$$

$\underline{\lambda \in \mathbb{C}^{Q_0} \cong \bigoplus_{i \in Q_0} \mathbb{C} \cdot I_{\alpha_i}}$ $= \bigoplus \mathfrak{gl}_{\alpha_i}$.

$\theta \in \mathrm{Hom}(GL_\alpha, \mathbb{C}^\times) \cong \mathbb{Z}^{Q_0}$ $\underline{\text{stability condition}}$.

Convention (because we could use PGL_α; the action of GL_α factors through PGL_α):

$$\underline{\theta \cdot \alpha = 0}.$$

To have X nonempty, need also $\lambda \cdot \alpha = 0$:

$\underline{\text{Exer}}$: explicitly, $\mu(\rho) = \sum_{a \in Q_1} [\rho(a), \rho(a^*)]$, $\Rightarrow$ $\boxed{\mathrm{tr}\, \mu(\rho) = 0.}$

But $\underline{\mathrm{tr}(\sum \lambda_i \cdot I_{\alpha_i}) = \sum \lambda_i \alpha_i = \lambda \cdot \alpha}$. $\Rightarrow X \cong \underline{(\Pi^\lambda\text{-mod of dim } \alpha)^{\theta\text{-s.s.}}} / \cong$ (Moduli interpretation)
Π^λ — deformed preprojective alg.

$\underline{\text{Note}}$: $\rho \in \mu^{-1}(\lambda)$ is θ-(semi)stable if $\forall$ subreps $\rho' \subseteq \rho$, $(\dim \rho') \cdot \theta \leq 0$ (semistable)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " < 0 (stable).

Ex: Jordan quiver, $\alpha=(n)$.

$\alpha=(n)$

$$\Rightarrow \mathfrak{M}_{0,0}(Q,(n)) = \underbrace{\{(X,Y)\in(\mathrm{Mat}_n\mathbb{C})^2 \mid [X,Y]=0\}}_{\text{commuting variety } \mu^{-1}(0)} /\!/ GL_n$$

$\| \quad T^*\mathfrak{gl}_n /\!/\!/ GL_n$ (as discussed)

[unknown (?): is it reduced?]

$\cong$ Joseph, Gan-Ginzburg $\mathbb{C}^{2n}/S_n$

~~Then $\mu^{-1}(0)^{\theta\text{-}ss}=$~~ ~~$\theta\neq 0$~~

Can't vary θ,λ, since only one vertex.

Trick to resolve/deform: add framing:

$$1 \xrightarrow{} n \circlearrowleft$$

$\hat{Q}$ "framed Jordan quiver"

Now $\mathfrak{M}_{00}(\hat{Q},(1,n)) \cong \mathfrak{M}_{0,0}(Q,(n)) \cong \mathbb{C}^{2n}/S_n$.

But ~~PGL~~ $(GL_1\times GL_n)/\mathbb{C}^\times \cong GL_n$ has characters!

$\mathfrak{M}_{\lambda,0}(\hat{Q},(1,n)) \cong CM_\lambda$ "Calogero-Moser space"

$:= \{(X,Y)\in(\mathrm{Mat}_n\mathbb{C})^2 \mid [X,Y]=$ matrix with char. poly. $x^{n-1}(x-\lambda)\} /\!/ GL_n$

Smoothing, related to

$\mathfrak{M}_{0,\theta}(\hat{Q},(1,n)) = \{(X,Y) \mid \cdot\, [X,Y]=0$, $\cdot\,\exists$ "cyclic vector" v, i.e., X^aY^bv spans $\mathbb{C}^n\} /\!/ GL_n$

i.e., an n-dim. quotient ring of $\mathbb{C}[x,y]$

Symplectic resolution

Nakajima $\cong \mathrm{Hilb}^n\mathbb{C}^2$

"{codim n ideals}/$\cong$ = {dim $= n$ quotient rings of $\mathbb{C}[x,y]$}"

(10/12)
(31)

Note $\mathcal{M}_{0,\theta}(\widetilde{Q},(1,n)) \cong \mathrm{Hilb}^n \mathbb{C}^2$
$\theta \neq 0$

$$\begin{array}{ccc} \mathcal{M}_{0,\theta}(\widetilde{Q},(1,n)) & \cong & \mathrm{Hilb}^n \mathbb{C}^2 \\ \downarrow & & \Downarrow \\ \mathcal{M}_{0,0}(\widetilde{Q},(1,n)) & \cong & \mathrm{Sym}^n \mathbb{C}^2 \end{array}$$

vertical arrows are S.R.'s.

Thm (Nakajima): $\mathcal{M}_{0,\theta}(\widetilde{Q},(r,n)) \cong$
$\theta \neq 0$

{moduli space of framed torsion-free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ with $c_2(\mathcal{F}) = n$, of rank r.}
$c_2 \in \mathbb{Z} = H^4(\mathbb{P}^2)$

- framed means $\mathcal{F}|_{l_\infty} \cong \mathcal{O}_{l_\infty}$, $l_\infty \subseteq \mathbb{P}^2$ is the line ($\mathbb{P}^1$) at infinity.
  $\Downarrow$
  $c_1(\mathcal{F}) = 0$.

- torsion-free means every local section has full support, i.e., $\nexists\ U \subseteq \mathbb{P}^2$, $m \in \mathcal{F}(U)$, $f \in \mathcal{O}(U)$, $f \cdot m = 0$
  [suffices to take $U \in \{\mathbb{A}^2_0, \mathbb{A}^2_1, \mathbb{A}^2_2\}$ = open affine cover]

- torsion-free $\Rightarrow$ on some open subset with codim 2 complement, we are locally free $\Rightarrow$ have well-defined rank
  [assuming the variety is normal + irreducible]
  here $= \mathbb{P}^2$

Note: If $p \in \mathrm{Hilb}^n \mathbb{C}^2$, have ideal $I_p \subseteq \mathcal{O}(\mathbb{C}^2)$, codim $I_p = n$. To this we can associate $\mathcal{I}_p$ = ideal sheaf in $\mathcal{O}_{\mathbb{P}^2}$, $\mathcal{I}_p|_{l_\infty} = \mathcal{O}_{l_\infty}$ since $\mathrm{supp}(p) \subseteq \mathbb{C}^2 = \mathbb{P}^2 \setminus l_\infty$.

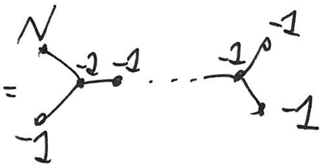
Exercise: $\underbrace{\dim \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_p)}_{= n} = c_2(\mathcal{I}_p) \Rightarrow c_2(\mathcal{I}_p) = n$.
So we recover the previous description.

Extended Dynkin case:

Thm Let Q = extended Dynkin, dim vector δ as before [$\delta \in \ker(C)$, $\delta_i = 1$ for some i]:

then $\mathfrak{M}_{0,0}(Q,\delta) \cong \mathbb{C}^2/\Gamma$, $\Gamma < SL_2\mathbb{C}$ finite, Q = McKay quiver

$$\mathfrak{M}_{0,\theta}(Q,\delta) \cong \widetilde{\mathbb{C}^2/\Gamma}$$

$\underline{\theta \neq 0}$. [nice choice: $\theta =$ 

$\rightsquigarrow$ $\theta_i = -1$ $\forall i$ except i = extending vertex. ($\theta \cdot \delta = 0$)]

(Nakajima's choice for more general dimensions (all -1 except for the "framing vertex")

$\boxed{\text{Why does this happen?}}$

$\mathbb{C}^2/\Gamma = \{\Gamma\text{-orbits in } \mathbb{C}^2\} = \{\mathbb{C}[x,y] \rtimes \Gamma\text{-modules which are } \cong \mathbb{C}[\Gamma] \text{ as } \Gamma\text{-modules}\}$ (*)

Now $\mathbb{C}[x,y] \rtimes \Gamma \underset{\text{(Crawley-Boevey–Holland)}}{\overset{\text{Morita equiv}}{\sim}} \Pi^0_{Q_\Gamma}$, Q_Γ = McKay quiver

$TV \rtimes \Gamma$
$\cong$
$T(\text{McKay}(Q))_{kQ_0}$
$e\,\mathbb{C}\Gamma e = \mathbb{C}Q_0$
$eV_e = \mathbb{C}Q_1$

$\dim e_i W e_i = \mathrm{mult}_W(\rho_i)$
$W = \mathbb{C}[G]$: get $\dim = \dim \rho_i$

$\Longrightarrow$ (*) $\cong$ $\Pi^0_{Q_\Gamma}$-mods of dim $= \delta$

i.e., $\mathfrak{M}_{0,0}(Q_\Gamma, \delta)$.

$\Pi^\lambda := \mathbb{C}\overline{Q} \Big/ \sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \lambda_i p_i$

p_i = 0-length path at i.

[Recall: (Π^λ_Q θ-s.s.-mods of dim α)$/\cong$ is isom. to $\mathfrak{M}_{\lambda,\theta}(Q,\alpha)$.]

(What is the Morita equiv? $A \overset{\text{Mor}}{\sim} B \iff A\text{-mod} \overset{\text{equiv}}{\simeq} B\text{-mod}$)

ex of this: $B = eAe$, where $e \in A$ is idempotent ($e^2 = e$) s.t. $AeA = A$.

Here take $e \in \mathbb{C}[\Gamma]$, $e = \sum e_i$, $\mathbb{C}[\Gamma]e_i \cong i^{\text{th}}$ irrep of Γ.

Geometric Connection with Kac-Moody algebras:

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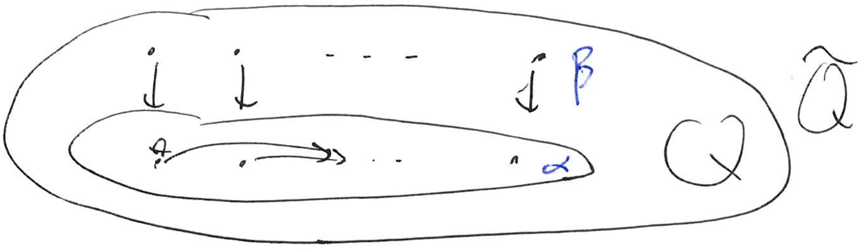
Let Q = a quiver, $\alpha \in \mathbb{N}^{Q_0}$ dimension vector.

Consider also a framed quiver $\widetilde{Q}$,



add a new vertex and arrow pointing to each vertex of Q.

$\widetilde{Q}_0$ = two copies of Q_0.

Let $\beta \in \mathbb{N}^{Q_0}$ be another dimension vector.

$\Rightarrow (\alpha, \beta) \in \mathbb{N}^{\widetilde{Q}_0}$.

<u>Nakajima quiver variety:</u> $\mathcal{M}_{\lambda,\theta}(Q, \alpha, \beta) :=$

$$T^* Rep_{(\alpha,\beta)}(\widetilde{Q}) /\!/\!/_{\lambda,\theta} GL_\alpha$$

$$\left(= \mu_\alpha^{-1}(\lambda)^{\theta\text{-}ss} /\!/ GL_\alpha\right), \quad \lambda \in \mathbb{C}^{Q_0}, \quad \theta \in \mathbb{Z}^{Q_0}.$$

<u>Thm (Nakajima):</u> The "middle" cohomologies ← $\boxed{Q \text{ has no loops}}$

$$\bigoplus_\alpha H^{\dim \mathcal{M}_{\lambda,\theta}(Q,\alpha,\beta)}\left(\mathcal{M}_{\lambda,\theta}(Q,\alpha,\beta)\right)$$ ← the top degree in which cohomology is nonzero, by Kaledin's semismallness

form an "integrable" highest weight rep of $\mathfrak{g}_Q$
(h.w. = β, if β suitable (not zero)) (Kac-Moody Lie alg)

$\rightsquigarrow$ can compute all these dimensions at once!

Thm (Varagnolo): Extend above to <u>all</u> cohomology, $\mathfrak{g}_Q \rightsquigarrow$ <u>Yangian</u>.

Idea behind Nakajima / Varagnolo:

maps $\mathfrak{M}_{\lambda,\theta}(Q,\alpha,\beta) \times \mathfrak{M}_{\lambda,\theta}(Q,\alpha',\beta)$

$$\downarrow \oplus$$

$$\mathfrak{M}_{\lambda,\theta}(Q,\alpha+\alpha',\beta)$$

Hecke correspondence:

$$\begin{tikzcd} & \{(U \subseteq V)\} = \text{pairs}, \; U \in \mathfrak{M}_{\lambda,\theta}(Q,\alpha,\beta), \; V \in \mathfrak{M}_{\lambda,\theta}(Q,\alpha+\alpha',\beta) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathfrak{M}_{\lambda,\theta}(Q,\alpha,\beta) = \{U\} & & \{V/U\} = \mathfrak{M}_{\lambda,\theta}(Q,\alpha',\beta) \end{tikzcd}$$

Now can apply $(p_2)_* p_1^*$, or $(p_1)_* p_2^*$.

For $\alpha' = \alpha + e_i$: get operators E_i, F_i generating the Kac–Moody algebra.

<u>Check</u>: Satisfies the relations of Kac–Moody.

Maulik–Okounkov: Give a new construction [?] of a Yangian Y_Q acting on the equivariant cohomologies:

$$\bigoplus_{\alpha} \overbrace{H^*_{G_\beta}(\mathfrak{M}_{\lambda,\theta}(Q,\alpha,\beta))}^{H(\alpha)} \circlearrowleft Y_Q$$

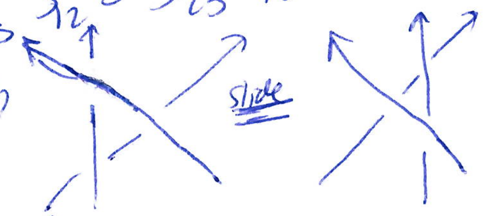
via Faddeev–Reshetikhin–Takhtajan construction: with

Defn QYBE (Quantum Yang–Baxter Equation) with spectral parameters: $R(z): V\otimes V \to V \otimes V$

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u)$$

$R_{ij} = R$ acting in i,j-th entries

[Getting rid of u,v, setting $S := R\circ\sigma$, $\sigma(v\otimes w) = (w \otimes v)$, then get "Braid eqn", $S_{12}\, S_{23}\, S_{12} = S_{23}\, S_{12}\, S_{23}$.]
satisfied by braids



Maulik–Okounkov construct R-matrices
$R_{\alpha,\alpha'}(u) \in \mathrm{End}(H(\alpha)\otimes H(\alpha'))$ (for all u)
via stable envelopes
"reasonably canonical" maps $\boxed{\mathrm{Stab}_C : H^*_A(X^A) \longrightarrow H^*_A(X)}$, A = torus

takes cohom. deg $= \dim X^A$ to $\dim(X)$
"middle degree"

Note: opposite direction is restriction,
$\iota^* : H^*_A(X) \to H^*_A(X^A)$ (restriction).

Ingredients: • $\mathfrak{a} := \mathrm{Lie}\, A$ partitioned into chambers $= \mathfrak{a} \setminus$ root hyperplanes (a root hyperplane in $\mathfrak{a}$ is the vanishing of a weight of N_{X^A} (normal bundle)) (8)

- Given a chamber $C \subseteq \mathfrak{a}$, call $x \in X$ <u>C-stable</u> if $\lim_{z\to 0} \sigma(z)\cdot x \in X^A$ <u>exists</u>, for $\sigma \in C$ (<u>open</u> chamber)
- Partition C-stable locus into <u>attracting sets</u> for components of X^A
- Define $\mathrm{Stab}_C(\gamma)$, supported in attracting locus for γ and on the component Z of γ in X^A, $\mathrm{Stab}_C|_Z = \pm e(N_-)\,V_-$.
- $R_{C',C}(u) = \mathrm{Stab}_{C'}^{-1} \circ \mathrm{Stab}_C \in \mathrm{End}(H_A^*(X^A))[\mathfrak{a}^{-1}]$. $u \in \mathfrak{a}^*$

<u>Maulik–Okounkov</u>: <u>This satisfies the QYBE</u>, in case $X = \mathfrak{M}_{\lambda,\theta}(Q, \alpha+\alpha', \beta)$, $\supseteq X^A = \mathfrak{M}_{\lambda,\theta}(Q,\alpha,\beta) \times \mathfrak{M}_{\lambda,\theta}(Q,\alpha',\beta)$

$\overset{FRT}{\rightsquigarrow}$ Yangian is spanned by operators

$\underline{T_V(M,u)} := \mathrm{tr}_V\left(\underset{\underset{\mathrm{End}(W)}{\cap}}{(M\otimes 1)\, R(u)}\right)$, $M \in \mathrm{End}(V)$, $R(u) \in \mathrm{End}(V\otimes W)$,

$\rightsquigarrow$ contains "Maulik–Okounkov" Lie alg of Q, which (if no loops in Q) contains the Kac–Moody.

<u>Big commutative subalgebra of Y_Q</u> (Baxter or Bethe subalg):

Fix $M \in$ centralizer of <u>all</u> R-matrices ($(M\otimes 1) R = R(M\otimes 1)$, etc.).

$\rightsquigarrow T_V(M,u)$ as V, u vary generate subalg.

<u>Conjecture</u>: This Baxter subalg = subalg generated by modified equivariant quantum mult.

Relationship with Crawley-Boevey/Kac conj:

- If ρ_a = irrep of Q, then
  $\bar{Q} := Q \sqcup Q^*$ add a reverse arrow a^* for every $a \in Q_1$

$\overset{CB}{\rightsquigarrow}$ indecomposable reps of Q often can be extended to simple. representations of $\Pi_Q = k\bar{Q} / \sum_{a \in Q_1} ([a, a^*])$.

$\rightsquigarrow$ the real roots give rise to indecomposable reps, since reflections s_i on α can be implemented for reps.

Illustration of why $\bar{Q}$, Π_Q are better?
With just Q, can't define a reflection in general. But can for Π_Q (and $\bar{Q}$).

Also: • given an ADE Dynkin quiver, can construct positive part of $U\mathfrak{g}_Q$ ← f.d. semisimple. using Hall algebras

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