Symplectic Resolutions + Singularities, lecture 8. (1/12)

Coulomb branches.

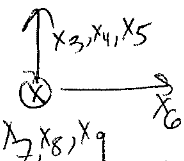
· Seiberg-Witten '95: moduli of gauge fields in $\mathbb{R}^{2,1}$

Origin: IIB string theory (10-dimensional) $\mathbb{R}^{9,1}$ (9 spatial, 1 time)

· Brane diagrams (Hanany–Witten '96) — 5-branes (of the other type)

(☆)

5-branes (of one type)



Everything extended in x_0 (time), x_1, x_2.

Rotation: Interchange x_3, x_4, x_5 coords with x_7, x_8, x_9 coords

X ⟷ |

Higgs, Coulomb branches are components of the moduli space of brane configurations of a certain type (can move branes, put different fields on them) following certain rules

= moduli of vacua (possible vacuum states in IIB string theory.)

Associated to
· Group G (reductive)
· Symplectic representation V

Rotation interchanges these.

Translation to quivers of type A:

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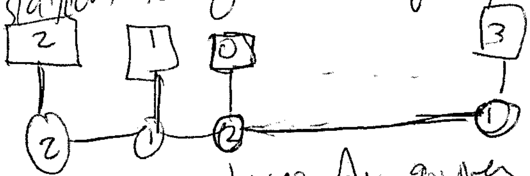


type A_4 quiver

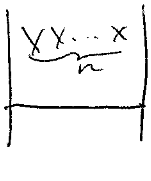
(in general can have type A_n quiver = n circles (gauge nodes) (vertices of original quiver), n squares (framing nodes).)

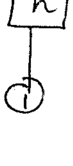
Here: · group $= GL_2 \times GL_1 \times GL_2 \times GL_1$ "gauge fields"

· representation $= T^*\big(\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^1,\mathbb{C}^1) \oplus \mathrm{Hom}(\mathbb{C}^3,\mathbb{C}^1)$ vertical arrows

horizontal arrows $\oplus \mathrm{Hom}(\mathbb{C}^2,\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C},\mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}^2,\mathbb{C})\big)$

Symplectic duality / mirror symmetry for type A quivers (includes $S_{\lambda\mu} \leftrightarrow S_{\lambda^t\mu^t}$)

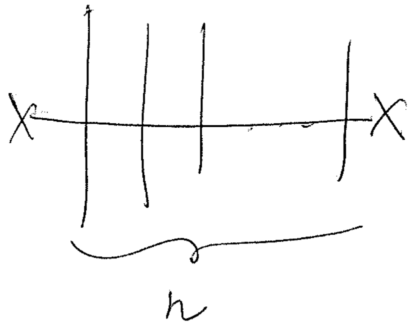


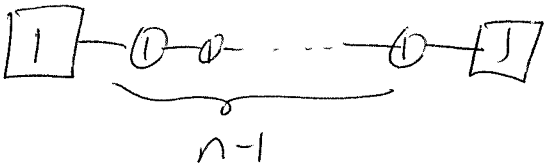


Higgs: the Nakajima q. var $= T^*\mathbb{C}^n /\!/\!/ \mathbb{C}^\times = \min(\mathfrak{sl}_n) \Leftarrow T^*\mathbb{P}^{n-1}$

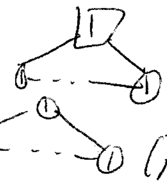
Coulomb: Slodowy slice to subregular, $\mathbb{C}^2/\mathbb{Z}_n$ (in $\mathfrak{sl}_n$). $T^*\mathbb{P}^{n-1}$

Rotate: Get





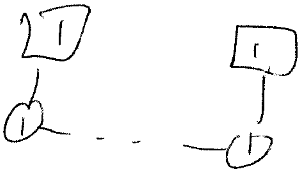
or: 

Higgs: same as , ($\hat{A}_{n-1}$): Slodowy slice $\mathbb{C}^2/\mathbb{Z}_n$.

Coulomb: $\min(\mathfrak{sl}_n)$.

$\boxed{\Rightarrow \text{Higgs}(Q) \cong \text{Coulomb}(Q^!),\ \text{Coulomb}(Q) \cong \text{Higgs}(Q^!)\ !}$

Balance: Gauge nodes where $\sum(\text{adjacent dims}) = 2\cdot(\text{dim at node})$

In 

, all gauge nodes balanced.

Physics says:
$$\mathfrak{g}_{(\text{quiver of balanced nodes})} \subseteq \mathcal{O}(\text{Coulomb}), \quad \{-,-\} \text{ Poisson br.}$$
~~Poisson subalg~~ $\boxed{\text{Lie subalg}}$

How do they know?

Monopole formula (Cremonesi, Hanany, Zaffaroni) 2013: The Hilbert series of $\mathcal{O}(M_C)$, M_C = Coulomb branch for group G, rep. V, is:

$$\sum_{\lambda\in\text{Dominant coweight for } G} t^{2\Delta(\lambda)} P_G(t,\lambda), \qquad P_G(t,\lambda) = \prod_{i=1}^{r} \frac{1}{1-t^{2d_i}}$$

r = rank G := dim(max torus H)
d_i = degrees of generators of $\mathbb{C}[\mathfrak{h}^*]^{W_\lambda}$, W_λ = Weyl group of $\mathrm{Stab}_G(\lambda)$.
Centraliser of λ in G.

$$\Delta(\lambda) = -\sum_{\alpha\in\Phi^+} |\alpha(\lambda)| + \frac{1}{4}\sum_{\mu} \mu(\lambda) \dim V_\mu$$

positive root; weight; weight space μ.

Note that if a gauge node i is balanced, then (Exercise!): $\Delta(\mathbb{C}^\times \cdot Id \subseteq GL_{n_i}) = 1 \Rightarrow$ get element in degree two (corr. to c_i) in $\mathcal{O}(\mathcal{M}_C)$. $\{-,-\}$ has degree -2, so degree two elements span a Lie subalgebra of $\mathcal{O}(\mathcal{M}_C)$.

Mathematical defn of Coulomb branch (Braverman–Finkelberg–Nakajima)

Convolution and Borel–Moore Homology → Refs: • Chriss–Ginzburg §2.6?; • Savage GRT notes, §3.

Borel–Moore homology $H_*^{BM}(X)$ of a topological space is defined like ordinary homology, but using infinite (but locally finite) chains. X must be locally compact.

Assume X is homotopic to a finite CW complex $\Rightarrow H_*^{BM}(X) \cong H_*(\hat{X}, \{\infty\})$, $\hat{X} = X \cup \{\infty\}$ is the one-point compactification.

Example: $H_i^{BM}(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & \text{if } i=n \\ 0, & \text{otherwise} \end{cases}$ $\left(\text{compare: } H_i(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & i=0 \\ 0, & \text{otherwise} \end{cases}\right)$

This is better for dealing with noncompact varieties for many reasons:

- Poincaré duality: for $X \subseteq M$ = smooth oriented manifold, $H_i^{BM}(X) \cong H^{\dim_{\mathbb{R}} M - i}(M, M \setminus X)$
- Any (closed) subvariety $Z \subseteq X$ has a fundamental class $[Z] \in H_{\dim Z}^{BM}(X)$; the irreducible components $[X_1], \ldots, [X_m]$ of X of $\dim = \dim X$ give a basis for $H_{\dim X}^{BM}(X)$.
- Proper pushforward: For $f: X \to Y$ proper, have $f_*: H_*^{BM}(X) \to H_*^{BM}(Y)$,
- Smooth pullback: For $p: X \to Y$ smooth of relative (real) dimension d, have $p^*: H_*^{BM}(Y) \to H_{*+d}^{BM}(X)$.
- Intersection pairing: for $X, X' \subseteq M$ smooth, $\cap: H_i(X) \times H_j(X') \longrightarrow H_{i+j-\dim_{\mathbb{R}} M}(X \cap X')$, obtained from $\cup$ on relative cohomology of M via Poincaré duality.

[Also: Künneth formula $H^{BM}_*(X\times Y)\cong H^{BM}_*(X)\otimes H^{BM}_*(Y)$, "restriction with supports": $i: N\subseteq M$ clsd embedding of smooth mflds $\Rightarrow i^*: H_i(X)\to H_{i+\dim_{\mathbb{R}} N-\dim_{\mathbb{R}} M}(X\cap N)$.]
$\to$ ~~Chriss–Ginzburg~~

Convolution product:

M_1, M_2, M_3 = smooth oriented manifolds
$p_{ij}: M_1\times M_2\times M_3 \to M_i\times M_j$ projection

$Z\subseteq M_1\times M_2$, $Z'\subseteq M_2\times M_3$ clsd; assume
$p_{13}: p_{12}^{-1}(Z)\cap p_{23}^{-1}(Z')\to M_1\times M_3$ proper (preimage of compact is compact)

$Z\circ Z' := \operatorname{im}(p_{13})$ [Composition of correspondences: in case Z, Z' are graphs of functions, this is the graph of the composite function.]

Defn $*: H_i(Z)\times H_j(Z')\to H_{i+j-\dim_{\mathbb{R}} M_2}$,

$$\boxed{c * c' = (p_{13})_*(p_{12}^* c\cap p_{23}^* c')}.$$

[There $p_{12}^* c$ can explicitly be written as $c\boxtimes[M_3]$ using Künneth, etc.]

Example:

Ginzburg: Considered "Steinberg" variety,

$$Z := T^*G/B \times_{\mathrm{Nil}(\mathfrak{g})} T^*G/B \underset{\text{singular}}{\subseteq} T^*G/B\times T^*G/B$$

Prop • Z = Union of conormal bdles to G-orbits in $G/B\times G/B$ ($\cong B$-orbits on G/B)
$\Rightarrow$ Lagrangian, labeled by Weyl group (Bruhat cells)

• $Z\circ Z = Z \Rightarrow H^{BM}_*(Z)$ is associative algebra with unit,
$H^{BM}_*(\rho^{-1}(x))$ = module, $\forall x\in \mathrm{Nil}(\mathfrak{g})$
(e.g., $x=0\Rightarrow \rho^{-1}(0)=G/B$.)

Equivalent version for $G\times\mathbb{C}^\times$: degenerate affine Hecke algebra "dAHA"

Chriss–Ginzburg Thm $H^{BM}_*(Z,\mathbb{Q})\cong \mathbb{Q}[W]$ group algebra of Weyl gp!

$\rightsquigarrow$ Can consider "Steinberg" for any symplectic resolution, $\rho:\tilde{X}\to X$, $Z:=\tilde{X}\times_X\tilde{X}$, this is Lagrangian by Kaledin's theorem (i.e. ρ is semismall)

$\rightsquigarrow H^{BM}_*(Z)$ is an algebra generalising the group algebra of the Weyl group.

• Preprint of McBreen–Nevins: Namikawa Weyl group embeds in $H^{BM}_*(Z)$.

Mathematical definition of Coulomb branch

Assume $V = T^*N$ "cotangent type" CBFN] :

$$\mathcal{T} := G_K \times_{G_\mathcal{O}} N_\mathcal{O} \xrightarrow{\pi} N_K$$

⚠ This is notation $X \times_G Y = (X \times Y)/G$ diagonal action — NOT fiber product.

$\mathcal{O} := \mathbb{C}[[t]] \subseteq \mathbb{C}((t)) =: K$

so for $G = GL_n$, $G_\mathcal{O} = GL_n(\mathcal{O})$, $G_K = GL_n(K)$, etc.

This one IS fiber prod

$$\mathcal{O}(M_C) := H_*^{BM, G_K}(\mathcal{T} \times_{N_K} \mathcal{T}) \text{ under convolution.}$$

equivariant.

[doesn't literally make sense: G_K not reductive, $\mathcal{T} \times_{N_K} \mathcal{T}$ not alg...]

Because it's equivariant cohomology, can rewrite as:

Case $N=0$: $H_*^{BM, G_K}(G_K/G_\mathcal{O} \times G_K/G_\mathcal{O})$:

$G_K/G_\mathcal{O} = Gr_G$ "Affine Grassmannian"

have isomorphism $Gr_G \times Gr_G \leftrightarrows G_K \times_{G_\mathcal{O}} Gr_G$

$(g, gx) \longleftarrow\!\shortmid (g, x)$

$$\Rightarrow H_*^{BM, G_K}(Gr_G \times Gr_G) \cong H_*^{BM, G_\mathcal{O}}(Gr_G)$$

$G_\mathcal{O}$ much better than G_K

"ind-scheme of ind-finite type" = inductive limit of schemes of finite type.

Thm (Bezrukavnikov–Finkelberg–Mirković): $M_C \cong$ universal (regular) centraliser in $G^\vee :=$ Langlands dual group

defined via Kostant section $c := \mathfrak{g}^\vee /\!/ G^\vee \overset{\kappa}{\hookrightarrow} \mathfrak{g}^\vee$, (image $= e + \ker(\mathrm{ad}\, f)$, e = regular nilpotent, $(e,h,f) = \mathfrak{sl}_2$-triple)

has property:

- $\mathfrak{g}^\vee/\!/G^\vee \hookrightarrow \mathfrak{g}^\vee \twoheadrightarrow \mathfrak{g}^\vee/\!/G^\vee$ is Id
- $\kappa(x)$ is regular $\forall x$ (i.e., $\dim G_x := \{g \mid \mathrm{Ad}\, g(x) = x\} = \mathrm{rk}\, G = \dim \mathfrak{h}$) minimum possible

Now $\mathcal{M}_C \cong \{(x, g) \in \mathrm{im}(\kappa) \times G \mid \mathrm{Ad}_g(x) = x\}$

i.e., g centralises x.

Thm This $\mathcal{M}_C$ is a symplectic manifold,

$\mathcal{M}_C \twoheadrightarrow \mathrm{im}(\kappa) \cong \mathfrak{g}^\vee /\!/ G^\vee$ is a Lagrangian abelian group fibration, ...

$\boxed{\text{Case } G = T \text{ turns: } \mathcal{M}_C = \mathfrak{t} \times T^\vee}$

General fix: $\mathcal{T} \underset{N_K}{\times} \mathcal{T} = G_K \underset{G_O}{\times} \mathcal{R}$,

~~$\mathcal{R} = \{(g, n) \in Gr_G \times N_O \mid n \in g^{-1} N_O\}$~~

~~Why? $\mathcal{T} = G_K \underset{G_O}{\times} N_O \xrightarrow{\pi} Gr_G \times N_K$, $\mathcal{R} = \pi^{-1}(Gr_G \times N_O)$~~

~~$(g, n) \longmapsto (g G_O, gn)$~~

~~$\Rightarrow \mathcal{T} \underset{N_K}{\times} \mathcal{T} \subseteq Gr_G$~~

$\mathcal{R} := \{[g, n] \in \mathcal{T} = G_K \underset{G_O}{\times} N_O \mid n \in g^{-1} N_O\}$

$$\boxed{\begin{aligned} \mathcal{T} \underset{N_K}{\times} \mathcal{T} \ni ([g, n], [g', n']) &\longmapsto [g', [(g')^{-1} g, n]] \in G_K \underset{G_O}{\times} \mathcal{R} \\ gn = g'n' \qquad & \\ ([g'\tilde{g}, n], [g', \tilde{g} n]) &\longmapsfrom [g', [\tilde{g}, n]], \end{aligned}}$$

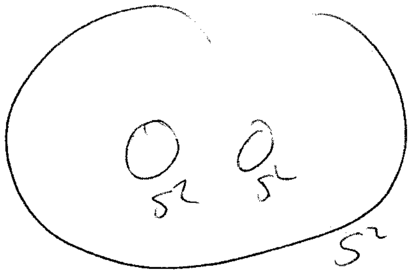
$$\Rightarrow H_*^{BM, G_K}\left(\mathcal{T} \underset{N_K}{\times} \mathcal{T}\right) \text{ "=" } \underbrace{H_*^{BM, G_O}(\mathcal{R})}$$

How [BFN] really define it.

(7/10)

In terms of conjectural 3D topological field theory (Rozansky–Witten theory, also from '96):

$\mathcal{O}(\mathcal{M}_e) = \mathcal{H}_{S^2}$, state space attached to S^2



gives mult $\mathcal{H}_{S^2} \times \mathcal{H}_{S^2} \to \mathcal{H}_{S^2}$

Also: "E_3 algebra" (3-balls) $\Rightarrow$ <u>Poisson</u>

~~Equivariance along~~ an axis $S^1 \subseteq S^3$: get quantization (E_1-alg) $=$ associative.

Now make an S^2 "very small":

$$\bigcirc \sqcup \bigcirc = \mathrm{Spec}\,\mathbb{C}[[t]] \sqcup \mathrm{Spec}\,\mathbb{C}[[t]]$$

formal disc, $\otimes$ punctured formal disc

$\mathrm{Spec}\,\mathbb{C}((t))$

$\Delta :=$ formal disc, $\Delta^* :=$ punctured formal disc

A G-bundle on this version of S^2 is a ~~pair~~ tuple $(P_1, P_2, \varphi_{\Delta^*})$ (P_1, P_2 are G-bundles on Δ, ~~P_3 a G-bundle on Δ^*~~ $\varphi : P_1|_{\Delta^*} \xrightarrow{\sim} P_2|_{\Delta^*}$ isomorphism)

Since P_1, P_2 are trivial, this means we have

$$\text{tuples}/\cong \; = \{\varphi \in G_K\} \Big/ \left(\varphi = \varphi\circ g,\ \varphi = g\circ\varphi,\ g \in G_{\mathcal{O}}\right) = \underbrace{G_{\mathcal{O}} \backslash G_K / G_{\mathcal{O}}}_{\text{stack}}$$

$G_K = \mathrm{Hom}(P_1|_{\Delta^*}, P_2|_{\Delta^*})$

$$H_*(\underbrace{G_{\mathcal{O}} \backslash G_K / G_{\mathcal{O}}}_{\text{stack}}) = H_*^{G_{\mathcal{O}}}(G_K / G_{\mathcal{O}}) = \mathcal{O}(\mathcal{M}_C)$$

for <u>$N = 0$</u>.

Generalisation to $N \neq 0$ gives $\mathcal{M}_C$ (of <u>cotangent</u> type).

More examples of $\mathcal{M}_C$:

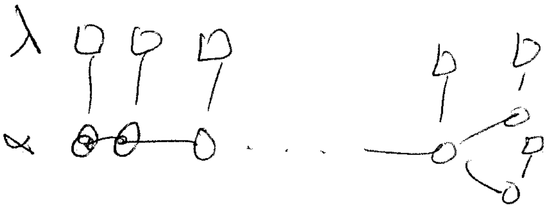
- Case $G = T$ = torus: get Gale dual to $T^*N /\!/\!/_{T}$, i.e., $T^*N^* /\!/\!/_{T^\perp}$

On Lie algebras: $0 \to t \to m \to m/t \to 0$

$t = \mathrm{Lie}\,T$ $\qquad 0 \leftarrow t^* \leftarrow m^* \leftarrow \underbrace{(m/t)^*}_{\text{Lie alg of } T^\perp} \leftarrow 0$

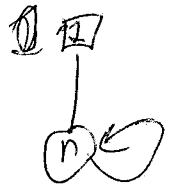
$m = \mathrm{Lie}$(max torus acting on N)
(diagonal matrices $\in \mathrm{End}\,N$)

- ADE Quiver: (Dynkin)



Get slice to G_O-orbit in $Gr_G = G_K/G_O$ labelled by t^α inside orbit labelled by λ.

[λ = dominant coweight, each $\boxed{m_i}$ means $m_i \cdot i^{th}$ fund. coweight,
$\alpha = \sum n_i \underbrace{\alpha_i}_{\text{simple roots}}$ $\quad \alpha_1 \; \alpha_2 \cdots \alpha_n$]

- 

$\mathcal{M}_C = S^n \mathbb{C}^2$ self-dual: also $\cong \mathcal{M}_H$

- $N = \mathfrak{g}$ = adjoint rep: $\mathcal{R} \cong$ "affine Steinberg variety"
$\quad$ = union of conormal bundles to G_O-orbits in Gr_G.

Get $\mathcal{M}_C = (T^\vee \times t)/W$.

- A $\mathcal{M}_C$ comes with: · deformations $\quad$ · quantisation

$\quad$ deformations $\leftrightarrow$ Equivariant cohomology wrt framing vertices

$\quad$ quantisation $\leftrightarrow$ ~~dilations~~ $\mathbb{C}^*$-equivariant cohomology

Properties: • $\mathcal{M}_C$ supposed to be "singular Hyperkähler" (↓ Physics), don't really know what that means.

Case $\Delta(\lambda) \geq \frac{1}{2}$ $\forall \lambda \neq 0$ "good or ugly" case ?? then $\mathcal{M}_C$ is conical (if $\Delta(\lambda) \geq 1$ "good", then there is a most singular point, the unique cone point.)

• $\mathcal{M}_C$ is reduced + normal [BFN], generally symplectic [BFN], expected (not proved!) to be a symplectic singularity.

• Integrable system map $\mathcal{M}_C \longrightarrow \mathfrak{t}/W$, Lagrangian fibers

[Note: the mirror construction for type A quivers shows why (morally) they are always normal $\Rightarrow$ all nilpotent orbit closures (and S3 varieties $\mathfrak{sl}_n$) are normal in $\mathfrak{sl}_n$.]

• Equipped with canonical deformations, quantization, (partial) crepant resolutions

vary framing (equivariant) params — vary gauge parameters (Kähler params)

Reversal from Higgs branch.

⚠ For Higgs, as we saw, NOT always normal or even reduced.

General strategy for identifying $\mathcal{M}_C$: use integrable system $\mathcal{M}_C \longrightarrow \mathfrak{t}/W$

Show $\mathcal{M}_C$ is isomorphic outside codim 2 in $\mathfrak{t}/W$ to a candidate $\mathcal{M}$. (over $(\mathfrak{t}/W)^\circ$)

$\mathcal{M}_C$ normal, affine $\Rightarrow \mathcal{M}_C \cong \mathrm{Aff}(\mathcal{M}|_{(\mathfrak{t}/W)^\circ}) \cong \mathcal{M}$, if $\mathcal{M}$ also normal + affine.

What codim 2 closed subset to use? (10/10)

Generalised roots for (G, N):

(I) nonzero weight of N which is NOT a multiple of a root of $\mathfrak{g}$

(II) roots of $\mathfrak{g}$

t° = complement of points in 2 or more such hyperplanes.

Outside all hyperplanes we just get:
fibers of $M_C \longrightarrow t/W$ are $T^\vee$, not much to show.
Calculation boils down to what happens generically along a hyperplane: type (I): reduces to G = torus case
(II): reduces to G lower rank!
(eg SL_2).