Defn $\mathrm{Diff}_{\leq n}(M,N) = \{\phi : M \to N$ $\mathcal{O}$-linear,

$m \mapsto \phi(bm) - b\phi(m)$
is $\in \mathrm{Diff}_{\leq n-1}\}$

$M = N \Rightarrow$ get filtered algebra $\mathrm{Diff}(M)$.

<u>Globalisation</u> $M, N \rightsquigarrow$ sheaves $\mathcal{F}, \mathcal{G} / X$
$B \rightsquigarrow \mathcal{O}_X$.

If $\mathcal{F} = \mathcal{G} =$ <u>line bundle</u>, $\mathrm{Diff}(\mathcal{F})$ locally $\simeq \mathcal{D}_X$.

Lemma If $\mathcal{F} = \mathcal{G}$ = line bdle, (X = smooth var)

$$\mathrm{gr}\,\mathrm{Diff}(\mathcal{F}) \underset{\text{global.}}{\cong} \mathrm{Sym}_{\mathcal{O}_X} T_X = \pi_* \mathcal{O}_{T^*X}.$$

Proof of claim: We have SES

$$0 \to \mathcal{O}_X = \mathrm{Diff}_0(\mathcal{F}) \to \mathrm{Diff}_{\le 1}(\mathcal{F}) \xrightarrow{\sigma} T_X = 0$$

where $\sigma(\xi)(f) := \xi \circ f - f \circ \xi \in \mathcal{O}_X$

$f \in \mathcal{O}_X = \mathrm{Diff}_0$

Surjectivity is local, $\mathcal{F}$ locally $\mathcal{O}_X$. $\square$

Now, because $\mathcal{D}iff(\mathcal{F})$ locally $\simeq \mathcal{D}_X$, gen. by $\mathcal{D}iff_{\leq 1}$, locally this realises

$\boxed{X \text{ smooth}}$ $\operatorname{gr} \mathcal{D}iff(\mathcal{F}) \cong \operatorname{Sym}_{\mathcal{O}_X} T_X$, by a globally well-defined map. $\square$

(Work w/ X = smooth var.)

<u>Defn</u> The class in $\operatorname{Ext}(T_X, \mathcal{O}_X) = H^1(X, \Omega_X^1)$ called "Atiyah class"

<u>Defn</u> $\boxed{X \text{ smooth}}$ A sheaf of TDOs "twisted diff ops" is

a sheaf of filtered algebras $\mathcal{D}$ s.t. $\mathrm{gr}\,\mathcal{D} \cong \pi_* \mathcal{O}_{T^*X}$

e.g. $\mathrm{Diff}(\mathcal{L})$, $\mathcal{L}$ = line bdle.

as Poisson algebras

Notice: $\mathrm{Diff}(\mathcal{L})$ is a locally trivial TDO : $\cong \mathcal{D}_X$ locally. Not all like that.

One can show (Ginzburg "...D-modules ...")

- locally trivial TDOs classified by $H^1(\Omega^1_{cl}) \to H^1(\Omega^1)$

  closed 1-forms? Atiyah class

- All TDOs classified $\mathbb{H}^1(\Omega^1 \xrightarrow{\text{dR}} \Omega^2_{cl})$.

Holomorphic: All TDOs are loc. trivial.

What is a Poisson algebra?

Given $\phi \in \mathcal{D}_{\le m}$, $\psi \in \mathcal{D}_{\le n}$, then $\phi \circ \psi - \psi \circ \phi \in \mathcal{D}_{\le m+n-1}$ ($\mathrm{gr}\,\mathcal{D}$ comm.)

$$\Rightarrow \{\mathrm{gr}_m \phi, \mathrm{gr}_n \psi\} := \mathrm{gr}_{m+n-1}(\phi \circ \psi - \psi \circ \phi).$$

Lie bracket, which is a biderivation

<u>Defn</u> A <u>Poisson algebra</u> is a comm alg. equipped with such $\{-,-\}$.

Recap: TDO is $\mathcal{D} \to X$ s.t. $\mathrm{gr}\,\mathcal{D} \simeq \mathrm{gr}\,\mathcal{D}_X$

X smooth e.g. $\mathcal{D}iff(\mathcal{L})$, $\mathcal{L}$ = line bdle.

General setup: A = filtered algebra ($A_0 \subseteq A_{\leq 1} \subseteq \dots$)

$\mathrm{gr}\,A = B$ commutative, inherits Poisson $\{-,-\}$.

Call A a "filtered quantisation" of $(B, \{-,-\})$.

TDOs are the filt. q's of $\pi_* \mathcal{O}_{T^*X}$.

$\mathcal{D}_X$ is gen by $(\mathcal{D}_X)_{\leq 1}$, in fact by T_X.

<u>More general</u> : $\mathfrak{g}$ = <u>Lie algebroid</u>

(relative/global version of Lie alg) vector bdle, $[-,-]$, $\to T_X$.

"Universal enveloping algebroid"

$U\mathfrak{g}$, $\mathrm{gr}\, U\mathfrak{g} \underset{\text{PBW}}{\cong} \mathrm{Sym}_{\mathcal{O}_X} \mathfrak{g} = \mathcal{O}_{T_X \mathfrak{g}^\vee}$

quant. Poisson total space of $\mathfrak{g}^\vee$.

Over $X = \mathrm{Spec}\, \mathbb{C}$: $U\mathfrak{g}$, $\mathfrak{g}$ = Lie alg

q. of $\mathrm{Sym}\, \mathfrak{g}$ = Poisson.

<u>Ex</u>: $B = (\mathrm{Sym}\, V)^G$, V = symplectic vector space,

$G < Sp(V)$, linear sympl. auts.

<u>Quantisation</u>: $A = \mathrm{Weyl}(V)^G$

$$\boxed{\begin{array}{l} f(x) = \Gamma(X, f) \\ \mathcal{D}(X) = \Gamma(X, \mathcal{D}_X). \end{array}}$$

(if $V = U \oplus U^*$, U, U^* Lagrangian,

$\mathrm{Weyl}(V) \cong \mathcal{D}(U)$, in basis $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$

independent of U. $\left(TV/(xy - yx - (x,y))\right)$

$[x_i, y_j] = \delta_{ij}$

$[x_i, x_j] = [y_i, y_j] = 0$

How to think of $\mathrm{Weyl}(V)$ as $\mathcal{D}(U)$?

Define $\mathrm{Weyl}(V) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{O}(U))$

$\Phi \longmapsto (f \longmapsto \Phi \cdot f)$ $= \mathcal{D}(U) \cdot T_v \Big/ \sum_i \mathcal{D}_i \cdot T_v \simeq \mathcal{O}_U$

$y_i \longmapsto -\frac{\partial}{\partial x_i}$.

If G = finite group (or reductive group)

$W \longmapsto W^G$ exact, W G-rep.

$\Rightarrow \mathrm{gr}\,(\mathrm{Weyl}(V))^G \cong (\mathrm{Sym}\,V)^G$.

filtered q.

G finite: by theorem of Losev '16, can classify all q's of $B = (\mathrm{Sym}\,V)^G$

"Spherical symplectic reflection algebras"

of Etingof–Ginzburg (Drinfeld)

(case $V = \mathbb{C}^2$, $G = \{\pm \mathrm{Id}\} \cong \mathbb{Z}/2$: $B \cong \mathcal{O}(\mathrm{Nil}_{2\times 2} = \mathrm{Nil}(\mathfrak{sl}_2))$)

$$B = \mathbb{C}[x,y]_{\text{even}} = \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[u,v,w]\big/(v^2-uw)$$

$$\cong \mathbb{C}[\mathrm{Nil}_{2\times 2}]$$

$$\text{''} \quad \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a^2+bc=0 \right\}$$

All quants are: $U\mathfrak{sl}_2 \big/ (C-\lambda)$, $C \in Z(U\mathfrak{sl}_2)$

$$\text{''} \quad ef+fe+\tfrac{1}{2}h^2$$

"Casimir"

Exer: what is Weyl?

Ex 2: $\mathfrak{g}$ = semisimple Lie algebra.

$\chi : Z(U\mathfrak{g}) \longrightarrow \mathbb{C}$ character.

$$A_\chi := U\mathfrak{g} \big/ (\operatorname{Ker} \chi)$$

filtered, $\operatorname{gr} A_\chi \simeq \operatorname{Sym} \mathfrak{g} \big/ \operatorname{gr}(\operatorname{Ker} \chi)$.

By Harish-Chandra ($Z(U\mathfrak{g}) \xrightarrow{\sim} (\operatorname{Sym} \mathfrak{h})^W$)

+ Kostant: $B \simeq \mathbb{C}[\operatorname{Nil} \mathfrak{g}]$, $\operatorname{Nil} \mathfrak{g} = \{x \in \mathfrak{g} \mid (\operatorname{ad} x)^N = 0, N \gg 0\}$.

That is, get picture:

- By Losev, A_χ = univ. quant. of $\mathrm{Nil}\,\mathfrak{g}$ (i.e. of $\mathbb{C}[\mathrm{Nil}\,\mathfrak{g}]$)
- B-B: $\Gamma(G/B, \mathcal{D}^\lambda) \cong A_\chi$ (λ dominant)
- Kostant + Springer: $\Gamma(G/B, \pi_* \underset{\mathrm{Sym}\, T_{G/B}}{\underset{\|}{T^*G/B}}) \cong B$.

$$\begin{tikzcd} \mathcal{D}^\lambda_{G/B} \arrow[r, mapsto, "\Gamma"] \arrow[d, squiggly] & A_\chi \arrow[d, squiggly, "gr"] \\ T^*_{G/B} \arrow[r, mapsto] & B \end{tikzcd}$$

Ex 3: Hamiltonian / symplectic reduction.

Motivation: Let G = positive-dim alg gp.

$G \curvearrowright X$ $\quad \dim X/\!/G \overset{\text{exp.}}{=} \dim X - \dim G$

$$\dim T^*(X/\!/G) \overset{\text{exp}}{=} 2\dim X - 2\dim G$$

$$\dim (T^*X)/\!/G \overset{\text{exp}}{=} 2\dim X - \dim G.$$

X = variety, $\mathfrak{g}$ = Lie alg acting on X, $\mathfrak{g} \to \mathrm{Vect}(X)$

Get a map $\mu : T^*X \longrightarrow \mathfrak{g}^*$

Global functions: $\Gamma(\mathrm{Sym}\, T_X) \longleftarrow \mathrm{Sym}\, \mathfrak{g}$

$T^*X /\!/\!/ \mathfrak{g} \underset{X \text{ affine}}{=} \mathbb{C}[\mu^{-1}(0)]^G$

(or $/\!/\!/ G$, G group, $\mathrm{Lie}\, G = \mathfrak{g}$)

In general, $\mu^{-1}(0) /\!/ G$ or $\mathfrak{g}$.

<u>Quantum</u>: X affine: $\left(\mathcal{D}(X) / \mu^*(\mathfrak{g}) \mathcal{D}(X) \right)^G$. "QHR"

Poisson setting: X = Poisson variety,

$\mu^*: \mathfrak{g} \longrightarrow \mathcal{O}(X)$ Lie

extend: $\mu^* \; \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathcal{O}(X)$

$X \longrightarrow \mathfrak{g}^*$, do same.

If X quantised by A (X affine),

$\left(A / \mu^*(\mathfrak{g})A\right)^G =$ "QHR".

Specific cases: · Hypertoric varieties:

$X = V = \mathbb{C}$ v.s., $G = (\mathbb{C}^\times)^m \curvearrowright V$ linear

$T^*V /\!/\!/ (\mathbb{C}^\times)^m$

$\circlearrowright$

$(\mathbb{C}^\times)^{n-m}$

"flavour" Hamiltonian

$n = \dim V$

NOT toric.

sympl. analogue of:

$V /\!/ (\mathbb{C}^\times)^m \circlearrowleft (\mathbb{C}^\times)^{n-m}$

toric var

Quiver varieties: $G = \prod_{i=1}^{m} GL_{n_i}$

Quiver := "directed graph"

Vertices: $1, \ldots, m$

r_{ij} arrows $i \xrightarrow{r_{ij}} j$

$$V = \bigoplus_{i,j=1}^{m} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})^{r_{ij}}$$

These are all cones: analogues of $V/\!/G$ or $\mathrm{Nil}(\mathfrak{g})$,

Deformations, resolutions.

To deform: replace $\mu^{-1}(0)$ by $\mu^{-1}(\lambda)$

$$\lambda \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* \subseteq \mathfrak{g}^*.$$

$$\rightsquigarrow G\cdot\{\lambda\} = \{\lambda\}.$$

Many cases: $\mu^{-1}(\lambda)/\!/G$ smooth:

To resolve: Geometric invariant theory

$$\underset{(\text{or } \lambda)}{\mu^{-1}(0)} \rightsquigarrow \mu^{-1}(0)^{\theta-ss} \underset{\text{open}}{\subseteq} \mu^{-1}(0), \quad \theta : G \to \mathbb{C}^\times \text{ character.}$$

θ-ss: $\exists\, f \in \mathbb{C}[X]^{m\theta}$, $f(x) \neq 0$, (X_f affine)

$x \in X^{\theta\text{-ss}} \iff$

In this case, $\mu^{-1}(0)^{\theta\text{-ss}} /\!\!/ G = \mathrm{Proj} \bigoplus_{m \geq 0} \mathbb{C}[\mu^{-1}(0)]^{m\theta}$

$\left(\mathbb{C}[X]^{m\theta} = \{ f \in \mathbb{C}[X] \mid g \cdot f = (m\theta)(g) f \} \right)$

<u>Ex</u>: $V = \mathbb{C}^2$, $T^*V /\!\!/\!\!/ \mathbb{C}^\times = \mathrm{Nil}_{2\times 2}$

$$\begin{array}{ccc} T^*V /\!\!/\!\!/ \mathbb{C}^\times & = & \mathrm{Nil}_{2\times 2} \\ \uparrow {\scriptstyle \theta \neq 0} & & \uparrow \\ \mu^{-1}(0)^{\theta\text{-ss}} /\!\!/ \mathbb{C}^\times & = & T^*\mathbb{P}^1 \end{array}$$

Saw 3 ways to realise $T^*\mathbb{P}^1 \longrightarrow \mathcal{N}il_{2\times 2}$:

- Hypertoric GIT
- $T^* G/B \twoheadrightarrow \mathcal{N}il_{2\times 2}$ (Springer)

$$\begin{array}{ccc} T^* G/B & \twoheadrightarrow & \mathcal{N}il_{2\times 2} \\ \| & & \cap \\ G \times_B \mathfrak{n} & \longrightarrow & \mathfrak{g} \\ (g,x) & \longmapsto & g\cdot x \end{array}$$

$\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$

- $\widetilde{\mathbb{C}^2/\{\pm I\}} \twoheadrightarrow \mathbb{C}^2/\{\pm I\}$

Symplectic resolutions of s. sings. $\rightsquigarrow$ Quantisations

Nontrivial ex: G = reductive or ss alg gp
$\mathfrak{g}$ = Lie alg.

What is $T^*\mathfrak{g} /\!/\!/ G$? $(\star)$ G acts by adjoint.

$G = GL_n$: $\mu^{-1}(0) \cong \{(X, Y) \mid [X, Y] = 0\}$

NOT known if reduced.

Joseph: $(\star)_{red} \cong T^*\mathfrak{h}/W = \operatorname{Spec} \mathbb{C}[T^*\mathfrak{h}]^W$

Gan–Ginzburg: proved $\star$ reduced for $\mathfrak{g} = \mathfrak{gl}_n$.

Thm (Levasseur, Stafford):

$$\mathcal{D}_{\mathfrak{g}} /\!/\!/ G := \left(\mathcal{D}(\mathfrak{g}) \Big/ \mu^*(\mathfrak{g}) \cdot \mathcal{D}(\mathfrak{g}) \right)^G$$

$$\cong \mathcal{D}(\mathfrak{h})^W .$$

H-C
radial parts.