Symplectic Representation Theory: Sheet 1

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Apologies for the delay in posting this assignment.

The starred exercises should be handed in in time for you to receive credit for the course, or by Friday, 3 May, whichever is earlier.

All vector spaces and algebraic varieties (and schemes) are over $\mathbf{C}$.

1. $\star$ Beilinson–Bernstein with symmetries. Let G be a semisimple algebraic group and $\mathfrak{g}$ its Lie algebra. For any subgroup $H < G$, with Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$, we can consider the category of (weakly or strongly) H-equivariant $\mathcal{D}$-modules on the flag variety G/B: see Sheet 1 and Hotta et al for the relevant definitions. Let $\mathcal{D}_H(G/B)$ be the category of strongly H-equivariant $\mathcal{D}_{G/B}$-modules.

   Recall that a Harish-Chandra $(\mathfrak{g}, H)$-module is a $\mathfrak{g}$-module together with an action of the group H such that the induced actions of $\mathfrak{h}$ are identical.

   Example 1. (Not needed for the problem.) This includes the theory of (unitary) real representations, in the case where $G = G'_{\mathbf{C}}$, the complexification of a real reductive group G', with maximal compact subgroup $K' < G$, and $H := K'_{\mathbf{C}}$: namely, unitary real representations are equivalent to unitary $(\mathfrak{g}, H)$-modules (for more general representations, some inequivalent real representations can yield equivalent $(\mathfrak{g}, H)$-modules, and those that do are called "infinitesimally equivalent").

   (a) If H is connected, prove that the action of H on a Harish-Chandra $(\mathfrak{g}, H)$-module is uniquely determined. Deduce that the Harish-Chandra modules form a full subcategory of all $\mathfrak{g}$-modules in this case.

   (b) Prove that the Beilinson–Bernstein equivalence induces another equivalence, from $\mathcal{D}_H(G/B)$ to the category of Harish-Chandra $(\mathfrak{g}, H)$-modules with the same central character as the trivial representation.

   (c) Generalize to arbitrary central characters for which the Beilinson–Bernstein equivalence holds.

2. $\star$ Given a du Val singularity $\mathbf{C}^2/\Gamma$, we can form a resolution via

   $$\mathrm{Hilb}^\Gamma(\mathbf{C}^2) := \{Z \subseteq \mathbf{C}^2 \text{ a subscheme} \mid \mathcal{O}(Z) \cong \mathbf{C}[\Gamma] \text{ as } \Gamma\text{-representations}\} \subseteq \mathrm{Hilb}^{|\Gamma|}(\mathbf{C}^2).$$

In the case $\Gamma = \mathbf{Z}/n$, explicitly verify that the locus of subschemes in $\mathrm{Hilb}^{\mathbf{Z}/n}(\mathbf{C}^2)$ concentrated at the origin consists of $n-1$ copies of $\mathbf{P}^1$ glued together, with intersection graph given by the diagram A_{n-1}.

3. Given an algebraic group G with a finite-dimensional representation V, we call $K < G$ a *parabolic* subgroup if there exists a vector $v \in V$ such that $\mathrm{Stab}_G(v) = K$.

   Let $K < G$ be a parabolic subgroup and $v \in V$ with $\mathrm{Stab}_G(v) = K$.

   If G is finite, show that a formal neighborhood of $G \cdot v \in V/G$ is isomorphic to $V/V^K \times V^K/K$.

   If moreoever V is symplectic and $G < \mathrm{Sp}(V)$, conclude that, V/G admits a symplectic resolution, so does V^K/K.

   (Bonus: Using Luna's slice theorem (see e.g., Drezet's notes on the subject), show that the same result holds if G is reductive and we replace ordinary quotients by Hamiltonian reductions.)

4. Let A be an algebra and $e \in A$ an idempotent ($e^2 = e$). Let $B := eAe$ (a subalgebra of A) and $V := eA$ (a B, A-bimodule).

   (i) Prove that $\mathrm{End}_{A^{\mathrm{op}}}(V) = B$, and that $Z(B) = \mathrm{End}_{B \otimes A^{\mathrm{op}}}(V)$.

   (ii) Consider the map $\phi : Z(A) \to B, a \mapsto za$. (This is sometimes called the "Satake map", after Lusztig and Etingof–Ginzburg.) Show that the image is contained in $Z(B)$. If A is prime (this means that, for all nonzero $a, b \in A$, then $aAb \neq 0$: a noncommutative generalization of integral domain), prove that ϕ is injective.

   (iii) Suppose that the "double centralizer property" holds: $\mathrm{End}_B(A) = V$. Prove that ϕ is an isomorphism onto $Z(B)$. Hint: show similarly that $\mathrm{End}_{B \otimes A^{\mathrm{op}}}(V) = Z(A)$.

   (iv) Give an example of a prime ring A such that the map $\phi : Z(A) \to Z(B)$ is not surjective (and hence the double centralizer property does not hold). Hint: Try the path algebra of a quiver.

5. $\star$ Symplectic reflection algebras: Given a symplectic vector space (V, ω) and a finite subgroup $G < \mathrm{Sp}(V)$, we call an element $g \in G$ a *symplectic reflection* if $\operatorname{codim} V^g = 2$ (this is the minimum codimension for $g \neq 1$ since V^g is symplectic). Let $S \subseteq G$ be the subset of symplectic reflections. Given $t \in \mathbf{C}$ and a conjugation-invariant function $c : G \to \mathbf{C}$, Drinfeld and Etingof–Ginzburg defined the *symplectic reflection algebra*:

$$H_{t,c} := T(V) \rtimes G/(v \otimes w - w \otimes v - t\omega(v, w) - \sum_{s \in S} c(s)\omega(\pi_s(v), \pi_s(w)), \qquad (2)$$

   where for $g \in G$, $\pi_g : V \to (V^g)^\perp$ is the orthogonal projection with kernel V^g (explicitly, if $g^n = 1$, then $\pi_g(v) = v - \frac{1}{n}\sum_{i=0}^{n-1} g(v)$).

   Let $e \in \mathbf{C}[G]$ be the symmetrizer idempotent, $e := |G|^{-1}\sum_{g \in G} g$. The subalgebra $eH_{t,c}e$ is called the *spherical symplectic reflection algebra*.

(a) Prove that, for $t = 0$ and $c = 0$, then $eH_{t,c}e = \mathcal{O}(V/G)$.

(b) In the simplest case $G = C_2 < \mathrm{SL}_2(\mathbf{C})$, with $V = \mathbf{C}^2$, show that $eH_{0,c}e$ is isomorphic to the algebra of functions on the locus of trace-zero matrices of determinant λ_c, and compute λ_c as a function of c.

(c) In the same case as before, prove that $H_{t,c}$ satisfies the *PBW* property: as a vector space, we have the decomposition $H_{t,c} \cong \mathrm{Sym}(V) \otimes \mathbf{C}[G]$. Put another way, we have an algebra isomorphism, $\mathrm{gr}(H_{t,c}) \cong \mathrm{Sym}(V) \rtimes \mathbf{C}[G]$ as algebras, using the filtration generated by $|V| = 1$ and $|G| = 0$. Show the same thing for the spherical subalgebras: $\mathrm{gr}(eH_{t,c}e) \cong \mathrm{Sym}(V)^G$ (using the subspace filtration).

NOTE: The above holds generally for all G, and implies that $H_{t,c}$ is a *flat deformation* of $H_{0,0}$, and indeed of H_{t_0,c_0} for any fixed t_0, c_0, and the same for the spherical subalgebras. It is a theorem of Etingof–Ginzburg that these deformations are *universal* for $t_0 \neq 0$. (It follows from Losev's theorem that a similar statement holds even at $t_0 = 0$, at least for the spherical subalgebra, if one restricts to Poisson deformations and quantizations.)

(d) There is a deep result, the *double centralizer theorem*, that $\mathrm{End}_{eH_{t,c}e}(eH_{t,c}) = H_{t,c}$. It is a theorem of Etingof and Ginzburg that $eH_{0,c}e$ is commutative for all c. Put together, using also the preceding exercise, show that $eH_{0,c}e \cong Z(H_{0,c})$. Give an example to show this is not true replacing 0 by nonzero t.

Note: The spectrum of $eH_{0,c}e \cong Z(H_{0,c})$ is called the *generalized Calogero-Moser space* (the ordinary space is the special case $G = S_n < \mathrm{GL}_n < \mathrm{Sp}_{2n}$).

6. Recall from lectures: it is a consequence of results of Ginzburg–Kaledin and Namikawa that the quotient V/G, for $G < \mathrm{Sp}(V)$ finite, admits a symplectic resolution if and only if the generalized Calogero–Moser space $\mathrm{Spec}\, eH_c(G)e$ is smooth for some (equivalently, generic) c. (More generally, one has that a conical symplectic singularity admits a symplectic resolution if and only if it admits a symplectic smoothing.)

   (a) Recover from this result and the preceding exercises that G admits a resolution only if it is generated by symplectic reflections (sometimes called Verbitsky's theorem). Such a group is called a *symplectic reflection group*.

   (b) Now let G be a symplectic reflection group. The representation V is called symplectically irreducible if there does not exist a symplectic subspace $U \subseteq V$ which is invariant under the group G. (This is weaker than the usual notion of irreducible, which here can be called "complex irreducible".)

   Prove that, if $G < \mathrm{Sp}(V)$ is a symplectic reflection group, then there is a decomposition $V = V_1 \oplus \cdots \oplus V_m$ into symplectically irreducible subspaces (the easy part) and moreover a decomposition $G = G_1 \times \cdots \times G_m$ with $G_i < \mathrm{Sp}(V_i)$ (the harder part). Thus $V/G \cong \prod_i V_i/G_i$, and the study of such quotients V/G (which can admit resolutions) reduces to the study of the symplectically irreducible ones.

7. Classification of simply-laced Dynkin and extended Dynkin quivers: a perhaps better way using the McKay correspondence:

(a) Let Q be an undirected graph obtained from a subgroup $\Gamma < \mathrm{SL}_2(\mathbf{C})$ via the McKay correspondence (note that a quiver is obtained from this by arbitrarily orienting the edges). Prove that the adjacency matrix of Q (a symmetric matrix) A has the vector $\delta = (\dim \rho_i)$ as an eigenvector of eigenvalue two, for ρ_i the irreducible representations of Γ.

(b) Using the Perron–Frobenius theorem for A (see below for a partial recollection), prove that the Cartan matrix $2I - A$ is positive semidefinite (i.e., all the eigenvalues are at least zero) and that $\mathrm{rk}(2I - A) = |Q_0| - 1$ (one less than the number of rows of the matrix). Deduce that for a proper subquiver, the Cartan matrix is positive definite.

   Recall: in particular, the Perron–Frobenius theorem says that, for a matrix with nonnegative entries such that the associated graph is (strongly) connected, there is a unique maximum real eigenvalue, it has multiplicity one, and every eigenvector with all positive entries has this as its eigenvalue.

(c) Now try to classify all extended Dynkin and Dynkin diagrams in the following way: by the McKay correspondence, the extended Dynkin diagrams are those admitting a positive integral vector such that each entry equals one-half the sum of the adjacent entries. Find all of these, and verify that they recover the extended ADE diagrams you already knew.

(d) Prove (using the classification) that every graph is either a proper subset of an extended Dynkin graph or contains an extended Dynkin graph.

(e) Prove furthermore that the Cartan matrix is either positive definite (a proper subset of an extended Dynkin quiver), positive semidefinite (the extended Dynkin case), or indefinite (non-Dynkin, non-extended Dynkin). This implies that the Dynkin diagrams are precisely the proper subsets of extended Dynkin diagrams.

(f) Finally prove that every Dynkin diagram can be obtained uniquely from an extended Dynkin one by chopping off a vertex where the entry equals one, called an extending vertex. Show that the symmetries of the diagram act transitively on the set of extending vertices, which implies that it does not matter which vertex we chop off.