Talk: Deformations of toric Poisson structures

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Based on joint work with Pym, Matviichuk–Pym

I. Motivation + setting

II. Holonomic Poisson structures (or log symplectic)

III. Normal crossings case (log toric)

IV. Examples $(\mathbb{C}^4, \mathbb{P}^4)$

Slides are on my Imperial page ("Talks and Lectures"):

https://www.imperial.ac.uk/people/t.schedler/page/talks-and-lectures.html

I. Motivation and Setting

Goal: To understand moduli of holomorphic Poisson manifolds (or Poisson $\mathbb{C}$-alg. varieties).

↳ gives new examples of Poisson structures
↳ elucidates ones we know, classifies their deformations
↳ Also applies to symplectic / log symplectic manifolds

Example: Harder preprint on toric Poisson degenerations of hyperkähler manifolds

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Moduli space: $\mathcal{M} := \{ (X, \pi)$ holomorphic Poisson manifold (or $\mathbb{C}$ alg. variety) $\}$

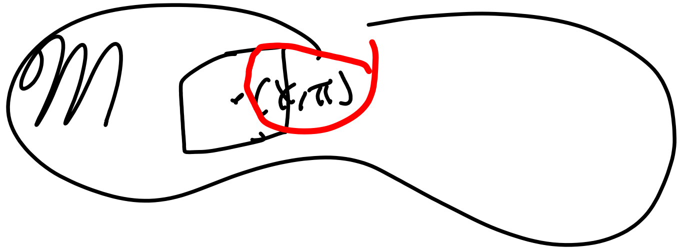
. $\pi \in \Gamma(X, \Lambda^2 T_X)$ Poisson bivector (manifold case).

Difficult to understand. ←

E.g.:
- Fixing $X = \mathbb{P}^3$, Cerveau–Lins Neto proved: $\mathcal{M}$ has 6 irreducible components.
- Using techniques from this project, we find ~ 40 (new) components for $X = \mathbb{P}^4$.

⚠ In general, X can also vary in $\mathcal{M}$; but, there are no deformations of $X = \mathbb{P}^n$.

Local approach:



$T_{(X,\pi)}\mathcal{M} \cong \boxed{HP^2(X,\pi)}$

2nd Poisson cohomology.

Formal neighbourhood: Governed by dgla (via Maurer–Cartan formalism)

differential graded Lie algebra

$$\mathbb{H}(\mathcal{L}^\bullet_{X,\pi}) := \left(\Lambda^\bullet T_X,\ d_\pi := [\pi, -]_{\text{Schouten–Nijenhuis}}\right) = \text{sheaf of dglas (shifted)}$$

$$\| \quad HP^\bullet(X)$$

BACKGROUND:
(Schouten–Nijenhuis bracket extends Lie bracket $\Lambda^1 T_X = T_X \times T_X \longrightarrow T_X$ so as to be a (graded) derivation: $[\xi, \eta\wedge\eta'] = [\xi,\eta]\wedge\eta' + (-1)^{|\eta||\eta'|}[\xi,\eta']\wedge\eta$.)

Reason: A formal deformation of π:

$$\pi' = \pi + \hbar\pi_1 + \hbar^2\pi_2 + \dots$$

Jacobi identity $\Longleftrightarrow$
$[\pi', \pi'] = 0$

$\cdot\ [\pi, \pi_1] = 0 \Longleftrightarrow \pi_1 \in \mathcal{L}^2_{X,\pi}$ d_π-closed

$\cdot\ \frac{1}{2}[\pi_1, \pi_1] + [\pi, \pi_2] = 0 \Longleftrightarrow [\pi_1, \pi_1] \subset \mathcal{L}^3_{X,\pi}$ coboundary

$\vdots$

II. Holonomic Poisson structures

Moral: Deformation theory is easier / more controllable the more non-degenerate (X,π) is:

Symplectic case: $\mathcal{M}_{(X,\pi)}$ is smooth at (X,π), $T_{(X,\pi)}\mathcal{M} = H^2(X)$.

Goto: A log symplectic structure is a closed meromorphic two-form $\omega \in \Gamma(X, \Omega^2(\log D))$, $D \subseteq X$ divisor = hypersurface (reduced)

closed

s.t.

$$\Omega^1_X(\log D) \underset{\omega^\#}{\overset{i_\pi}{\rightleftarrows}} T_X(-\log D) \quad \cong$$

nondegeneracy

$\Rightarrow (X,\pi)$ Poisson

Example: $X = \mathbb{P}^n$, $D = \mathbb{P}^{n-1}_1 \cup \dots \cup \mathbb{P}^{n-1}_k$, $k \leq n$ gen. smooth

$$\Rightarrow \Omega^1_X(\log D) = \mathrm{Span}_{\mathcal{O}_X}\left(\frac{dx_i}{x_i} \,\middle|\, i \leq k,\ dx_i \,\middle|\, i > k\right)$$

$$T_X(\log D) = \mathrm{Span}_{\mathcal{O}_X}\left(x_i\partial_i \,|\, i \leq k,\ \partial_i \,|\, i > k\right)$$

E.g.: $\omega = \sum_{i=1}^{n/2} \underbrace{\frac{dp_i}{p_i}}_{d\log p_i} \wedge dq_i$, $\pi = \sum_{i=1}^{n/2} p_i \partial_{p_i} \wedge \partial_{q_i}$, $D = \bigcup_{i=1}^{n/2} \{p_i = 0\}$.

Proposition (Goto): (X,π) log symplectic $\iff$ $D :=$ degeneracy locus is $\boxed{\text{reduced}}$. (7/17)

(= generically smooth)

$$D = Z\left(\mathrm{Pf}(\pi) = \wedge^{\dim X/2}\pi\right)$$

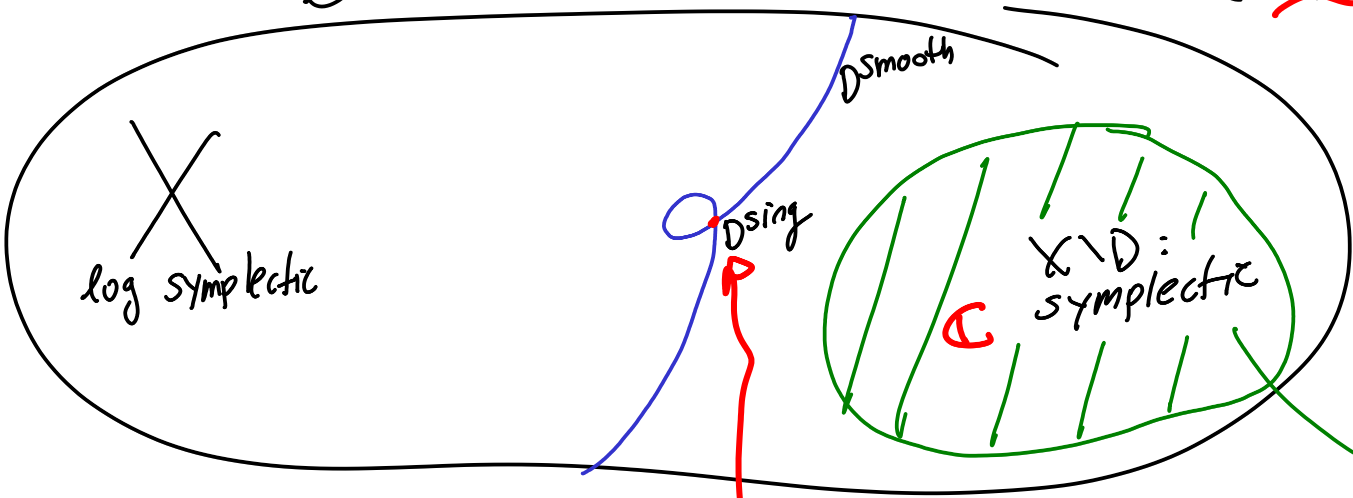
no repeated factors

section of anticanonical bundle

Proposition: In this case, $\mathcal{L}_{(X,\pi)}$ is locally finite-rank

on D^{smooth}: $\mathcal{L}_{(X,\pi)}\big|_{X\setminus D^{sing}} \simeq \left(\Omega^{\bullet}_X(\log D)\big|_{X\setminus D^{sing}}, d_{DR}\right)$

locally finite-dimensional



would be ∞ rank

$$\mathcal{L}_{X\setminus D} \simeq \Omega^{\bullet}_{X\setminus D} \simeq \mathbb{C}_{X\setminus D}$$

(de Rham cohomology of B^n is $\mathbb{C}$)

Hierarchy: (X, π) is: $(X$ connected$)$

- Generically symplectic $(D = Z(\mathrm{Pf}\,\pi) \subsetneq X)$ $\Longleftrightarrow$ $\mathcal{L}_{X,\pi}$ finite-rank outside codim ≥ 1 (D) $\Rightarrow$ $HP^0(X) = \mathbb{C}$ finite-dimensional
- log symplectic (D reduced) $\Longleftrightarrow$ $\mathcal{L}_{X,\pi}$ finite-rank outside codim ≥ 2 (D^{sing}) $\Rightarrow$ $HP^1(X) = \langle \xi_{\log g} \rangle$, $Z(g) \leq D$, finite-dimensional

$\vdots$

$\xi_f = \{f, -\}$

Defn (Pym–S.) (X, π) is holonomic if D-module version $M_{X,\pi} := \mathrm{Diff}(\mathcal{O}_X, \mathcal{L}_{X,\pi})$ has holonomic cohomology

ie, Lagrangian support $\subseteq T^*X$

$\xrightarrow{\text{Kashiwara}}$ $\mathcal{L}_{X,\pi}$ constructible $\Rightarrow$ $\mathcal{L}_{X,\pi}$ locally of finite rank (all strata)

Prop (Pym–S.) In this case, $\mathcal{H}^i(M_{X,\pi}) = 0$, $i \neq 0$ $\xrightarrow{\text{Kashiwara}}$ $\mathcal{L}_{X,\pi}[\dim X]$ perverse.

In particular, X is stratified,

$\mathcal{L}_{X,\pi}$ built out of local systems $\mathcal{L}_i[-d_i]$ on strata of codim d_i.

$X \supsetneq D \supsetneq D^{sing} \supsetneq \cdots$ (refines iterated singular loci)

Prop (Pym–S.) This includes symplectic leaves $Z \subseteq X$ such that $\xi_{\log g} = \Delta_\pi$ tangent to Z ($\{g=0\}=D$ locally)

modular vector field.

Defn: "characteristic symplectic leaves"

Conjecture (Matviichuk–Pym–S.)

X holonomic $\Longleftrightarrow$ # characteristic symplectic leaves $< \infty$.

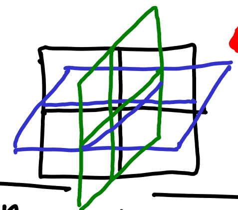
By Prop, $\Longrightarrow$ holds.

Theorem (Matviichuk–Pym–S.)

Conjecture holds if D has normal crossings

i.e. $\forall z \in D$, $\widehat{D_z} \cong \{x_1 x_2 \cdots x_k = 0\} \subseteq \widehat{\mathbb{C}^n}$

$n = \dim X$



$n=k=3$: $\{xyz=0\} \subseteq \widehat{\mathbb{C}^3}$

E.g. toric Poisson $\Rightarrow$ normal crossings, $X \setminus D = (\mathbb{C}^\times)^n$, strata are tori.

III. Normal crossings case (eg toric)

(10/17)

$X \supseteq D =: D^1 \supseteq D^2 \supseteq D^3 \supseteq \cdots$

$D^k :=$ D locally $\cong \{x_1 \cdots x_k = 0\}$.

Question: is there a log symplectic manifold nonholonomic in codim 2?

- $D \setminus D^1$: symplectic
- At $x \in D^1 \setminus D^2$: a neighbourhood $\cong$ symplectic $\times$ log symplectic surface
  - symplectic: dim $n-2$, $^{\frac{n}{2}-1}$
  - log symplectic surface: dim 2, D
- At $x \in D^2 \setminus D^3$ [Ran] : Either: (a) As above ($^{\frac{n}{2}-1}$ $\times$ D), OR
  (b) $^{\frac{n}{2}-2}$ $\times$ $\times$

Characteristic symplectic leaf

$\Rightarrow$ holonomic outside D^3.

Proposition (Matviichuk–Pym–Schedler):

- $HP^{\le 2}(X) \cong \underbrace{H^{\le 2}(X \setminus D)}_{\text{symplectic}} \oplus \bigoplus_{Z} H^0(\overline{Z}, \mathcal{L}_Z^{nr})[-2]$

  Z characteristic codim 2 leaf ($^{\frac{n}{2}-1} \times \cdot$ in (a))

  $\mathcal{L}_Z^{nr}$: local system in $\mathcal{L}_{X,\pi}$ ("nonresonantly" extended to $\overline{Z}$)

  $H^0(\overline{Z}, \mathcal{L}_Z^{nr}) = \begin{cases} \mathbb{C}, & \text{if } \mathcal{L}_Z \text{ trivial, "nonresonant" "Z smoothable"} \\ 0, & \text{if } \mathcal{L}_Z \text{ nontrivial or resonant} \end{cases}$

- If $X \setminus D^4$ holonomic,

  $HP^{\le 3}(X) \cong H^{\le 3}(X \setminus D) \oplus \bigoplus H^{\le 1}(Z, \mathcal{L}_Z)[-2]$

(13/17)

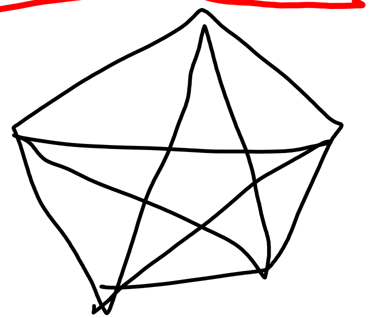
Diagrams: Given (X,π) normal crossings, have

dual graph $\Sigma(X)$: · vertices = components $D_i \subseteq D = D^1$

· k-cells = components of D^{k-1}

e.g.: $X = \mathbb{P}^4$, toric structure:

$$D = \mathbb{P}^3_0 \cup \cdots \cup \mathbb{P}^3_4, \quad \mathbb{P}^3_i = \{x_i = 0\}$$

dual graph = , complete graph since $\mathbb{P}^3_{i_1} \cap \cdots \cap \mathbb{P}^3_{i_p} \cong \mathbb{P}^{4-p} \neq \emptyset$, irreducible.

Color an edge red if it corresponds to a smoothable stratum (characteristic leaf with trivial L_Z)

Prop: $Z \subseteq D^\circ_{ij}$ smoothable $\iff \forall k \notin \{i,j\}$, $Y \subseteq \bar{Z} \cap D_k$ stratum,

biresidues of ω along Y $= B_Y = \begin{array}{c|ccc} & i & j & k \\ \hline i & 0 & a & b \\ j & -a & 0 & c \\ k & -b & -c & 0 \end{array}$ for $\boxed{\frac{c-b}{a} \in \mathbb{Z}_{\geq 0}}$

Color an angle red if it corresponds to a k with $\frac{c-b}{a} \geq 1$, opposite a red edge. (14/17)

Thm (MPS): For (X, π) toric, dim n, $D = D_1 \cup \dots \cup D_m$

$$\widehat{\mathcal{M}}_{(X,\pi)} \cong \bigcup V_I \,/\, (\mathbb{C}^\times)^n \rtimes \Gamma_I$$

$I \subseteq \{\text{smoothable strata} = \text{red edges in graph}\}$

smooth these strata

$$V_I \cong \underbrace{\mathbb{C}^I}_{\substack{\text{smooth} \\ \text{components in } I}} \times \underbrace{H^2(X \setminus D)^I}_{\substack{\text{Deformations of } \omega \text{ leaving} \\ I \text{ smoothable}}} \quad \text{linear space} \subseteq HP^2(X, \pi)$$

$\Gamma_I = \{$permutations of codim two strata coming from $\mathrm{Aut}(X, \pi, I)\}$.

To find components of $\mathcal{M}$: suffices to let $I =$ all red edges (up to moving π in $H^2(X \setminus D)^I$)

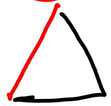
$\rightsquigarrow$ just count colorings of dual graph $/\cong$!

IV. $\mathbb{C}^4$, dual graph Δ $\left(D=\{x=0\}\cup\{y=0\}\cup\{z=0\}\right)$: (16/17)

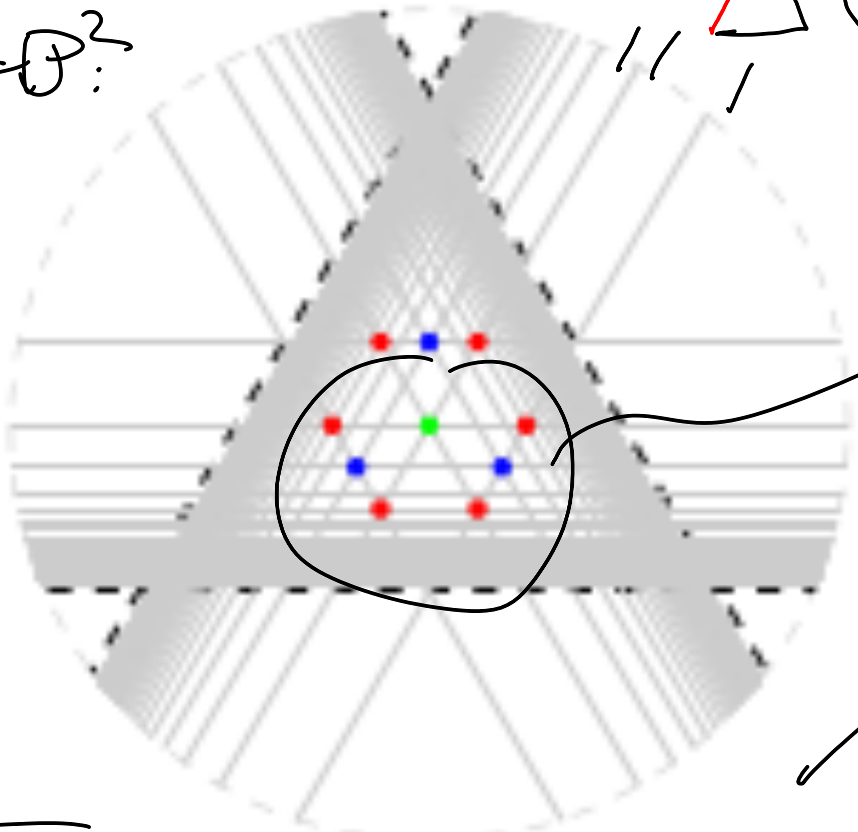
$\mathcal{M}_{\mathbb{C}^4,\pi}$ = union of linear spaces over $H^2(\mathbb{C}^4\setminus D)\cong\mathbb{C}^3$

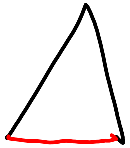
Depict over $\mathbb{P}(\mathbb{C}^3)=\mathbb{P}^2$:

$\mathcal{M}_{\mathbb{C}^4,\pi}$ =



One smoothing direction



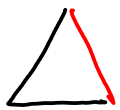


one smoothing direction

three smoothing directions

$/(\mathbb{C}^\times)^2\rtimes S_3$

acts on smoothing directions.



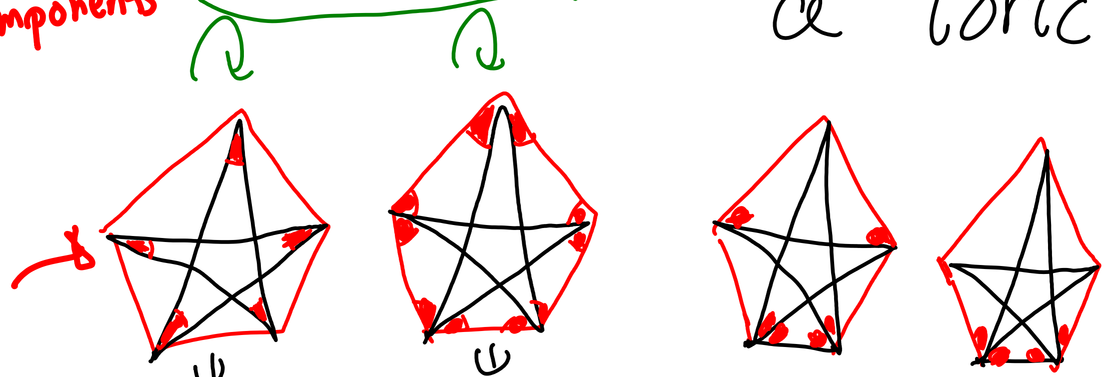
one smoothing direction

Same colour dots = same orbit under S_3

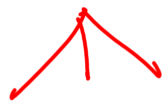
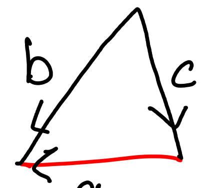
$\mathbb{P}^4$ case:

Thm (MPS): ~40 irreducible components of $\mathcal{M}$ for $X=\mathbb{P}^4$ containing a toric holonomic structure

characterises these components ⇝ dihedral symmetry

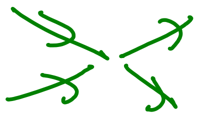


$q_{5,1}$ (Feigin–Odesskii) $\quad$ $q_{5,2}$ $\quad$ new $\quad$ new

Rules:
- No  (≤ 2 red edges at each vertex)
- Have to label edges by biresidues; given  $\Rightarrow \frac{b+c}{a} \in \mathbb{Z}_{\geq 0}$

Red angle: ≥ 1

Graph determines geometry of smoothed D. (which components glued, lower strata...)

$\mathbb{P}^{2n}$ case:
- at most two red angles opposite each red edge
- Kirchoff law holds:  Σ biresidues in $= \Sigma$ biresidues out
- ...