Algebraic Geometry, Spring 2020

What is Algebraic Geometry? (Introduction)

$\boxed{\text{The study of solutions to polynomial equations. Working over a field "}k\text{".}}$

$a_1x_1 + \dots + a_nx_n = 0$.

Example: (1) Linear equations define hyperplanes in $k^n = \{(x_1, \dots, x_n) \mid x_i \in k\}$

More generally: affine-linear: $a_1x_1 + \dots + a_nx_n = b$

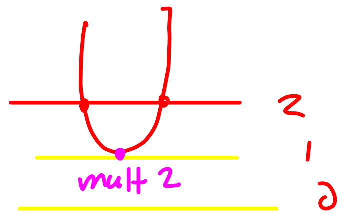
FACT (Linear Algebra): If we have m such equations, the intersection is:

- empty, OR
- affine-linear space of dim $\geq n-m$

if irredundant: $= n-m$

(2) In the plane k^2, a line and a conic (i.e. quadr(at)ic) intersect at ≤ 2 points.

e.g.: $\underbrace{y=0}_{m=1}$ $\quad$ $\underbrace{y = ax^2+bx+c}_{n=2}$ $\quad$ $1\cdot 2$



$\boxed{\text{BÉZOUT'S THEOREM: Two curves in } k^2 \text{ defined by degree } m, n \text{ polynomials intersect at } \leq m\cdot n \text{ points.}}$

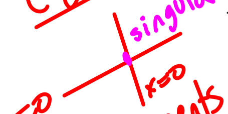
Can enhance to $=$, e.g.:

$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ has n roots with multiplicity over $\mathbb{C}$.

$\{f(x) = y\} \cap \{y = 0\}$ has size n, with mult.

Goal of module: Understand geometry of these intersections
(number of components, dimensions, singularities...)

e.g. $xy=0$

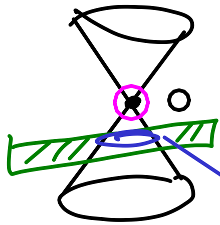


2 components

Example: $x^2+y^2+z^2=0$

$\subseteq \underbrace{k^3}_{3\text{-D}}$

Two-dimensional.



$z=c$ — 2-D plane

$x^2+y^2=c^2,\ z=c$ — 1-D circle

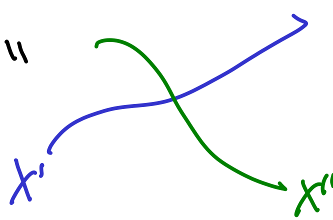
2-D singular cone — singular at 0 — has dilations

$x \mapsto \lambda x$
$y \mapsto \lambda y$
$z \mapsto \lambda z$

FACT (Algebraic Geometry): There is a notion of dimension, such that:

- $\dim(k^n)=n$
- $\dim(\{f=0\}) = n-1$ (f nonconstant). ⚠ What if $\{f=0\}=\emptyset$? (e.g. $x^2+y^2=-1$, $k=\mathbb{R}$)
- If $\dim X = m$, $X \subseteq k^n$, then, for $X_0 := X \cap \{f=0\}$,

if X is "irreducible" ($X \neq$ "



"), AND $\underbrace{X \not\subseteq \{f=0\}}_{f \text{ is irredundant}}$:

Theorem $\dim X_0 = \dim X - 1$. ←

a main goal of this module.
Generalises the linear algebra fact from before.

- $X \subseteq Y \Rightarrow \dim X \leq \dim Y$
- $\dim(X \cup Y) = \max(\dim X, \dim Y)$
- What about $\dim(X \cap Y)$?

FACT Can also define degree, generalise Bézout's theorem to:

$$\text{"}\deg X_0 \le (\deg X)(\deg f)\text{"}, \quad + \text{"equality refinement"}$$

Basic Idea of Algebraic Geometry:

Dictionary: (commutative) algebra $\longleftrightarrow$ Geometry

$k[x_1, \dots, x_n]$ $\qquad$ k^n

Applications:

(Commutative) algebra ← gives geometric approach.

- Complex analysis (polynomials are like holomorphic functions) ← $\mathbb{C}$-differentiable
- Number Theory:
  - Find solutions to equations over:
    - integers ("Diophantine")
    - number fields ($\mathbb{Q}, \mathbb{Q}[i], \dots$)
    - finite fields ($\mathbb{F}_p, \mathbb{F}_{p^m}, \dots$)
    - p-adic numbers $\mathbb{Z}_p, \mathbb{Q}_p, \dots$

    Can let k be one of these.
  - Important constructions using algebraic varieties: elliptic curves, Shimura varieties, ...
- Computer science: many applications of e.g. algebraic curves to cryptography, etc.
- See Wikipedia "Algebraic Geometry": tons of fields in industry use this.