Group Representation Theory — Lecture 1

Introduction:

Revise: Definition of a group (if you forgot).

Defn (informal): An n-dimensional representation ("rep" — shorthand.) of G is a way of writing (or "representing") group elements $g \in G$ as $n \times n$ matrices $\rho(g)$ such that:

over any field (mostly restricted to $\mathbb{C}$)

$$\boxed{\begin{array}{l} \cdot\ \rho(e) = I_n \\ \cdot\ \rho(gh) = \rho(g)\rho(h) \end{array}} \quad (\star)$$

Example: $\forall G$, let $\rho(g) = I_n$, $\forall g \in G$. "trivial n-dimensional ("dim") representation"

satisfies $(\star)$

If n is not specified, "trivial rep" = "trivial 1-dim rep".

Example: Let $\zeta \in \mathbb{C}$ s.t. $\zeta^m = 1$ "m-th root of 1", and $G = C_m = \{e, g, \ldots, g^{m-1}\}$, $g^m = e$.

$\rho(g^j) := (\zeta^j)$ (1×1 matrix) is a 1-dimensional (complex) representation.

Why does it satisfy $(\star)$? Plug in and check. $\rho(g^i)\rho(g^j) = \zeta^i \zeta^j = \zeta^{i+j} = \rho(g^{i+j})$. Could be: $i+j > m$: $\rho(g^{i+j}) = \rho(g^{i+j-m}) = \zeta^{i+j-m} = \zeta^{i+j}$

Note: for $\zeta = 1$, we recover the trivial rep. (works $\forall m$)

Example: $G = S_n$, $\rho(\sigma) =$ permutation matrix of σ "P_σ"

Recall: P_σ defined by: $\boxed{P_\sigma e_i = e_{\sigma(i)}}$, $e_i :=$ i^{th} elementary vector $= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ entry.

Dimension of ρ: n

Why does it satisfy ✪? $\cdot P_e = I$, $\cdot P_\sigma P_\tau \overset{?}{=} P_{\sigma\tau}$

$P_\sigma(P_\tau e_i) = P_\sigma e_{\tau(i)} = P_{\sigma\tau} e_i$ ✓ True $\forall i \Rightarrow P_\sigma P_\tau = P_{\sigma\tau}$

Example: $G = S_n$, $\rho(\sigma) = (\text{sign}(\sigma))$.

Recall: $\text{sign}(\sigma) = \begin{cases} 1, & \text{if } \sigma = \text{product of an } \underline{\text{even}} \text{ \# of transpositions}_{(ij)}, \\ -1, & \text{" " " " " } \underline{\text{odd}} \text{ " " "} \end{cases}$.

Revise: this is well-defined.

$$\left.\begin{array}{l} (-1)^{\#\text{ transpositions product to } \sigma} \\ \cdot(-1)^{\#\text{ " of } \tau} \\ = (-1)^{\#\text{ for } \sigma\tau} \end{array}\right\}$$

$\text{Dim}(\rho) = 1$. Why does it satisfy ✪? $\rho(e) = \text{sign}(e) = 1$

$\rho(\sigma)\rho(\tau) = \text{sign}(\sigma)\text{sign}(\tau) = \text{sign}(\sigma\tau)$

(Revise: $\text{sign}(\sigma) = \det(P_\sigma)$ (note: $\det(a_{ij}) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, a_{i\sigma(i)}$.))

Example: $G = D_n$ = dihedral group of order $2n$

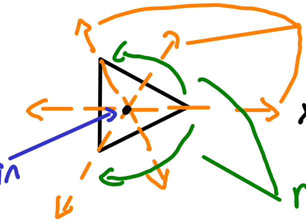
(⚠ Sometimes this is called D_{2n}. I will use the "geometric" convention D_n. That way, $C_n < D_n < S_n$, and note $|S_n| = n! \neq n$.)

Revise: definition of D_n = rotational + reflectional symmetries of a regular n-gon.

- Take $\rho(g)$ = the 2×2 matrix which acts on a fixed regular n-gon centred at the origin. Assume that it is symmetric about the x-axis.

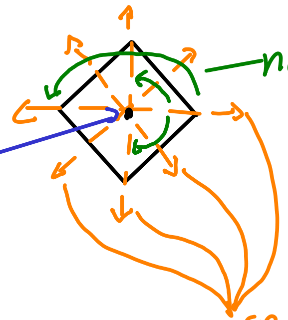
Small n:

$n=3$:



Exercise: work out the matrices for these ($n=3,4$) from scratch.

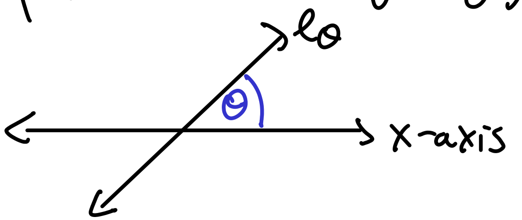
$n=4$:



Matrices:

ρ(rotation by θ counter-clockwise): $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

ρ(reflection by ℓ_θ): $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ ← $\theta = 180° = \pi$ ✓ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$\theta = 0$: ← $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\theta = \frac{\pi}{2}$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\theta = \frac{\pi}{4}$ ↗ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Motivation + History of Group Representation Theory:

- Born in letter from Dedekind to Frobenius in late 1800s:

  Let: • $G = \{x_1, \dots, x_n\}$ = group of size n.

  • A = $n \times n$ multiplication table (as $n \times n$ matrix).

$\begin{pmatrix} x_1 & \cdots & x_n \\ \vdots & \cdot x_i \cdots & \\ x_n & & \end{pmatrix}$

$\cdot f := \det A$, considered as polynomial $\in k[x_1,\dots,x_n]$, degree $= n$.

Observation (Dedekind): f factors into irreducible polynomials as follows:

$f = f_1^{d_1} \cdots f_m^{d_m}$, f_i irreducible, such that:

$$\boxed{d_i = \deg f_i} \quad (\star)$$

Examples: $\cdot\ G = C_3$: $A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{pmatrix} \Rightarrow f = \det A = 3x_1x_2x_3 - x_1^3 - x_2^3 - x_3^3$

Factorisation of f into irreducibles: (use: $\zeta = e^{2\pi i/3}$, primitive cube root of 1):

$-(x_1+x_2+x_3)(x_1+\zeta x_2+\zeta^2 x_3)(x_1+\zeta^2 x_2+\zeta x_3)$

$\star$ holds b.c. $d_i = 1 = \deg f_i$ linear, mult one.

Exercise: do this for C_n, general n.

Observe: all irreducible factors are linear.

Deep fact: this holds iff G is abelian.

$\cdot\ G = S_3$: (Computer) Exercise:

$f = (x_1+x_2+\dots+x_6)(x_1-x_2+x_3-x_4+x_5-x_6)\cdot g^2$

"sign rep"

Here: x_1, x_3, x_5 are even, x_2, x_4, x_6 are odd (transpositions).

$x_1+\dots+x_n$ ALWAYS factor (trivial rep)

$g =$ irreducible polynomial of degree 2. $\rightarrow$ a 2-dim rep. (Q: which?)

Explaining + proving this led Frobenius to invent group representation theory.

$\rightarrow$ proved $\star$ in 1896.

- Felix Klein 1872: "Erlangen program": group theory arises as <u>symmetries</u> of geometric spaces ($\rightsquigarrow$ matrices as in D_n case above)
- Burnside 1904: proved the following using <u>representation theory</u>:

$$\boxed{\begin{array}{l}\text{If } |G| = p^r q^s, \ p, q \text{ prime}, \ r+s \geq 2, \text{ then:} \\ \quad \exists \text{ normal subgroup } N \trianglelefteq G, \ N \neq \{e\}, \ N \neq G \\ \quad (G \text{ is NOT "simple"}).\end{array}}$$

we will prove this at end of course (nonexaminably).

- Number Theory: Representations of Galois groups, especially in "number field" case:

$$\overline{F}/F, \quad F \supseteq \mathbb{Q}, \quad [F:\mathbb{Q}] < \infty.$$

$\rightsquigarrow$
- "Class Field Theory" = study of the 1-dim reps (now well-understood).
- ≥ 2-dim reps: "classical Langlands correspondence"

$\hookrightarrow$ Special case of dim 2: Taniyama–Shimura conjecture
(reps coming from elliptic curves)

$\Downarrow$

Wiles' proof of Fermat's Last Theorem

<u>Theorem (Breuil–Conrad–Diamond–Taylor)</u>

"Geometric Langlands": a geometric analogue

$\hookrightarrow$ "a kind of grand unified theory of mathematics"

(Edward Frenkel, Author of "Love and Math", director of "Rites of Love and Math"
(homage to Mishima "Patriotism, or Rites of Love and Death"))

- Chemistry: G = symmetry group of a molecule/compound/crystal
  ρ = action by rotations + reflections

- Quantum Mechanics: Spherical symmetry gives rise to discrete ("quantised") energy levels, orbitals, etc, via representation theory of ρ via 3D rotations.

- Differential Equations: G = symmetry group of an equation
  $\Rightarrow$ the vector space of solutions is a representation of G.

- Quantum Chromodynamics (physics): Representations of basic rotation groups give rise to Gell-Mann's "Eightfold Way" (name from Buddhism)
  ↳ this led to the development of the quark model.
  ↳ now, representation theory explains this via flavour symmetry groups between types of quarks, and their representations.