Group representation theory, Lecture Notes

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1 References

These notes are for courses taught by the author at Imperial College. They follow somewhat the previous iterations by Bellovin (`http://wwwf.imperial.ac.uk/~rbellovi/teaching/m3p12.html`), Newton (`https://nms.kcl.ac.uk/james.newton/M3P12/notes.pdf`), Segal (`http://www.homepages.ucl.ac.uk/~ucaheps/papers/Group%20Representation%20theory%202014.pdf`), and Towers (`https://sites.google.com/site/matthewtowers/m3p12`).

I will mostly base the course on the above notes and will not myself look much at textbooks. That said, it may be useful for you to read textbooks to find additional exercises as well as further details and alternative points of view. The main textbook for the course is:

- G. James and M. W. Liebeck, *Representations and characters of groups.*

Other recommended texts include (listed also on the library page for the course):

- P. Etingof et al, *Introduction to representation theory.* This is also available online at `http://math.mit.edu/~etingof/replect.pdf`. Note that the approach is more advanced and ring-theoretic than we will use, but nonetheless we will discuss some of the results in Sections 1–3 and maybe touch on (non-examinably) the definition of categories (Section 6).

- J.-P. Serre, *Linear representations of finite groups.* Only Chapters 1,2,3,5, and 6 are relevant (and maybe a bit of Chapters 7 and 8).

- M. Artin, *Algebra.* Mainly Chapter 9 is relevant (also you should be familiar with most of Chapters 1-4 and some of Chapter 7 already). You should also be familiar with some of Chapter 10 (be comfortable with rings; no algebraic geometry is needed).

- W. Fulton and J. Harris, *Representation theory : a first course.* Only Sections 1–3 are relevant; Sections 4 and 5 might be interesting as examples but we won't discuss it in the course (beyond a brief mention perhaps).

- J. L .Alperin, *Local representation theory.* The first two chapters are introductory and relevant to the course; beyond that is irrelevant for us, but interesting if you are curious what happens working over positive characteristic fields.

1.1 Limitations on scope of the exam and course

In this course we will exclusively use the *complex field* in all assessed work and in all material you are responsible for on the exam. You can also expect that you can restrict to the case where the representations are *finite-dimensional complex vector spaces.* There are a few results stated in greater generality, but you don't need to remember the details of what extends and what doesn't to the infinite-dimensional case: if not then simply say in your solutions (e.g., to the recollection of a definition or theorem) that you will take the vector spaces involved to be finite-dimensional. Therefore for purposes of following the course and preparing for the exam, you may restrict your attention to finite-dimensional complex vector spaces if you prefer.

Additionally, we will almost exclusively be dealing with *representations of finite groups*: that is, the group G that we take representations of will almost always be finite. You won't need to learn a lot about what happens outside of this case, although a few results are stated in greater generality (because the hypothesis that G is finite is not needed) and I have provided a few examples where G is infinite that I think you can and should understand. Any assessed work (including the exam) will keep the use of representations of infinite groups to a minimum.

In these notes I have included several *non-examinable* remarks. The main purpose of these is to make sure you have some exposure to more general situations than the ones we are primarily concerned with in this course: typically the case where the field is not $\mathbf{C}$, or where the vector spaces (representations) are infinite-dimensional. You should glance at these to have an idea for what happens, but only read in more detail if you are curious.

I stress that *there is no obligation to look at these at all*, and I will not expect you to understand or remember these.

2 Introduction and fundamentals

2.1 Groups

Definition 2.1.1. A group is a set G together with an associative multiplication map $G \times G \to G$ (written $g \cdot h$) such that there is an identity element $e \in G$ (i.e., $e \cdot g = g \cdot e = g$ for all $g \in G$) and, for every element $g \in G$, an inverse element g^{-1} satisfying $g \cdot g^{-1} = e = g^{-1} \cdot g$.

A group G is called *finite* if G is a finite set.

2.2 Representations: informal definition

Informally speaking, a representation of a group G is a way of writing the group elements as square matrices of the same size (which is multiplicative and assigns $e \in G$ the identity matrix). The *dimension* of the representation is the size (number of rows = number of columns) of the matrices. **Before getting to the formal definitions**, let us consider a few examples. Given $g \in G$, let $\rho(g)$ denote the corresponding matrix.

Example 2.2.1. If G is any group, we can consider the 1-dimensional representation $\rho(g) = (1)$ for all $g \in G$. This is called the *trivial representation*.

Example 2.2.2. Let ζ be any n-th root of unity (e.g., $\zeta = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$, or more generally $\zeta = e^{2\pi ik/n}$ for any $k \in \{0, 1, \ldots, n-1\}$). Then, if $G = C_n = \{1, g, g^2, \ldots, g^{n-1}\}$ is the cyclic group of size n, then we can consider the 1-dimensional representation $\rho(g^m) = (\zeta^m)$. Notice that for $\zeta = 1$ we recover the trivial representation.

Example 2.2.3. Let $G = S_n$. Consider the n-dimensional representation where $\rho(g)$ is the corresponding $n \times n$ permutation matrix.

Example 2.2.4. Let $G = S_n$ and consider the one-dimensional representation where $\rho(g) = (\mathrm{sign}(g))$, the sign of the permutation g.

Remark 2.2.5. The previous two examples are related: applying the determinant to the permutation matrix recovers the sign of the permutation.

Finally one geometric example:

Example 2.2.6. Let $G = D_n$, the dihedral group of size $2n$ [Caution: in algebra this group is often denoted D_{2n}! We decided to use the geometric D_n notation since C_n and D_n naturally are subgroups of S_n, whereas $|S_n| = n \neq n!$.] We have a 2-dimensional representation where $\rho(g) = A$ is the matrix such that the map $v \mapsto Av, v \in \mathbf{R}^2$ is the associated reflection or rotation in $\mathbf{R}^2$. Therefore, for g a (counterclockwise) rotation by θ,

$\rho(g) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. For g a reflection about the line which makes angle θ with the x-axis, $\rho(g) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$. For example, if $n = 4$ so $|G| = 8$, we can list all eight elements $g \in G$ and their matrices $\rho(g)$:

- $g = 1$: $\rho(1) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $g =$ (counterclockwise) rotation by $\pi/2$: $\rho(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $g =$ rotation by π: $\rho(g) = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
- $g =$ rotation by $3\pi/2$: $\rho(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- $g =$ reflection about x-axis: $\rho(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $g =$ reflection about $x = y$ line: $\rho(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $g =$ reflection about y-axis: $\rho(g) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- $g =$ reflection about $x = -y$ line: $\rho(g) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

2.3 Motivation and History

Group representation theory was born in the work of Frobenius in 1896, triggered by a letter from Dedekind (which made the following observation, which I take from Etingof et al: Let the elements of a finite group G be variables $x_1, \ldots, x_n$, and consider the determinant of the multiplication table, a polynomial of degree n. Then Dedekind observed that this factors into irreducible polynomials, each of whose multiplicity equals its degree.) The history of group representation theory is explained in a book by Curtis, *Pioneers of representation theory*.

This theory appears all over the place, even before its origin in 1896:

- In its origin, group theory appears as symmetries. This dates at least to Felix Klein's 1872 Erlangen program characterising geometries (e.g., Euclidean, hyperbolic, spherical, and projective) by their symmetry groups. These include as above rotational and reflectional matrices.

- In 1904, William Burnside famously used representation theory to prove his theorem that any finite group of order $p^a q^b$, for p, q prime numbers and $a, b \geq 1$, is not simple, i.e., there exists always a proper nontrivial normal subgroup. (By induction on a and b this implies that these groups are solvable). This is in the course textbook of James and Liebeck and it requires not much more than what is done in this course (we could try to get to it at least in non-assessed coursework).

- In number theory it is of crucial importance to study representations of the absolute Galois groups of finite extension fields of $\mathbf{Q}$. The case of one-dimensional representations is called *(abelian) class field theory*, and was a top achievement of algebraic number theory of the 20th century. A generalisation to higher dimensional representations was formulated by Robert Langlands in the late 1960s, and is called the *(classical) Langlands correspondence*. In the two-dimensional case for certain representations coming from elliptic curves, this statement becomes the Taniyama–Shimura conjecture, which is now a theorem, which implies Fermat's Last Theorem (Wiles and Taylor–Wiles proved the latter in 1995 by proving the necessary cases of the conjecture, and in 2001 the full Taniyama–Shimura conjecture was proved by Breuil–Conrad–Diamond–Taylor). In general the classical Langlands correspondence remains a wide-open conjecture.

- Partly motivated by an attempt to break through its difficulty, the Langlands correspondence has a (complex) geometric analogue, thinking of the Galois group as automorphisms of a geometric covering (of a complex curve = Riemann surface), where representations yield local systems. One obtains the "geometric Langlands correspondence" (the first version was proved in the one-dimensional case by Deligne, in the two-dimensional case by Drinfeld in 1983, stated for dimensions ≥ 3 by Laumon in 1987, and proved in 2001 and 2004 by Frenkel–Gaitsgory–Vilonen and Gaitsgory). There are by now many deeper and more general statements, many of which are proved.

- Representation theory appears in chemistry (going back at least to the early 20th century). For instance, G is the symmetry group of a molecule (rotational and reflectional symmetries) and $\rho(g) = A$ is the 3×3-matrix such that $v \mapsto Av$ is the rotation or reflection.

- In quantum mechanics, spherical symmetry of atoms give rise to orbitals and, in atoms, is responsible for the discrete ("quantised") energy levels, momenta, and spin of electrons. By the Pauli exclusion principle (formulated by Wolfgang Pauli in 1925) this is responsible for the structure of electron shells and ultimately all of chemistry.

- More generally, given any differential equations that have a symmetry group G, the solutions to the system of equations form a representation of G. Applied to the Schrödinger equation this recovers the previous example in quantum mechanics.

- The study of quantum chromodynamics (quarks and gluons and their strong nuclear interactions) is heavily based in the representation theory of certain basic Lie groups. For example, Gell-Mann famously proposed in 1961 the "Eightfold Way" (of physics, not Buddhism after which it was named) which states that the up, down, and strange quarks form a basis for the three-dimensional representation of SU(3) (the "special unitary group" of matrices A such that $\det(A) = 1$ and $A^{-1} = \overline{A^t}$). This explained why the particles appear in a way matching precisely this group's representation theory.

2.4 Representations: formal definition

Now we give the main definition of the course. We will need to work with vector spaces and linear algebra: so you are assumed to be familiar with this.

Given a vector space V, let $\mathrm{GL}(V)$ denote the group of invertible linear transformations from V to itself ("GL" stands for "General Linear"). If V has a basis $v_1, \ldots, v_n$, then in terms of the basis we can identify $\mathrm{GL}(V)$ with the more familiar group GL_n of invertible $n \times n$ matrices: this follows by the correspondence between linear maps $V \to V$ and $n \times n$ matrices (which you should know).

Definition 2.4.1. A representation of a group G is a pair (V, ρ) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

Remark 2.4.2. We can also think of (V, ρ) as a pair of V and a map $G \times V \to V, g \cdot v := \rho(g)(v)$. Then, the axioms that $G \times V \to V$ should satisfy are: (1) it is a group action (the action is associative and $e \cdot v = v$ for all $v \in V$), and (2) the map $V \to V$, $v \mapsto g \cdot v$ is linear for all $g \in G$. That is, a representation is nothing but a *linear action of G on V*.

You may have noticed that whenever a new notion is introduced in mathematics, there is a corresponding notion of the important functions, or "morphisms," between such objects. Then, the "isomorphisms" are the invertible morphisms. For example:

Objects	Morphisms	Isomorphisms
Groups	Homomorphisms	Isomorphisms
Vector spaces	Linear maps	Isomorphisms
Topological spaces	Continuous maps	Homeomorphisms
Rings	Ring homomorphisms	Ring isomorphisms
Group representations	?	?

The question marks are the following:

Definition 2.4.3. A *homomorphism of representations* $T : (V, \rho) \to (V', \rho')$ is a linear map $T : V \to V'$ such that

$$T \circ \rho(g) = \rho'(g) \circ T \tag{2.4.4}$$

for all $g \in G$. T is an *isomorphism* if T is invertible. Two representations are *isomorphic* if there exists an isomorphism between them.

As is the case in many places in mathematics, condition (2.4.4) can equivalently be expressed by requiring that the following diagram *commute*, i.e., all paths with the same endpoints give the same function (in this case, both compositions from the top left corner to the bottom right are equal):

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V' \arrow[r, "\rho'(g)"] & V' \end{tikzcd} \tag{2.4.5}$$

Exercise 2.4.6. (i) If T is a homomorphism of representations and T is invertible *as a linear transformation* then the inverse T^{-1} is also a homomorphism of representations. (ii) Using linear algebra and (i), prove that the following are equivalent: (a) T is an isomorphism of representations; (b) T is a bijective homomorphism of representations.

In this course we will always take the underlying field of the vector space to be $\mathbf{C}$, the field of complex numbers, and will not further mention it. However, much of what we say will not require this, and in your future lives you may be interested in more general fields (e.g., the real field in geometry, and finite extensions of $\mathbf{Q}$ as well as finite and p-adic fields in number theory). Moreover, we will exclusively be concerned with the case where G is a finite group. Thus the subject of this course is properly *complex representations of finite groups*, and especially studying them up to isomorphism.

Suppose that V is n-dimensional, and pick a basis $\mathcal{B} = (v_1, \ldots, v_n)$ of V. As above, we can identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbf{C})$, the group of invertible $n \times n$ matrices with complex coefficients. Precisely, we have the map $T \mapsto [T]_{\mathcal{B}} = T_{\mathcal{B},\mathcal{B}}$, where $[T]_{\mathcal{B}}$ is the matrix of T in the basis $\mathcal{B}$ (and more generally $[T]_{\mathcal{B},\mathcal{B}'}$ is the matrix of T in the pair of bases $\mathcal{B}, \mathcal{B}'$). Thus Definition 2.4.1 becomes more concrete:

Definition 2.4.7. A (complex) n-dimensional representation of G is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbf{C})$.

Although equivalent to Definition 2.4.1, Definition 2.4.7 is *inferior* since the equivalence requires choosing a basis of V. Let us define some notation for the result of such a choice:

Definition 2.4.8. Given (V, ρ) with $\dim V = n$ and a basis $\mathcal{B}$ of V, let $\rho^{\mathcal{B}} : G \to \mathrm{GL}_n(\mathbf{C})$ be the map

$$\rho^{\mathcal{B}}(g) = [\rho(g)]_{\mathcal{B}}. \tag{2.4.9}$$

Different choices of basis give rise to different homomorphisms to $\mathrm{GL}_n(\mathbf{C})$. This follows directly from the change-of-basis formula from linear algebra. Below we will need the following basic formula: Let W_1, W_2, and W_3 be vector spaces with bases $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$. Then given linear maps $T : W_1 \to W_2$ and $S : W_2 \to W_3$,

$$[S \circ T]_{\mathcal{B}_1,\mathcal{B}_3} = [S]_{\mathcal{B}_2,\mathcal{B}_3}[T]_{\mathcal{B}_1,\mathcal{B}_2}. \tag{2.4.10}$$

Now let $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{B}' = (v'_1, \ldots, v'_n)$ are two bases of V. They are related by some invertible matrix $P \in \mathrm{GL}_n(\mathbf{C})$. Precisely,

$$P = [I]_{\mathcal{B}',\mathcal{B}}, \tag{2.4.11}$$

or equivalently

$$(v_1, v_2, \ldots, v_n)P = (v'_1, v'_2, \ldots, v'_n). \tag{2.4.12}$$

Exercise 2.4.13. Verify that (2.4.11) and (2.4.12) are equivalent.

Solution (omitted from lecture): Note that (2.4.12) means that, for $P = (p_{ij})$, then $v'_i = \sum_j v_j P_{ji}$. We claim that, for $v = \sum_i a_i v_i$ in terms of $\mathcal{B}$, then in terms of $\mathcal{B}'$, we have $v = \sum_i a'_i v'_i$, where

$$(a'_1, a'_2, \ldots, a'_n)^t = P^{-1}(a_1, \ldots, a_n)^t. \tag{2.4.14}$$

To see that this follows from (2.4.12), we can do the computation

$$v = (v_1, \ldots, v_n) \cdot (a_1, \ldots, a_n) = (v_1, \ldots, v_n)(a_1, \ldots, a_n)^t = (v'_1, \ldots, v'_n)P^{-1}(a_1, \ldots, a_n)^t, \tag{2.4.15}$$

using the dot product of vectors in the second expression. We conclude from (2.4.14) that $P^{-1} = [I]_{\mathcal{B},\mathcal{B}'}$. Similarly, $P = [I]_{\mathcal{B}',\mathcal{B}}$.

Lemma 2.4.16. The representations $\rho^{\mathcal{B}}$ and $\rho^{\mathcal{B}'}$ are related by *simultaneous* conjugation:

$$\rho^{\mathcal{B}'}(g) = P^{-1}\rho^{\mathcal{B}}(g)P. \tag{2.4.17}$$

Proof. By (2.4.10): $\rho^{\mathcal{B}'}(g) = [\rho(g)]_{\mathcal{B}',\mathcal{B}'} = [I]_{\mathcal{B},\mathcal{B}'}[\rho(g)]_{\mathcal{B},\mathcal{B}}[I]_{\mathcal{B}',\mathcal{B}} = P^{-1}\rho^{\mathcal{B}}(g)P$. □

This motivates:

Definition 2.4.18. Two n-dimensional representations $\rho, \rho' : G \to \mathrm{GL}_n(\mathbf{C})$ are equivalent if there is an invertible matrix $P \in \mathrm{GL}_n(\mathbf{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in G$.

Proposition 2.4.19. Two finite-dimensional representations (V, ρ) and (V', ρ') are isomorphic if and only if $\dim V = \dim V'$ and, for any (single) choices of bases of V and V', the resulting $(\dim V)$-dimensional representations ρ and ρ' are equivalent.

Here (V, ρ) is called finite-dimensional if V is finite-dimensional.

Proof. First, it is obvious that the two representations can be isomorphic only if they have the same dimension. Now the statement follows from the next important lemma. □

Lemma 2.4.20. Let (V, ρ) and (V', ρ') be two representations. Let $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{B}' = (v'_1, \ldots, v'_n)$ be bases of V and V', respectively. Then a linear map $T : V \to V'$ is a homomorphism of representations if and only if

$$[T]_{\mathcal{B},\mathcal{B}'} \circ \rho^{\mathcal{B}}(g) = (\rho')^{\mathcal{B}'}(g) \circ [T]_{\mathcal{B},\mathcal{B}'}. \tag{2.4.21}$$

It is an isomorphism if and only if $[T]_{\mathcal{B},\mathcal{B}'}$ is additionally an invertible matrix, so that it conjugates $\rho^{\mathcal{B}}(g)$ to $(\rho')^{\mathcal{B}'}(g)$.

Proof. This follows immediately from the linear algebra rule for matrices in terms of bases (2.4.10). □

Therefore, Definitions 2.4.1 and 2.4.7 are not so different if we look only at representations up to isomorphism in the first case and up to equivalence in the second case. This is what we are mostly interested in in this course.

Generalising Example 2.2.2, we can understand all representations of cyclic groups:

Exercise 2.4.22. Let $G = \langle g \rangle$ be a cyclic group. (i) Prove that a representation (V, ρ) of G is equivalent to a pair (V, T) of a vector space V and a linear transformation $T \in \mathrm{GL}(V)$ such that, if g has finite order $m \geq 1$, then $T^m = I$. The equivalence should be given by $T = \rho(g)$. (ii) Prove that, if $G = \{1\}$, then representations are equivalent to vector spaces.

2.5 Going back from matrices to vector spaces

We just discussed how to go from vector spaces (Definition 2.4.1) to matrices (Definition 2.4.7). It is useful to go back, especially in order to work with the examples from 2.2 via vector spaces. This is actually easier than the other way: Given an $n \times n$ matrix A, let $\widetilde{A} : \mathbf{C}^n \to \mathbf{C}^n$ be the linear map given by $\widetilde{A}(e_i) = Ae_i$. The map $A \mapsto \widetilde{A}$ is an isomorphism $\mathrm{GL}_n(\mathbf{C}) \xrightarrow{\sim} \mathrm{GL}(\mathbf{C}^n)$. (There is no danger in thinking of $\mathrm{GL}_n(\mathbf{C})$ and $\mathrm{GL}(\mathbf{C}^n)$ as the same in this way, but I will use the tildes to avoid possible confusion.)

Example 2.5.1. Let $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ be a group homomorphism. Then we can define the corresponding homomorphism $\widetilde{\rho} : G \to \mathrm{GL}(\mathbf{C}^n)$, given by $\widetilde{\rho}(g) = \widetilde{\rho(g)}$ for all $g \in G$. Then $(\mathbf{C}^n, \widetilde{\rho})$ is a group representation in the sense of Definition 2.4.1.

We apply this to Example 2.2.3 and it rewrites in the following subtly different way:

Example 2.5.2. We take $G = S_n$ and $\rho : S_n \to \mathrm{GL}_n(\mathbf{C})$ assigning to a permutation its permutation matrix. Then we get $(\mathbf{C}^n, \widetilde{\rho})$ where $\widetilde{\rho}(\sigma)(e_i) = e_{\sigma(i)}$.

2.6 Representations from group actions

The next construction can be thought of as a wide generalisation of Example 2.5.2:

Definition 2.6.1. If X is a finite set, define the complex vector space $\mathbf{C}[X]$ of linear combinations of the elements of X:

$$\mathbf{C}[X] := \{\sum_{x \in X} a_x x \mid a_x \in \mathbf{C}\} \tag{2.6.2}$$

Here, addition and scalar multiplication is done coeffiecientwise:

$$\lambda \sum_{x \in X} a_x x + \mu \sum_{x \in X} b_x x := \sum_{x \in X} (\lambda a_x + \mu b_x) x, \tag{2.6.3}$$

for all $\lambda, \mu, a_x, b_x \in \mathbf{C}$.

For ease of notation, we can drop the summands where $a_x = 0$. For instance, we write, for $X = \{x_1, x_2, x_3\}$,

$$2x_1 + 0x_2 + 3x_3 = 2x_1 + 3x_3. \tag{2.6.4}$$

Example 2.6.5. Let X be a finite set and $G \times X \to X$ a group action, $(g, x) \mapsto g \cdot x$. Then we can consider the representation $(\mathbf{C}[X], \rho)$ given by $\rho(g)(x) = g \cdot x$ for all $g \in G$ and $x \in X$, linearly extended to all of $\mathbf{C}[X]$: $\rho(g)(\sum_{x \in X} a_x x) = \sum_{x \in X} a_x (g \cdot x)$.

Remark 2.6.6 (Non-examinable)**.** The above can be done also if X is infinite, although then we have to modify the definition of $\mathbf{C}[X]$ to include only linear combinations of *finitely many* elements of X (or equivalently, all but finitely many of the a_x have to be zero). We will not need to use this and this remark is non-examinable.

2.7 The regular representation

The most important representation obtained as before is the *regular representation*:

Example 2.7.1. Let G be a finite group. Let X be the set $X := G$ together with the action $G \times X \to X$ given by left multiplication, $g \cdot x = gx$. Then the resulting representation $(\mathbf{C}[X], \rho)$ is called the *regular representation*.

When there is no confusion, we will drop the X and simply write $(\mathbf{C}[G], \rho)$ in this case.

Remark 2.7.2. Actually, $\mathbf{C}[G]$ as above also admits a ring structure with the same multiplication $g \cdot h = gh$ for $g, h \in G$, extended linearly. This will become important later and we will call it the *group algebra* or group ring. **Caution** that you should still think of the ring $\mathbf{C}[G]$ and the representation $\mathbf{C}[G]$ as different types of objects, although the underlying set is the same.

Example 2.7.3. Let $G = C_n$ be the cyclic group. Then the regular representation $(\mathbf{C}[G], \rho)$, in terms of its basis $\mathcal{B} = (1, g, g^2, \dots, g^{n-1})$, has the following form:

$$\rho^{\mathcal{B}}(g) = P_{(1,2,\dots,n)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2.7.4}$$

and $\rho^{\mathcal{B}}(g^m) = P_{(1,2,\dots,n)^m} = P^m_{(1,2,\dots,n)}$. In other words, $\rho^{\mathcal{B}}(g^m)$ is the permutation matrix such that the ones appear in the entries $(i+m, i)$, with i taken modulo n.

For example, when $n = 3$, we have

$$\rho^{(1,g,g^2)}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho^{(1,g,g^2)}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho^{(1,g,g^2)}(g^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.7.5}$$

Exercise 2.7.6. Let (V, ρ_V) be a representation of G. Show that, for every vector $v \in V$, there is a unique homomorphism of representations $\mathbf{C}[G] \to V$ sending $e \in G \subseteq \mathbf{C}[G]$ to $v \in V$. Conversely, show that all homomorphisms are of this form. Letting $\mathrm{Hom}_G(\mathbf{C}[G], V)$ denote the set of homomorphisms of representations $\mathbf{C}[G] \to V$, deduce that there is a bijection of sets $\phi : \mathrm{Hom}_G(\mathbf{C}[G], V) \xrightarrow{\sim} V$, $\phi(T) = T(e)$. Moreover, prove that this is linear, in the sense that $\phi(aS + bT) = a\phi(S) + b\phi(T)$ for all $a, b \in \mathbf{C}$ and all homomorphisms of representations $S, T : \mathbf{C}[G] \to V$. (In fact, as we will explain in multiple ways in Section 2.14, $\mathrm{Hom}_G(\mathbf{C}[G], V)$ is a vector space under addition and scalar multiplication. Thus, we proved that ϕ is an isomorphism of vector spaces.)

2.8 Subrepresentations

Definition 2.8.1. A subrepresentation of a representation (V, ρ) is a vector subspace $W \subseteq V$ such that $\rho(g)(W) \subseteq W$ for all $g \in G$. Such a W is also called a *G-invariant subspace.* Thus, $(W, \rho|_W)$ is a representation, where by definition $\rho|_W : G \to \mathrm{GL}(W)$ is defined by $\rho|_W(g) := \rho(g)|_W$ **(Caution: we are not restricting the domain of ρ, but rather the domain of $\rho(g)$ for all $g \in G$.)** We call the subrepresentation *proper* if $W \neq V$ and *nonzero* if $W \neq \{0\}$.

Exercise 2.8.2. For W finite-dimensional, show that $\rho(g)(W) \subseteq W$ implies $\rho(g)(W) = W$. **Harder:** Show that this is *not* true if W is infinite-dimensional, although it is still true that $\rho(g)(W) \subseteq W$ *for all* $g \in G$ implies $\rho(g)(W) = W$ *for all* $g \in G$. So we could have stated the definition using an equality.

Remark 2.8.3. Caution: probably the terminology "proper nontrivial" exists in the literature instead of "proper nonzero" and I might even say this by mistake, but I think it could be confusing since the trivial representation (or a trivial representation) is not the same thing as the zero representation. I will try to avoid it.

Definition 2.8.4. A representation (V, ρ) is *irreducible* (or *simple*) if it is nonzero and there does not exist any proper nonzero subrepresentation of V. It is *reducible* if it has a proper nonzero subrepresentation.

Remark 2.8.5. Caution that the nonzero condition here is *not redundant*: it will be convenient for us not to consider the zero representation to be irreducible. For one thing, we will want to say how many irreducibles we need to build a general representation, so we obviously don't want to count zero. It's like not including one as a prime number.

On the other hand, we should neither call zero a *reducible* representation: just as for the integer 1, which is neither prime nor composite, we call the zero representation neither reducible nor irreducible.

We will only be interested in finite-dimensional representations in this course. It turns out that, for finite groups G (the main case of interest in this course), all irreducible representations are automatically finite-dimensional:

Proposition 2.8.6. If G is a finite group and (V, ρ) an irreducible representation, then V is finite-dimensional.

Proof. Let $v \in V$ be a nonzero vector. Let W be the span of $\rho(g)(v)$. This is a subrepresentation of V, since $\rho(h) \sum_{g \in G} a_g \rho(g) v = \sum_{g \in G} a_g \rho(hg) v$. But W is a span of finitely many vectors and hence finite-dimensional. Since v was nonzero, W is nonzero. By irreducibility, $V = W$, which is finite-dimensional. $\square$

Remark 2.8.7 (Non-examinable)**.** For general infinite groups G, there can be infinite-dimensional irreducible representations (for example, V could be an infinite-dimensional vector space and $G = \mathrm{GL}(V)$, under which V is an irreducible representation). For an example where G is abelian, see Remark 2.12.5.

Exercise 2.8.8. (i) Every one-dimensional representation is irreducible. In particular this includes the sign representation of S_n. (ii) The two-dimensional representations of D_n for $n \geq 3$ explained in Example 2.2.6 are irreducible. (iii) The permutation representation of S_n is not irreducible for $n \geq 2$.

Solution to (ii): This will be for problem class.

Solution to (iii): There is a nonzero subrepresentation $\{(a, a, \ldots, a) \mid a \in \mathbf{C}\} \subseteq \mathbf{C}^n$ (let us write $\mathbf{C}^n$ as n-tuples now, where to translate to column vectors we take transpose). That is, $\widetilde{\rho}(\sigma)(a, a, \cdots a) = (a, a, \cdots, a)$ for all $\sigma \in S_n$, since multiplying a permutation matrix by a column vector whose entries are all equal does not do anything.

Another subrepresentation of the permutation representation is $\{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n = 0\} \subseteq \mathbf{C}^n$, called the *reflection representation*:

Definition 2.8.9. The reflection representation of S_n is the subrepresentation $\{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n\} \subseteq \mathbf{C}^n$ of the permutation representation.

Remark 2.8.10 (Non-examinable)**.** This is called the *reflection representation* because of its appearance associated to the special linear group $\mathrm{SL}_n(\mathbf{C})$ of $n \times n$ matrices of determinant one: it is the representation of the Weyl group on the Lie algebra of the maximal torus (see an appropriate textbook on Lie groups or Lie algebras for details).

We obtain that, not only is the permutation representation $(\mathbf{C}^n, \widetilde{\rho})$ not irreducible, but it is a *direct sum* of a trivial representation $\{(a, a, \ldots, a)\}$ and the reflection representation.

Proposition 2.8.11. If $T : (V, \rho) \to (V', \rho')$ is a homomorphism of representations, then $\ker(T)$ and $\mathrm{im}(T)$ are subrepresentations of V and V', respectively.

Proof. If $T(v) = 0$ then $T(g \cdot v) = g \cdot T(v) = 0$ for all $g \in G$, thus $\ker(T)$ is a subrepresentation of V. Also, for all $v \in V$, $g \cdot T(v) = T(gv) \in \mathrm{im}(T)$, so $\mathrm{im}(T)$ is a subrepresentation of V'. □

This is related to quotient groups and vector spaces. We have the following definition parallel to that of subrepresentations:

Definition 2.8.12. A *quotient representation* of a representation (V, ρ_V) is one of the form $(V/W, \rho_{V/W})$ for $W \subseteq V$ a subrepresentation and $\rho_{V/W}(g)(v + W) := \rho(g)(v) + W$.

Another way to think of this is via the following analogue of the first isomorphism theorem:

Proposition 2.8.13. If $T : (V, \rho) \to (V', \rho')$ is a homomorphism of representations, then for $W := \ker T$ and $W' := \mathrm{im}\, T$, we have an isomorphism $\overline{T} : (V/W, \rho_{V/W}) \xrightarrow{\sim} (W', \rho'|_{W'})$, given by $\overline{T}(v + W) = T(v)$.

Proof. By linear algebra $\overline{T} : V/W \to W'$ is an isomorphism. We only need to observe it is G-linear:

$$\overline{T} \circ \rho_{V/W}(g)(v + W) = T \circ \rho(g)(v) = \rho'(g) \circ T(v) = \rho'(g) \circ \overline{T}(v + W). \tag{2.8.14}$$

□

To apply the above, let us recall the following standard definition from linear algebra. First let us fix notation:

Definition 2.8.15. For V and W (complex) vector spaces, by $\mathrm{Hom}(V,W)$ and $\mathrm{End}(V) = \mathrm{Hom}(V,V)$ we always mean the vector space of linear maps (not the set of all group homomorphisms: only the linear ones!).

Caution: We will sometimes also use Hom to denote group homomorphisms, but never when the inputs are vector spaces (unless specified).

Now we can recall the definition of projection operators.

Lemma 2.8.16. The following are equivalent for $T \in \mathrm{End}(V)$: (i) $T^2 = T$; (ii) For all $w \in \mathrm{im}(T)$, we have $T(w) = w$. In the case these assumptions hold, we have a direct sum decomposition $V = \ker(T) \oplus \mathrm{im}(T)$.

Proof. (i) implies (ii): For $w \in \mathrm{im}(T)$, write $w = T(v)$. Then $T(w) = T^2(v) = T(v) = w$. (ii) implies (i): By (ii), $T(T(v)) = I(T(v)) = T(v)$.

For the last statement, first we show that $\ker(T) \cap \mathrm{im}(T) = \{0\}$. If $w = T(v) \in \mathrm{im}(T)$ and $T(w) = 0$, then $T^2(v) = 0$, so $T(v) = 0$ as well. But then $w = 0$. Next we show that $V = \ker(T) + \mathrm{im}(T)$. To see this, $v = T(v) + (v - T(v))$ and $T(v - T(v)) = T(v) - T^2(v) = 0$. $\square$

Definition 2.8.17. Let $T \in \mathrm{End}(V)$. Then T is a projection (operator) if either of the two equivalent conditions of Lemma 2.8.16 hold.

For convenience let us give an adjective for the property of being a homomorphism of representations:

Definition 2.8.18. A linear map $T : V \to W$ is called *G-linear* if it is a homomorphism of representations.

This is also suggestive, since the property is saying in terms of the G-action that $T(g \cdot v) = g \cdot T(v)$, which looks like linearity over G.

Corollary 2.8.19. Suppose that $T : V \to V$ is a G-linear projection operator. Then we get that (V, ρ) is a direct sum of two subrepresentations, $\ker(T)$ and $\mathrm{im}(T)$.

In the next subsection, we will formalise the phenomenon in the preceding examples into the notion of decomposability.

2.9 Direct sums

We first recall the relevant notion from linear algebra. If V and W are two vector spaces, then $V \oplus W$ is the Cartesian product $V \oplus W = V \times W$ equipped with the addition and scalar multiplication, $a(v,w) = (av, aw)$ and $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$. This is called the *external direct sum*. If there is any confusion, let us denote this operation by $\oplus_{\mathrm{ext}}$.

There is another important standard notation which conflicts with this: if V is a vector space and V_1, V_2 two vector subspaces, such that $V_1 \cap V_2 = \{0\}$ and $V_1 + V_2 = V$ then we write $V = V_1 \oplus V_2$. This is called the *internal direct sum*. To avoid confusion with external direct sum, let us denote it for a moment as $\oplus_{\text{int}}$. [Generally, if $V_1, V_2 \subseteq V$, then the operation $V_1 \oplus_{\text{int}} V_2$ makes sense if and only if $V_1 \cap V_2 = \{0\}$, and in this case $V_1 \oplus_{\text{int}} V_2 = V_1 + V_2$, with the notation of direct sum indicating that $V_1 \cap V_2 = \{0\}$.]

There should be no confusion between the two notations because, if $V = V_1 \oplus V_2$, then the sum is the internal direct sum if and only if V_1 and V_2 are subspaces of V.

Remark 2.9.1 (Omit from lecture, non-examinable)**.** The relation between the two notions is the following: If $V_1, V_2 \subseteq V$ are two subspaces, then there is a map $V_1 \oplus_{\text{ext}} V_2 \to V$ defined by $(v_1, v_2) \mapsto v_1 + v_2$. It is an isomorphism if and only if $V = V_1 \oplus_{\text{int}} V_2$.

Thus when $V = V_1 \oplus_{\text{int}} V_2$ we have $V_1 \oplus_{\text{ext}} V_2 \cong V = V_1 \oplus_{\text{int}} V_2$. We conclude that V is an internal direct sum of its subspaces V_1, V_2 if and only if every vector $v \in V$ is uniquely expressible as a sum of vectors in V_1 and V_2, i.e., $v = v_1 + v_2$ for unique $v_1 \in V_1$ and $v_2 \in V_2$.

The two notions both carry over to representations: we just need to require that V, V_1, and V_2 are all representations of G (and in the internal case, that V_1 and V_2 are subrepresentations):

Definition 2.9.2. Given two representations (V_1, ρ_1) and (V_2, ρ_2), the external direct sum is the representation $(V_1, \rho_1) \oplus_{\text{ext}} (V_2, \rho) := (V, \rho)$ where $V = V_1 \oplus_{\text{ext}} V_2$ and $\rho(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$.

Definition 2.9.3. Given a representation (V, ρ) and subrepresentations $V_1, V_2 \subseteq V$, we say that (V, ρ) is the internal direct sum of $(V_1, \rho|_{V_1})$ and $(V_2, \rho|_{V_2})$ if $V = V_1 \oplus_{\text{int}} V_2$.

Definition 2.9.4. A nonzero representation is *decomposable* if it is a direct sum of two proper nonzero subrepresentations. Otherwise, it is *indecomposable*.

Remark 2.9.5. As in Remark 2.8.5, we exclude the zero representation from being irreducible or indecomposable, for the same reason as before. The zero representation is *neither decomposable nor indecomposable*.

Definition 2.9.6. A representation is *semisimple* or *completely reducible* if it is an (internal) direct sum of irreducible representations.

Example 2.9.7. Let V be an indecomposable representation. Then by definition, we see that V is semisimple if and only if it is actually irreducible. Indeed, if $V = \bigoplus_{i=1}^m V_i$ is a direct sum expression with the V_i irreducible, then by indecomposability, $m = 1$ and $V = V_1$ is irreducible.

Remark 2.9.8. As we will see shortly, by Maschke's theorem, when G is finite and V finite-dimensional (working over $\mathbf{C}$ as always), then V *is always semisimple*. Hence in the case of main interest, indecomposable is equivalent to irreducible. As the next example shows, this is *not* true without these assumptions.

Example 2.9.9. Let $G = \mathbf{Z}$ and consider the two-dimensional representation given by $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. This is reducible since the x-axis is a subrepresentation. However, this is the only subrepresentation: any nonzero subrepresentation is a line fixed by all of these matrices, i.e., an eigenline. But the only eigenline of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the x-axis. This shows that there cannot be a decomposition into irreducible subrepresentations, so ρ is indecomposable.

From now on, we will not include the notation "ext" and "int" in our direct sums, because which type of sum will be clear from context, and anyway the distinction does not change much. Here is another exercise to make this clear.

Exercise 2.9.10 (Non-examinable)**.** Show that $V \oplus_{\text{ext}} W = (V \oplus_{\text{ext}} 0) \oplus_{\text{int}} (0 \oplus_{\text{ext}} W)$.

2.10 Maschke's Theorem

In this section finally the assumption that we are working over $\mathbf{C}$, as well as the finiteness of G, becomes very important.

Definition 2.10.1. Let (V, ρ) be a representation of G and $W \subseteq V$ a subrepresentation. A complementary subrepresentation is a subrepresentation $U \subseteq V$ such that $V = W \oplus U$.

Theorem 2.10.2. (Maschke's Theorem) Let (V, ρ) be a finite-dimensional representation of a finite group G. Let $W \subseteq V$ be any subrepresentation. Then there exists a complementary subrepresentation $U \subseteq V$.

Proof. The idea of the proof is to construct a homomorphism $T : V \to V$ of representations which is a projection operator with image W, and set $U = \ker(T)$. Then we apply Corollary 2.8.19.

We begin by observing that, by linear algebra, there exists a complementary subspace $U \subseteq V$, i.e., a vector subspace such that $V = U \oplus W$. (Recall the proof: extend a basis $(v_1, \ldots, v_m)$ of W to a basis $(v_1, \ldots, v_n)$ of V, and set U to be the span of $v_{m+1}, \ldots, v_n$.) As a result, every vector $v \in V$ is uniquely expressible as a sum $v = w + u$ for $w \in W$ and $u \in U$.

Let $T : V \to V$ be the map defined by $T(w + u) = w$ for $w \in W$ and $u \in U$. Then it is immediate that $T^2 = T$, i.e., T is a projection. By definition $\text{im}(T) = W$. If T were a homomorphism of representations, we would be done by Corollary 2.8.19. However, it need not be.

The idea of the proof is to *average* T over the group G, in order to turn T into a homomorphism of representations. Namely, define

$$\widetilde{T} := |G|^{-1} \sum_{g \in G} \rho(g) \circ T \circ \rho(g)^{-1}. \qquad (2.10.3)$$

Let us check that the result is a homomorphism of representations:

$$\widetilde{T}(\rho(h)v) = |G|^{-1} \sum_{g \in G} \rho(g) \circ T \circ \rho(g)^{-1} \rho(h) v = |G|^{-1} \sum_{g'=h^{-1}g \in G} \rho(h)\rho(g') \circ T \circ \rho(g')^{-1} v = \rho(h) \circ \widetilde{T}(v). \tag{2.10.4}$$

It remains to check that $\widetilde{T}(w)$ is still a projection with image W. We first show that $\widetilde{T}(w) = w$ for all $w \in W$:

$$\widetilde{T}(w) = |G|^{-1} \sum_{g \in G} \rho(g) \circ T(\rho(g)^{-1}(w)) = |G|^{-1} \sum_{g \in G} \rho(g) \circ \rho(g)^{-1}(w) = w. \tag{2.10.5}$$

The crucial second equality comes from the fact that $\rho(g)^{-1}(w) = \rho(g^{-1})(w) \in W$, and $T(w') = w'$ for all $w' \in W$ (in particular $w' = \rho(g^{-1})(w)$).

Since $\mathrm{im}(T) = W$, and $\rho(g)(W) \subseteq W$ for all $g \in G$ (because W is a subrepresentation), the definition of $\widetilde{T}$ implies that $\mathrm{im}(\widetilde{T}) \subseteq W$. Since $\widetilde{T}(w) = w$ for all $w \in W$, we obtain that $\mathrm{im}(\widetilde{T}) = W$. Therefore T is a projection by Lemma 2.8.16. Now setting $U := \ker(T)$ we get a complementary subrepresentation to W by Corollary 2.8.19. □

Remark 2.10.6. Note that there is a clever formula in the proof which turns a general linear map $T : V \to V$ into a homomorphism of representations. In Section 2.14 we will generalise this.

By induction on the dimension we immediately conclude:

Corollary 2.10.7. If V is a finite-dimensional representation of a finite group, then V is a direct sum of some irreducible subrepresentations. That is, V is semisimple (completely reducible).

This immediately implies:

Corollary 2.10.8. If V is a finite-dimensional indecomposable representation of a finite group, then V is irreducible.

(Note that actually the finite-dimensional hypothesis is not necessary, as every indecomposable representation of a finite groups is also irreducible.)

Remark 2.10.9. See Example 2.9.9 above for an example where G is infinite and the conclusions of the corollaries fail (hence also of the theorem). In fact that example is indecomposable but not irreducible, so it is not semisimple by Example 2.9.7.

Example 2.10.10. If $G = \{1\}$ is trivial, then a representation V is the same thing as a vector space. There is only one irreducible representation of G up to isomorphism: the trivial (one-dimensional) representation, which is the same as a one-dimensional vector space. Indeed, every (finite-dimensional vector space V is a direct sum of one-dimensional subspaces $V = \bigoplus V_i$, but choosing such a decomposition is almost the same as choosing a basis. Precisely, it is choosing a little bit less than choosing a basis: each V_i is the same information as a vector up to nonzero scaling. So we are choosing a basis up to rescaling each element of the basis.

Here is a notation we will use from now on: $\mathbf{C}v$ (or $\mathbf{C}\cdot v$) denotes the span of the (single) vector v.

Exercise 2.10.11. Let's give a similar example where the decomposition is unique. Let $A \in \mathrm{GL}_n(\mathbf{C})$ be a diagonal matrix with distinct diagonal entries which are m-th roots of unity (for some $m \geq n$ of course). Then consider the n-dimensional representation of $C_m = \{1, g, \ldots, g^{m-1}\}$ given by $(\mathbf{C}^n, \rho)$ with $\rho(g) = A$. Show that: (i) the irreducible subrepresentations of $(\mathbf{C}^n, \rho)$ are precisely the subspaces $\mathbf{C}e_i \subseteq \mathbf{C}^n$, i.e., the coordinate axes; (ii) $\mathbf{C}^n = \mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_n$ is the unique decomposition of $\mathbf{C}^n$ into a direct sum of irreducible subrepresentations.

Example 2.10.12. Here is another example with a unique decomposition. Let $G = C_3 = \{1, g, g^2\}$ and $(\mathbf{C}[G], \rho)$ be the regular representation. Then there is an obvious subrepresentation, $\mathbf{C}\cdot v_1$ for $v_1 := 1 + g + g^2$, since $g \cdot v_1 = g \cdot (1 + g + g^2) = 1 + g + g^2 = v_1$. This is one-dimensional and hence irreducible. We can generalise this: if ζ is a cube-root of unity, let $v_\zeta := 1 + \zeta g + \zeta^2 g^2$. Then $g \cdot v_\zeta = g \cdot (1 + \zeta g + \zeta^2 g^2) = g + \zeta g^2 + \zeta^2 1 = \zeta^{-1} v_\zeta$. Hence, $g^i \cdot v_\zeta = \zeta^{-i} v_\zeta$ and the subspace $\mathbf{C}\cdot v_\zeta$ is a one-dimensional subrepresentation. For $\zeta := e^{2\pi i/3}$, we see that $\mathbf{C}[G] = \mathbf{C}\cdot v_1 \oplus \mathbf{C}\cdot v_\zeta \oplus \mathbf{C}\cdot v_{\zeta^2}$.

Exercise 2.10.13. Generalise the previous example: (i) For $G = C_n = \{1, g, g^2, \cdots, g^{n-1}\}$, take the regular representation $(\mathbf{C}[G], \rho)$. For every n-th root of unity ζ, let $v_\zeta := \sum_{i=0}^{n-1} \zeta^i g^i$. Show that $g \cdot v_\zeta = \zeta^{-1} v_\zeta$, and deduce that, when $\zeta = e^{2\pi i/n}$, then $\mathbf{C}[G] = \bigoplus_{i=0}^{n-1} \mathbf{C}\cdot v_{\zeta^i}$ is a decomposition into one-dimensional representations. (ii) Now for G an arbitrary finite group, we cannot say as much, but let $v_1 := \sum_{g\in G} g \in \mathbf{C}[G]$, and show that $\mathbf{C}\cdot v_1 \subseteq \mathbf{C}[G]$ is always a subrepresentation isomorphic to the trivial representation. Find a complementary subrepresentation.

Remark 2.10.14 (Non-examinable)**.** We used that the field was $\mathbf{C}$ in order for $|G|^{-1}$ to exist. As this is all we needed, we don't really need the field to be $\mathbf{C}$: any field in which $|G|$ is invertible (i.e., of characteristic not a factor of $|G|$) will do.

For an example where the conclusion of Maschke's theorem fails if $|G|$ is not invertible, let $G = C_2 = \{1, g\}$ and the field also be $\mathbf{F}_2$ (the field of two elements, i.e., $\mathbf{Z}/2\mathbf{Z}$). Then the two-dimensional representation defined by $\rho(g) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one one-dimensional subrepresentation, $\mathbf{F}_2 \cdot e_1$, so it is indecomposable but not irreducible (hence not semisimple by Example 2.9.7). This is really similar to Example 2.9.9 where the theorem fails because instead G is infinite (and the field is $\mathbf{C}$).

This motivates the following definition: *modular representation theory of a finite group* means the study of representations of the group over a field of characteristic dividing the order of the group. (For infinite groups, in general "modular representation theory" simply means working over any field of positive characteristic, especially in characteristics where there exist non-semisimple finite-dimensional representations, i.e., the conclusion of Maschke's theorem fails).

Remark 2.10.15 (Non-examinable)**.** The assumption that V be finite-dimensional is not really needed for Maschke's theorem, as long as you know that complementary subspaces exist in infinite dimensions as well. The key thing is that G is finite (and $|G|^{-1}$ is in the ground field).

2.11 Schur's Lemma

We now give a fundamental result about irreducible representations.

Lemma 2.11.1 (Schur's Lemma)**.** Let (V, ρ_V) and (W, ρ_W) be irreducible representations of a group G.

(i) If $T : V \to W$ is a G-linear map, then T is either an isomorphism or the zero map.

(ii) Suppose V is finite-dimensional. If $T : V \to V$ is G-linear then $T = \lambda I$ for some $\lambda \in \mathbf{C}$.

By Proposition 2.8.6, in part (ii), if we assume that G itself is finite then we can drop the assumption that V is finite-dimensional (it is automatic).

Remark 2.11.2 (Non-examinable)**.** Part (i) and its proof actually holds when $\mathbf{C}$ is replaced by a general field. Part (ii) requires the field to be $\mathbf{C}$ as we assume (or more generally an algebraically closed field). This is needed in order to have the existence of a (nonzero) eigenvector. (Recall the proof of this fact in linear algebra: take the characteristic polynomial $\chi_T(x)$ of T and finding a root $\lambda \in \mathbf{C}$, which exists by the fundamental theorem of algebra. Then an eigenvector with eigenvalue λ is any element of $\ker(T - \lambda I) \neq 0$. In general, a field with the property that every polynomial over a field has a root is called an *algebraically closed field*.)

Proof of Lemma 2.11.1. (i) If $T : V \to W$ is a homomorphism of representations, then $\ker(T)$ is a subrepresentation. If T is nonzero, this is not all of V, so by irreducibility, $\ker(T) = 0$. That is T is injective. Since V is nonzero, this means that $\mathrm{im}(T)$ is a nonzero subrepresentation of W. By irreducibility again, $\mathrm{im}(T) = W$. So T is also surjective, and hence an isomorphism.

(ii) We know from linear algebra that every linear transformation $T \in \mathrm{End}(V)$ has a nonzero eigenvector: a nonzero vector v such that $Tv = \lambda v$ for some $\lambda \in \mathbf{C}$.

We claim that $T = \lambda I$. Indeed, $T - \lambda I : V \to V$ is also a G-linear map, which is not injective. By (i) $T - \lambda I = 0$, which implies $T = \lambda I$ as desired. $\square$

Example 2.11.3. Let us take $G = S_n$ and $(\mathbf{C}^n, \rho)$ to be the permutation representation. Then we can obtain a G-linear map $T : \mathbf{C}^n \to \mathbf{C}^n$ which is not a multiple of the identity: $T(a_1, \ldots, a_n) = (a, a, \ldots, a)$ where $a = \frac{1}{n}\sum_{i=1}^{n} a_i$ is the average of $a_1, \ldots, a_n$ (this is the projection of $\mathbf{C}^n$ onto the trivial subrepresentation), so this representation is not irreducible by (the contrapositive of) Schur's Lemma.

Example 2.11.4 (Omit from lecture, non-examinable)**.** If $G = D_n$ is a dihedral group for $n \geq 3$ and $(\mathbf{C}^2, \rho)$ is the representation of Example 2.2.6, Schur's Lemma implies that every

G-linear map $T : \mathbf{C}^2 \to \mathbf{C}^2$ is a multiple of the identity. This can be proved parallel to how we showed that the representation is irreducible, using reflections: if we take any reflection $g \in G$, then $T(gv) = gT(v)$ shows that, if v is parallel to the axis of reflection, i.e., $gv = v$, then $T(v) = T(gv) = gT(v)$, so also $T(v)$ is parallel to the axis of reflection. Thus v is also an eigenvector of T. But then every vector parallel to a reflection axis is an eigenvector of T. This implies that T has at least three non-parallel eigenvectors, so T has to be a multiple of the identity (if it were not, then the eigenspaces of T would have to be one-dimensional and there could be at most two of these).

Example 2.11.5. Let $G = C_n = \{1, g, \ldots, g^{n-1}\}$ and let $(\mathbf{C}^2, \widetilde{\rho})$ be the two-dimensional representation where g^k acts as a rotation in the plane by angle $2\pi k/n$, i.e., the one given by

$$\rho(g^k) = R_{2\pi k/n}, \quad R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.11.6}$$

Then, every rotation $T = R_\theta$ is G-linear. For $\theta \notin \pi\mathbf{Z}$, these are not scalar multiples of the identity. By the contrapositive of Schur's Lemma, the representation is not irreducible.

Remark 2.11.7 (Non-examinable)**.** We see from this example that Schur's Lemma (part (ii)) does not hold when we work over the real field instead of the complex field, since the representation $(\mathbf{R}^2, \rho)$ of C_n given by rotations is actually irreducible, working over $\mathbf{R}$, when $n \geq 3$. Indeed, rotation matrices by θ only have real eigenvectors when $\theta = m\pi$ for $m \in \mathbf{Z}$, so the rotations by $2\pi k/n$ do not have real eigenvectors when $n \geq 3$. This means they have no real one-dimensional subrepresentations.

Exercise 2.11.8. Use Schur's Lemma to prove the following formula: Let $V = V_1 \oplus \cdots \oplus V_n$ where each of the V_i are irreducible and (pairwise) nonisomorphic (i.e., $V_i \ncong V_j$ for $i \neq j$). Prove that every linear map $V_i \to V$ is a scalar multiple of the inclusion map. Conclude that the decomposition $V = V_1 \oplus \cdots \oplus V_n$ is the unique one into irreducible representations.

Conversely, show that, if $V_i \cong V_j$ for some $i \neq j$, then there are infinitely many injective homomorphisms of representations $V_i \to V$. Conclude that the decomposition $V = V_1 \oplus \cdots \oplus V_n$ is unique *if and only if* all of the V_i are (pairwise) nonisomorphic.

2.12 Representations of abelian groups

Thanks to Schur's Lemma we can understand finite-dimensional irreducible representations of abelian groups up to isomorphism (and hence, by Maschke's theorem, also finite-dimensional representations of finite abelian groups):

Proposition 2.12.1. Let G be an abelian group (not necessarily finite). Then every finite-dimensional irreducible representation of G is one-dimensional.

By Maschke's theorem we immediately deduce:

Corollary 2.12.2. Let G be a finite abelian group. Then every finite-dimensional representation is a direct sum of one-dimensional representations.

The proposition will follow from the following basic observation:

Lemma 2.12.3. Let (V, ρ_V) be a representation of a group G. Let $z \in G$ be central, i.e., $zg = gz$ for all $g \in G$. Then $\rho_V(z) : V \to V$ is G-linear.

Proof. We have $\rho_V(z) \circ \rho_V(g) = \rho_V(zg) = \rho_V(gz) = \rho_V(g) \circ \rho_V(z)$. $\square$

Proof of Proposition 2.12.1. Assume G is abelian. Let (V, ρ_V) be a finite-dimensional irreducible representation. Since every $g \in G$ is central, by Lemma 2.12.3, $\rho_V(g)$ is G-linear. By Lemma 2.11.1.(ii), $\rho_V(g)$ is actually a multiple of the identity matrix. Now let $v \in V$ be any nonzero vector. Let $V_1 := \mathbf{C} \cdot v$, the span of v, which is one-dimensional. We have that V_1 is a subrepresentation, since $\rho_V(g)(v)$ is always a multiple of v. By irreducibility, $V = V_1$. $\square$

Remark 2.12.4. [Non-examinable, omit from lectures] The proof actually shows the following characterisation of the center of G: $z \in G$ is central only if $\rho_V(g)$ is a scalar matrix for all irreducible representations V. If G is finite, then the converse is also true: if $\rho_V(z)$ is a scalar matrix for all irreducible representations V, then $\rho_V(zg) = \rho_V(gz)$ for every irreducible representation V and every $g \in G$. By Maschke's theorem, for every finite-dimensional representation V, it follows that $\rho_V(zg) = \rho_V(gz)$ for all $g \in G$. Now let $V = \mathbf{C}[G]$, and then we get $zg = gz$ for all $g \in G$. Thus z is central.

Remark 2.12.5 (Non-examinable)**.** The hypothesis of finite-dimensionality is required, since the proof invokes Schur's Lemma (Lemma 2.11.1), part (ii). Here is a counterexample without this hypothesis (which also gives a counterexample to Lemma 2.11.1.(ii) when V is infinite-dimensional). Let $V = \mathbf{C}(x) = \{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0} \mid m, n \geq 0\}$ be the field of rational fractions of x (some of these expressions are equal: two fractions are equal if they can be simplified to the same fraction by canceling common factors from the numerator and denominator and deleting $0x^n$ from the numerator if $n > 0$; equivalently $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ if and only if $f_1 g_2 = f_2 g_1$ as polynomials). Let $G = \mathbf{C}(x)^\times := \mathbf{C}(x) \setminus \{0\}$ be the multiplicative group of nonzero rational fractions (i.e., the numerator is nonzero). Then V is an irreducible infinite-dimensional representation of G under the action $\rho(g)(f) = gf$, and G is abelian.

Corollary 2.12.6. For $G = C_n$, there are exactly n irreducible representations, up to isomorphism, which are one-dimensional and are the ones given in Example 2.2.2: $\rho_\zeta(g^i) = (\zeta^i)$, as ζ ranges over all n-th roots of unity.

Proof. Since G is finite, all irreducible representations are finite-dimensional by Proposition 2.8.6. By Proposition 2.12.1, all irreducible representations are therefore one-dimensional. By Proposition 2.4.19, up to isomorphism, they are given by representations $G \to \mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^\times$, i.e., multiplicative assignments of G to nonzero complex numbers. But the generator $g \in G$ has to map to an n-th root of unity, call it ζ. Then we obtain the representation ρ_ζ. Note that two one-dimensional matrix representations are conjugate if and only if they are identical, since one-by-one matrices commute. $\square$

Exercise 2.12.7. Classify all irreducible representations of finite abelian groups. Namely, from group theory, recall that every finite abelian group is isomorphic to a product $G = C_{n_1} \times \cdots \times C_{n_k}$. Extending the previous corollary, show that the irreducible representations of G are, up to isomorphism, precisely the one-dimensional representations of the form $\rho_{\zeta_1,\ldots,\zeta_k}$ sending the generators g_i of C_{n_i} to $(\zeta_i) \in \mathrm{GL}_1(\mathbf{C})$. Deduce that the number of irreducible representations up to isomorphism is $|G| = n_1 \cdots n_k$. We will generalise this statement to products of nonabelian groups (and higher-dimensional irreducible representations) in Proposition 2.20.5.

We now give some examples of *subrepresentations and decompositions* of representations of abelian groups.

Example 2.12.8. Let $G = \{\pm 1\} \times \{\pm 1\}$ be (isomorphic to) the Klein four-group and $(\mathbf{C}^2, \rho)$ be the two-dimensional representation defined by

$$\rho(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{2.12.9}$$

V is the direct sum of the subrepresentations $\mathbf{C} \cdot e_1$ and $\mathbf{C} \cdot e_2$. These two are isomorphic to the one-dimensional representations $\rho_1, \rho_2 : G \to \mathbf{C}^\times$ given by $\rho_1(a, b) = a$ and $\rho_2(a, b) = b$, respectively.

Note that, if G is infinite, even if it is abelian, Corollary 2.12.2 does not apply: see, for example, Example 2.9.9, with $G = \mathbf{Z}$. A similar example can be given with $G = \mathbf{C}$: we can use the same formula, $\rho(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. Again, the only proper nonzero subrepresentation is $\mathbf{C} \cdot e_1$ so the representation is indecomposable but not irreducible, exactly as in Example 2.9.9.

Example 2.12.10. Now let $G = \mathbf{C}^\times$ be the group of nonzero complex numbers under multiplication, and consider the two-dimensional representation $(\mathbf{C}^2, \widetilde{\rho})$, with

$$\rho(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{2.12.11}$$

This is easily verified to be a representation:

$$\begin{aligned} \rho(a + bi)\rho(c + di) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} &= \begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix} \\ &= \rho(ac - bd + (ad + bc)i) = \rho((a + bi)(c + di)). \end{aligned} \tag{2.12.12}$$

See the exercise below for a more conceptual way to verify this.

Since G is infinite we cannot apply Corollary 2.12.2, but can still apply Proposition 2.12.1 and we know there is at least one one-dimensional subrepresentation. In fact, there are two anyway: $V_+ := \mathbf{C} \cdot (1, i) = \{(a, ai) \mid a \in \mathbf{C}\}$ and $V_- := \mathbf{C} \cdot (1, -i) = \{(a, -ai) \mid a \in \mathbf{C}\}$. This means that actually $\mathbf{C}^2 = V_+ \oplus V_-$, and we do get a direct sum of irreducible representations. Concretely, in terms of the bases $(1, i)$ and $(1, -i)$ of V_+ and V_-, respectively [or any bases since these are one-dimensional], we have:

$$(V_+, \widetilde{\rho}|_{V_+}) \cong (\mathbf{C}, \rho_+), \quad (V_-, \widetilde{\rho}|_{V_-}) \cong (\mathbf{C}, \rho_-), \tag{2.12.13}$$

$$\rho_+(a + bi) = a - bi, \quad \rho_-(a + bi) = a + bi. \tag{2.12.14}$$

This is because $\rho(a + bi)(1, i) = (a - bi)(1, i)$ and $\rho(a + bi)(1, -i) = (a + bi)(1, -i)$.

In other words, $\rho_+ : \mathbf{C}^\times \to \mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^\times$ is the complex conjugation map, and $\rho_- : \mathbf{C}^\times \to \mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^\times$ is the identity map.

Explicitly (but equivalently), we can realise this decomposition by conjugating by the change-of-basis matrix $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$, with inverse $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$:

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} a - bi & 0 \\ 0 & a + bi \end{pmatrix} \tag{2.12.15}$$

Exercise 2.12.16. Here is a more conceptual way to verify that ρ in Example 2.12.10 is a representation. Show that $\rho(a + bi)|_{\mathbf{R}^2} : \mathbf{R}^2 \to \mathbf{R}^2$ is the complex multiplication by $a + bi$ when we identify $\mathbf{R}^2$ with $\mathbf{C}$. Using this show that $\rho(a+bi)\rho(c+di)|_{\mathbf{R}^2} = \rho((a+bi)(c+di))|_{\mathbf{R}^2}$. Since $\mathbf{R}^2$ spans $\mathbf{C}^2$ over the complex numbers, conclude that ρ is a homomorphism.

Example 2.12.17. Restricting the Example 2.12.10 from $\mathbf{C}^\times$ to the group of n-th roots of unity, $G = \{e^{2\pi i m/n}\} \subseteq \mathbf{C}^\times$, which is isomorphic to C_n (and to $\mathbf{Z}/n\mathbf{Z}$), we obtain the representation sending $e^{2\pi i m/n}$ to the rotation matrix by $2\pi m/n$. For $\zeta = e^{2\pi i/k}$, we can identify G with the cyclic group $C_n = \{1, g, \ldots, g^{n-1}\}$, via $g^k = \zeta^k = e^{2\pi i k/n}$. Then, the representation of C_n we obtain is the one of Example 2.11.5, which I reprint for convenience: $\rho(g^k) = R_{2\pi k/n}, \quad R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

The previous example then decomposes this as a direct sum of two one-dimensional representations. In the form of Example 2.2.2, these are the ones corresponding to $\zeta^{-1} = \overline{\zeta}$ and ζ respectively. Indeed, $\rho_+(\zeta^k) = (\zeta^{-1})^k$ and $\rho_-(\zeta^k) = \zeta^k$. Explicitly,

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \tag{2.12.18}$$

which we apply to the cases $\theta = 2\pi k/n$.

The above examples illustrate the general phenomenon:

Exercise 2.12.19. Let G be a group and $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ be an n-dimensional representation. Show that all matrices $\rho(g)$ are simultaneously diagonalisable (i.e., there exists a simultaneous eigenbasis for all of them) if and only if $(\mathbf{C}^n, \widetilde{\rho})$ decomposes as a direct sum of one-dimensional representations. (Hint: an eigenbasis $(v_1, \ldots, v_n)$ corresponds to a decomposition $\mathbf{C}v_1 \oplus \cdots \oplus \mathbf{C}v_n$.)

Remark 2.12.20. By linear algebra, one can directly see that, if G is finite abelian, then for every representation, the matrices $\rho(g)$ are simultaneously diagonalisable for all $g \in G$. Thus Exercise 2.12.19 provides a linear-algebraic proof of (and explanation for) Corollary 2.12.2.

2.13 One-dimensional representations and abelianisation

In terms of matrices, a one-dimensional representation is a homomorphism $\rho : G \to \mathrm{GL}_1(\mathbf{C}) \cong \mathbf{C}^\times$ (where $\mathbf{C}^\times$ is the multiplicative group of nonzero complex numbers under multiplication). Note that two distinct such representations are inequivalent, since one-by-one matrices commute. To avoid confusion, let us call a representation in terms of vector spaces, by Definition 2.4.7, an *abstract* representation. By Proposition 2.4.19, all one-dimensional abstract representations are isomorphic to exactly one one-dimensional matrix representation (as the latter are all inequivalent). So to find the one-dimensional abstract representations up to isomorphism is the same as finding all homomorphisms $\rho : G \to \mathbf{C}^\times$, i.e., all one-dimensional matrix representations.

Since $\mathbf{C}^\times$ is abelian, we can reduce the problem of one-dimensional matrix representations of G to a canonical abelian quotient of G, called its *abelianisation*, $G_{\mathrm{ab}} = G/[G, G]$: there will be a bijection between one-dimensional representations of G and of G_{ab}. Perhaps you have seen the construction of G_{ab} in group theory, but just in case you haven't, we will recall the construction in Section 2.13.1 below.

By Exercise 2.12.7, this implies that, when G_{ab} is finite, the number of one-dimensional matrix representations of G is precisely the size of this quotient G_{ab} (see Corollary 2.13.13 below).

However, this construction is mostly only of theoretical importance, and is not necessary in order to compute the one-dimensional representations. Let us give some examples, which we will then explain from the point of view of abelianisation.

Example 2.13.1. A one-dimensional matrix representation ρ of S_n, $n \geq 2$, must send $(1, 2)$ to ± 1, since $(1, 2)^2$ is the identity. However, every transposition (i, j) is conjugate to $(1, 2)$: $(1, i)(2, j)(1, 2)(2, j)(1, i) = (i, j)$. [More generally, conjugacy classes in S_n just depend on the cycle decomposition by the formula $\sigma(a_1, a_2, \ldots, a_m)\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_m))$ for all $\sigma \in S_n$ and $a_1, \ldots, a_m \in \{1, \ldots, n\}$.] Therefore, since conjugation of one-by-one matrices is trivial, we must have $\rho(i, j) = \rho(1, 2) = \pm 1$, a fixed number, for all i, j. The two choices of sign give rise to the trivial and the sign representation. Thus these two are precisely the one-dimensional (matrix) representations of S_n for $n \geq 2$. These are nonisomorphic (since they are distinct and conjugation is trivial), and every one-dimensional abstract representation is isomorphic to one of these.

Example 2.13.2. Let $G = D_n$. Let $x \in D_n$ be the counter-clockwise rotation by $2\pi/n$ and $y \in D_n$ be a reflection (about the x-axis, say), so that x and y generate D_n, and we have the relations $x^n = 1 = y^2$ and $yxy^{-1}(= yxy) = x^{-1}$. These relations actually define D_n, i.e., a homomorphism $\varphi : D_n \to H$ is uniquely determined by $a = \varphi(x)$ and $b = \varphi(y)$ subject to the conditions $a^n = 1 = b^2$ and $bab^{-1} = a^{-1}$.

Let ρ be a one-dimensional representation, i.e., a homomorphism $\rho : G \to \mathbf{C}^\times$. Then, again since $\mathbf{C}^\times$ is abelian (conjugation of one-by-one matrices is trivial), we have

$$a^{-1} = \rho(x^{-1}) = \rho(yxy^{-1}) = \rho(y)\rho(x)\rho(y^{-1}) = \rho(x) = a, \tag{2.13.3}$$

which implies that $a = \pm 1$. Also, $y^2 = 1$ implies that $b = \pm 1$ (not necessarily the same value as $\rho(x)$. So there can be at most four one-dimensional representations, depending on the choice of $a, b \in \{\pm 1\}$. Moreover, if n is odd, then $a^n = 1$ implies that $a = 1$, not -1. Conversely any choices of $a, b \in \{\pm 1\}$, subject to the condition that $a = 1$ if n is odd, will satisfy the conditions. So in fact there are *two* one-dimensional matrix representations when n is odd, and *four* one-dimensional matrix representations if n is even. As before, these are all inequivalent and hence every one-dimensional abstract representation is isomorphic to exactly one of these.

2.13.1 Recollections on commutator subgroups and abelianisation

Definition 2.13.4. Given two elements $g, h \in G$, the commutator $[g, h]$ is defined as $[g, h] := ghg^{-1}h^{-1}$. The commutator subgroup $[G, G] \subseteq G$ is the subgroup generated by all commutators $[g, h] = ghg^{-1}h^{-1}$.

Lemma 2.13.5 (Group theory)**.** The subgroup $[G, G]$ is normal.

Proof. Since $[g, h]^{-1} = [h^{-1}, g^{-1}]$, an arbitrary element of $[G, G]$ is a product of commutators (we don't need inverse commutators). Then the lemma follows from the standard formula $k(gh)k^{-1} = (kgk^{-1})(khk^{-1})$, since it implies $k[g, h]k^{-1} = [kgk^{-1}, khk^{-1}]$. □

Remark 2.13.6 (Non-examinable)**.** The above lemma is really unnecessary since we could simply define the commutator subgroup to be the *normal* subgroup generated by the commutators. The lemma explains though why this is the same as the ordinary subgroup so generated, which is the usual definition of $[G, G]$.

The subgroup $[G, G]$ is characterised by the following:

Proposition 2.13.7 (Group theory)**.** (i) The quotient $G/[G, G]$ is abelian. (ii) For an arbitrary normal subgroup $N \trianglelefteq G$, the quotient G/N is abelian if and only if $[G, G] \subseteq N$.

Proof. Note that (i) is a formal consequence of (ii), so we only prove (ii). This follows because $hg = gh[h^{-1}, g^{-1}]$, so that $ghN = hgN$ if and only if $[h^{-1}, g^{-1}] \in N$. So all elements in G/N commute if and only if N contains all commutators, and the statement follows. □

Definition 2.13.8. The quotient $G/[G, G]$ is called the *abelianisation* of G, and denoted G_{ab}.

A further justification for the term is the following. Let $q_{\mathrm{ab}} : G \to G_{\mathrm{ab}}$ be the quotient.

Corollary 2.13.9 (Group theory)**.** If $\varphi : G \to H$ is a homomorphism, then the image of φ is abelian if and only if $[G, G] \subseteq \ker(G)$. Thus, this is true if and only if φ factors through the quotient q_{ab}, i.e., $\varphi = \overline{\varphi} \circ q_{\mathrm{ab}}$ for some $\overline{\varphi} : G_{\mathrm{ab}} \to H$.

Proof. By the first isomorphism theorem, $\mathrm{im}(\varphi) \cong G/\ker(\varphi)$. Then the first statement follows from Proposition 2.13.7. The second statement then follows from the following basic lemma. □

Lemma 2.13.10. Let $\varphi : G \to H$ be a homomorphism and $K \trianglelefteq G$ a normal subgroup. Then φ factors through $q : G \to G/K$ if and only if $K \subseteq \ker(\varphi)$.

Proof. If $K \subseteq \ker(\varphi)$ then we can define $\overline{\varphi}(gK) = \varphi(g)$, so that $\varphi = \overline{\varphi} \circ q$. Conversely, given $\overline{\varphi}$ such that $\varphi = \overline{\varphi} \circ q$, for all $k \in K$ we have $\varphi(k) = \overline{\varphi}(K) = e$ and hence $K \subseteq \ker(\varphi)$. □

In particular, we will often use the following statement.

Corollary 2.13.11. Let G be a group and A an abelian group. Then the map $\mathrm{Hom}(G_{\mathrm{ab}}, A) \to \mathrm{Hom}(G, A)$, given by $\phi \mapsto \phi \circ q_{\mathrm{ab}}$ is an isomorphism.

Proof. The map is obviously injective, since q_{ab} is surjective. By Corollary 2.13.9, every homomorphism $\varphi \in \mathrm{Hom}(G, A)$ is of the form $\varphi = \phi \circ q_{\mathrm{ab}}$ for some ϕ. □

2.13.2 Back to one-dimensional representations

Since $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^\times$ is abelian, we obtain from Corollary 2.13.11 the following:

Proposition 2.13.12. The one-dimensional matrix representations of G are the same as the one-dimensional matrix representations of the abelian group G_{ab}.

Applying Exercise 2.12.7 we obtain:

Corollary 2.13.13. If G_{ab} is finite (e.g., if G is finite), then the number of one-dimensional matrix representations of G is equal to $|G_{\mathrm{ab}}|$.

Example 2.13.14. Suppose that $G = [G, G]$. This is called a *perfect group*. Then $G_{\mathrm{ab}} = \{e\}$, and it follows that there are *no* nontrivial one-dimensional (matrix) representations of G.

Example 2.13.15. As a special case of the previous example, suppose G is simple and nonabelian (the only abelian simple groups are the prime order p ones, i.e., C_p). Since $[G, G]$ is always a normal subgroup, and only trivial if G is abelian, it follows that $G = [G, G]$. That is, every nonabelian simple group is perfect, and hence has no nontrivial one-dimensional representations.

Example: the alternating group A_n for $n \geq 5$ (you might have seen that this is a simple group). There are lots of other examples of finite simple groups, and they are all classified: see `https://en.wikipedia.org/wiki/List_of_finite_simple_groups`.

The above example includes the "monster" group that I mentioned. We see that, for perfect groups (such as nonabelian simple groups) it is in general an interesting question to determine the minimal dimension of a nontrivial irreducible representation (since it cannot be one).

In the remainder of this section we compute the abelianisations of the symmetric and dihedral groups and recover the classification we already found of their one-dimensional representations. The main point is that $(S_n)_{\mathrm{ab}} \cong C_2$ and $(D_n)_{\mathrm{ab}} \cong \begin{cases} C_2, & \text{if } n \text{ is odd,} \\ C_2 \times C_2, & \text{if } n \text{ is even.} \end{cases}$ Therefore, there are exactly the number of one-dimensional representations we computed earlier.

Since the explicit verification of this doesn't contain any new representation theory, we will skip it in lecture and it is non-examinable.

Example 2.13.16 (Non-examinable, skip in lecture)**.** Let $G = S_n$. We already saw that this has precisely two one-dimensional representations. By the corollary, G_{ab} must have size two. We prove this directly here. We prove more precisely that $[G, G] = A_n$, and this is a subgroup of index two (for more on index two subgroups, if you are curious, see Remark 2.13.22 below).

We have the formula, for i, j, k distinct numbers in $\{1, \ldots, n\}$:

$$(i, j)(j, k)(i, j)(j, k) = (i, k, j) \tag{2.13.17}$$

Now the three-cycles are known to generate A_n (see Remark 2.13.20 for a proof of this). Hence $A_n \leq [G, G]$. Conversely, $[G, G] \subseteq A_n$ since $\mathrm{sign}([g, h]) = \mathrm{sign}(g)\mathrm{sign}(h)\mathrm{sign}(g^{-1})\mathrm{sign}(h^{-1}) = 1$ for all $g, h \in G$ (more abstractly, $\mathrm{sign} : G = S_n \to \{\pm 1\}$ has abelian image and hence $[G, G]$ is in the kernel by Corollary 2.13.9).

Example 2.13.18 (Non-examinable, skip in lecture)**.** Now, for $G = D_n$, by the preceding results, we get that G_{ab} must have size two if n is odd and size four if n is even. Let us prove this directly. As before let x be the counter-clockwise rotation by $2\pi/n$ and y be a reflection (say by the x-axis). Then $[G, G]$ contains the element $xyx^{-1}y^{-1} = x(yx^{-1}y^{-1}) = xx = x^2$. Therefore $[G, G]$ contains the cyclic subgroup H_n defined as follows:

$$H_n := \begin{cases} \{1, x^2, \ldots, x^{n-2}\} \cong C_{n/2}, & \text{if } n \text{ is even,} \\ \{1, x, \ldots, x^{n-1}\} \cong C_n, & \text{if } n \text{ is odd.} \end{cases} \tag{2.13.19}$$

This subgroup is clearly normal. Since $|G/H_n| \leq 4$, it follows that G/H_n is abelian, and hence $[G, G] \leq H_n$ by Corollary 2.13.9. Therefore $[G, G] = H_n$. Computing in more detail, if n is even, $G_{\mathrm{ab}} = G/H_n \cong C_2 \times C_2$ is isomorphic to the Klein four-group, and if n is odd, $G_{\mathrm{ab}} \cong C_2$.

Remark 2.13.20 (Group theory, non-examinable, skip in lecture)**.** Here is a proof that the three-cycles generate A_n. First, note that A_n is obviously generated by products of two transpositions, i.e., $(i, j)(k, \ell)$ for arbitrary $i, j, k, l \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq \ell$. We can assume that $\{i, j, k, \ell\}$ has size ≥ 3 (otherwise the element is the identity). In the case $|\{i, j, k, \ell\}| = 3$, then the product is a three-cycle. In the case i, j, k, ℓ are all distinct, the element is generated by three-cycles by the formula

$$(i, j)(k, \ell) = (i, j, k)(j, k, \ell). \tag{2.13.21}$$

Remark 2.13.22 (Group theory, non-examinable, skip in lecture)**.** The commutator subgroup is relevant to the study of the subgroups of a group of index two. Given any group G, every index-two subgroup $N \leq G$ of a group G contains $[G, G]$ since G/N is abelian, so by the first isomorphism theorem, index-two subgroups of G are in bijection with index-two subgroups of G_{ab}.

In the case of S_n, since G_{ab} has size two, we conclude that S_n has a unique subgroup of index two, namely $[G, G] = A_n$. There is also a direct way to verify this fact: every index-two subgroup contains the squares of all elements, which includes the three-cycles, which generate A_n by Remark 2.13.20 below.

Returning to the dihedral groups, recall that $G_{\mathrm{ab}} \cong C_2$ has a unique index-two subgroup if n is odd, but $G_{\mathrm{ab}} \cong C_2 \times C_2$ does not when n is even. Hence, when n is odd, then D_n has a unique index-two subgroup. This is its commutator subgroup H_n of rotations as above. However, when n is even, then D_n has multiple index-two subgroups: in addition to the subgroup of all rotations, we can take two subgroups consisting of half of the rotations (the ones in H_n by multiples of $4\pi/n$) and half of the reflections (there are two choices here, any of the two evenly-spaced collections of half of the reflection lines will do). This indeed corresponds to the three order-two subgroups of the Klein four-group $C_2 \times C_2 \cong G_{\mathrm{ab}}$. (This also gives another proof that G_{ab} is isomorphic to the Klein four-group rather than a cyclic group of size four, since in the latter case we would have only had one index-two subgroup of G.)

For general G, when G_{ab} is finite (e.g., if G is finite), there is a unique index-two subgroup if and only if $G_{\mathrm{ab}} \cong C_m \times H$ for m even and $|H|$ odd (we can take m to be a positive power of two if desired).

2.14 Homomorphisms of representations and representations of homomorphisms

Observe that if $S, T : (V, \rho_V) \to (W, \rho_W)$ are homomorphisms of representations, so is $aS+bT$ for any $a, b \in \mathbf{C}$. So the set of homomorphisms of representations forms a vector space:

Definition 2.14.1. If V and W are representations of G over F, then $\mathrm{Hom}_G(V, W)$ is the vector space of homomorphisms of representations $T : V \to W$. If $V = W$ we denote $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$.

This allows us to restate Schur's Lemma as follows: Let V and W be irreducible. Then

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1, & \text{if } V \cong W \\ 0, & \text{otherwise.} \end{cases} \tag{2.14.2}$$

Another way to see that $\mathrm{Hom}_G(V, W)$ is a vector space is given as below. We first consider the vector space of all linear maps, $\mathrm{Hom}(V, W)$, from V to W. This is a representation of G:

Definition 2.14.3. Given representations (V, ρ_V) and (W, ρ_W), define a linear action $G \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$, i.e., a homomorphism $\rho_{\mathrm{Hom}(V,W)} : G \to \mathrm{GL}(\mathrm{Hom}(V, W))$, by

$$\rho_{\mathrm{Hom}(V,W)}(g)(\varphi) = g \cdot \varphi := \rho_W(g) \circ \varphi \circ \rho_V(g)^{-1}. \tag{2.14.4}$$

Exercise 2.14.5. Verify that (2.14.4) really defines a representation (by Remark 2.4.2, verify it is a linear group action of G).

Solution to exercise: The fact that it is associative follows from

$$\begin{aligned} g \cdot (h \cdot \varphi) = \rho_W(g) \circ \rho_W(h) \circ \varphi \circ \rho_V(h)^{-1} \circ \rho_V(g)^{-1} = \rho_W(gh) \circ \varphi \circ \rho_V(h^{-1}) \circ \rho_V(g^{-1}) \\ = \rho_W(gh) \circ \varphi \circ \rho_V(h^{-1}g^{-1}) = \rho_W(gh) \circ \varphi \circ \rho_V(gh)^{-1}. \end{aligned} \tag{2.14.6}$$

The fact that $1 \cdot \varphi = \varphi$ is immediate. To see that the action is linear, we use the identities from linear algebra: $S \circ (a_1T_1 + a_2T_2) = a_1 S \circ T_1 + a_2 S \circ T_2$, and similarly $(a_1T_1 + a_2T_2) \circ S = a_1(T_1 \circ S) + a_2(T_2 \circ S)$. Thus $S_1 \circ (a_1T_1 + a_2T_2) \circ S_2 = a_1(S_1 \circ T_1 \circ S_2) + a_2(S_1 \circ T_2 \circ S_2)$. Apply this to $S_1 = \rho_W(g)$ and $S_2 = \rho_V(g)^{-1}$.

We will make use of the following definition later on:

Definition 2.14.7. Let (V, ρ_V) be a representation of G. Define the *G-invariant subspace* as $V^G := \{v \in V \mid \rho_V(g)(v) = v, \forall g \in G\}$.

Exercise 2.14.8. Verify that $V^G \subseteq V$ is indeed a linear subspace. You can do this in two ways: (i) Direct proof; (ii) by proving that V^G is the intersection, over all $g \in G$, of the eigenspace of $\rho(g)$ of eigenvalue one (i.e., $\ker(\rho(g) - I)$).

As a consequence, $V^G \subseteq V$ is in fact a subrepresentation, since V^G is obviously fixed under G ($\rho_V(g)(v) = v$ for all $v \in V^G$). For every $v \in V^G$, we have that $\mathbf{C} \cdot v$ is a subrepresentation isomorphic to the trivial representation, and conversely every such subrepresentation is contained in V^G (i.e., V^G is the sum of all trivial subrepresentations of V).

We now obtain an alternative proof that $\mathrm{Hom}_G(V, W)$ is a vector space as a consequence of the following:

Exercise 2.14.9. Show that $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$.

We can now explain the clever formula in Maschke's theorem, which turns a projection into a G-linear projection (i.e., a projection map which is a homomorphism of representations):

Proposition 2.14.10. Given any representation (V, ρ_V), there is a G-linear projection $S : V \to V^G$ given by the formula

$$S(v) = |G|^{-1} \sum_{g \in G} \rho_V(g)(v). \tag{2.14.11}$$

Note as in Maschke's theorem the necessity of being able to invert $|G|$ here (which we get by working over $\mathbf{C}$).

In the proof of Maschke's theorem we applied this proposition replacing V by $\mathrm{Hom}(V, V)$, v by a transformation $T : V \to V$, and $S(v)$ by $\widetilde{T}$.

Proof of Proposition 2.14.10. As in the proof of Maschke's theorem, note that $S(v) = v$ for all $v \in V^G$. Moreover, $\rho_V(g)(S(v)) = S(v)$ by the same argument as (2.10.5). So the image of S is exactly V^G and S is the identity restricted to V^G. By Lemma 2.8.16, we get that S is a projection map.

So see that S is G-linear, note that again the same argument as in (2.10.5) shows that $S(\rho_V(g)(v)) = S(v)$. So $\rho_V(g)(S(v)) = S(\rho_V(g)(v)) = S(v)$. □

2.15 The decomposition of a representation

Let G be a finite group and (V, ρ) a finite-dimensional representation. Since V is a direct sum of irreducible representations, up to isomorphism we can group together the isomorphic representations and say that

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}, \tag{2.15.1}$$

where (V_i, ρ_i) are *pairwise nonisomorphic* irreducible representations and $r_i \geq 1$. The purpose of this subsection is to explain one way to compute the r_i.

Proposition 2.15.2. $r_i = \dim \mathrm{Hom}_G(V_i, V) = \dim \mathrm{Hom}_G(V, V_i)$.

Proof. This follows from Schur's Lemma together with the following basic lemma. □

Lemma 2.15.3. For arbitrary representations $(V_1, \rho_1), (V_2, \rho_2), (W, \rho_W)$ of G, we have linear isomorphisms

$$\mathrm{Hom}_G(V_1 \oplus V_2, W) \cong \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W), \tag{2.15.4}$$

$$\mathrm{Hom}_G(W, V_1 \oplus V_2) \cong \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2). \tag{2.15.5}$$

Proof. Here are the linear maps:

$$S : \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W) \to \mathrm{Hom}_G(V_1 \oplus V_2, W), \quad S(\varphi_1, \varphi_2)(v_1, v_2) = \varphi_1(v_1) + \varphi_2(v_2), \tag{2.15.6}$$

$$T : \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2) \to \mathrm{Hom}_G(W, V_1 \oplus V_2), \quad T(\varphi_1, \varphi_2)(w) = (\varphi_1(w), \varphi_2(w)). \tag{2.15.7}$$

To verify these are isomorphisms, we explicitly construct their inverses. For $k \in \{1, 2\}$, let $i_k : V_k \to (V_1 \oplus V_2)$ be the inclusion and let $q_k : (V_1 \oplus V_2) \to V_k$ be the projection. Then we can write:

$$S^{-1}(\varphi) = (\varphi \circ i_1, \varphi \circ i_2), \quad T^{-1}(\varphi) = (q_1 \circ \varphi, q_2 \circ \varphi). \tag{2.15.8}$$

It is easy to see that S and T are linear and that S^{-1} and T^{-1} are inverses ($S \circ S^{-1} = I$ and $S^{-1} \circ S = I$, and similarly for T and T^{-1}). □

Remark 2.15.9. Taking $G = \{1\}$, we get the same statements as above without the G present (just isomorphisms of ordinary linear Hom spaces). Conversely, given the statements without the G, e.g., $\mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W) \cong \mathrm{Hom}(V_1 \oplus V_2, W)$, we get the statements with the G by taking G-invariants: $\mathrm{Hom}(V_1, W)^G \oplus \mathrm{Hom}(V_2, W)^G \cong \mathrm{Hom}(V_1 \oplus V_2, W)^G$, since in general $(U \oplus W)^G = U^G \oplus W^G$. Then recall that $\mathrm{Hom}(V, W)^G \cong \mathrm{Hom}_G(V, W)$ (Exercise 2.14.9).

Corollary 2.15.10. The decomposition (2.15.1) is unique up to replacing each (V_i, ρ_i) by an isomorphic representation.

Proof. This follows because, up to isomorphism, each irreducible representation V' must occur exactly $r' = \dim \mathrm{Hom}_G(V', V)$ times. If $r' > 0$, since the V_i are all assumed to be (pairwise) nonisomorphic, there is exactly one i such that $V' \cong V_i$ and $r_i = r'$.

In more detail, suppose that $V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m} \cong W_1^{s_1} \oplus \cdots \oplus W_n^{s_n}$ are two decompositions into nonisomorphic irreducible representations. Then

$$r_i = \dim \mathrm{Hom}_G(V_i, V) = \dim \bigoplus_{j=1}^{n} \mathrm{Hom}_G(V_i, W_j^{s_j}) = \sum_{j=1}^{n} s_j \dim \mathrm{Hom}_G(V_i, W_j). \tag{2.15.11}$$

Since $r_i > 0$, must exist some j such that $\mathrm{Hom}_G(V_i, W_j) \neq 0$. By Schur's Lemma, $W_j \cong V_i$, and since the W_j are nonisomorphic, this j is unique. Then $s_j = r_i$. Thus we get an injection $\sigma : \{1, \ldots, m\} \hookrightarrow \{1, \ldots, n\}$ such that $V_i \cong W_{\sigma(i)}$ and $r_i = s_{\sigma(i)}$. Swapping the roles of V_i and W_j we also have an injection $\tau : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $W_j \cong V_{\tau(j)}$ and $s_j = r_{\tau(j)}$. It is clear that σ and τ are inverse, so σ is a permutation. Since $V_i \cong W_{\sigma(i)}$ and $r_i = s_{\sigma(i)}$ for all i, we get the desired statement. □

This also allows us to form numerical criterion for the decomposition (2.15.1), and in particular to be irreducible:

Proposition 2.15.12. Suppose (V, ρ_V) decomposes as in (2.15.1). Then

$$\dim \mathrm{End}_G(V) = r_1^2 + \cdots + r_m^2. \tag{2.15.13}$$

In particular, (V, ρ_V) is irreducible if and only if $\dim \mathrm{End}_G(V) = 1$.

We remark that for the last statement, the "only if" is precisely Schur's Lemma (Lemma 2.11.1), part (ii).

Proof of Proposition 2.15.12. Decompose V as in (2.15.1). Applying Lemma 2.15.3, we get

$$\mathrm{End}_G(V) \cong \mathrm{Hom}_G(V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}, V) \cong \bigoplus_{i=1}^{m} \mathrm{Hom}_G(V_i, V)^{r_i}, \tag{2.15.14}$$

and the dimension of the RHS is clearly $r_1^2 + \cdots + r_m^2$. □

Exercise 2.15.15. Suppose that (V, ρ_V) is a representation of G with $\dim \mathrm{End}_G(V) \leq 3$. Then show that V is a direct sum of (pairwise) nonisomorphic irreducibles, and exactly $\dim \mathrm{End}_G(V)$ of them.

2.16 The decomposition of the regular representation

Again assume G is finite. Recall exercise 2.7.6. By this, we obtain:

$$\dim \operatorname{Hom}_G(\mathbf{C}[G], V) = \dim V, \tag{2.16.1}$$

for every representation (V, ρ). Applying this to irreducible representations, Proposition 2.15.2 implies:

Corollary 2.16.2. There are finitely many nonisomorphic irreducible representations of G. Calling them $(V_1, \rho_1), \ldots, (V_m, \rho_m)$ up to isomorphism, there exists a G-linear isomorphism

$$\mathbf{C}[G] \cong V_1^{\dim V_1} \oplus \cdots \oplus V_m^{\dim V_m}. \tag{2.16.3}$$

Taking dimensions, we obtain the following important identity:

Corollary 2.16.4. $|G| = \sum_i (\dim V_i)^2$.

Proof. This is because $\dim(V \oplus W) = \dim V + \dim W$ and $\dim V^r = r \dim V$. □

Corollary 2.16.5. Unless G is trivial, the dimension of every irreducible representation is less than $\sqrt{|G|}$.

Proof. For every irreducible representation (V, ρ) other than the trivial representation,

$$|G| = \dim \mathbf{C}[G] = \dim(\mathbf{C} \oplus V^{\dim V} \oplus \cdots) \geq \dim \mathbf{C} \oplus V^{\dim V} = 1 + (\dim V)^2. \tag{2.16.6}$$

□

2.17 Examples: S_3, dihedral groups and S_4

Here we classify irreducible representations of S_3, the dihedral groups, and of S_4 up to isomorphism.

The following result will be important. Recall that the *reflection representation* of $G = S_n$ is the representation (V, ρ) with $V = \{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n = 0\} \subseteq \mathbf{C}^n$ and ρ is obtained by restricting the permutation representation. Explicitly, $\rho(\sigma)(a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$.

Remark 2.17.1 (Omit from lecture, non-examinable)**.** Caution: the inverses are needed here since the formula really corresponds to permuting the components according to σ, i.e., putting the quantity that was before in the i-th entry into the $\sigma(i)$-th entry: setting $b_i = a_{\sigma^{-1}(i)}$ to be the new entries in the i-th position, then $b_{\sigma(i)} = a_i$. We can also explicitly verify that the formula above gives an action of S_n on $\mathbf{C}^n$:

$$\rho(\tau \circ \sigma)(a_1, \ldots, a_n) = \rho(\tau)(b_1, \ldots, b_n) = (b_{\tau^{-1}(1)}, \ldots, b_{\tau^{-1}(n)}) = (a_{\sigma^{-1}\circ\tau^{-1}(1)}, \ldots, a_{\sigma^{-1}\circ\tau^{-1}}(n)). \tag{2.17.2}$$

Proposition 2.17.3. The reflection representation (V, ρ) is irreducible.

Proof. Suppose $W \subseteq V$ is a nonzero subrepresentation and let $w = (a_1, \ldots, a_n) \in W$ be a nonzero vector. Since $a_1 + \cdots + a_n = 0$, there must be two unequal entries, say $a_i \neq a_j$. Then $w' := w - \rho((i,j))w = (a_i - a_j)(e_i - e_j)$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the standard basis vector with 1 in the i-th entry. Since W is a subrepresentation, $w' \in W$. Rescaling, we see $e_i - e_j \in W$. Also, for any $k \neq \ell$, then $\rho((i,k)(j,\ell))(e_i - e_j) = e_k - e_\ell \in W$ as well. But these vectors span V, hence $V = W$. $\square$

Remark 2.17.4 (Non-examinable)**.** Note that the proposition above is actually valid over *any field* of characteristic not dividing n (since this is enough to guarantee that $a_1 + \cdots + a_n = 0$ implies that not all of the a_i are equal).

Example 2.17.5. We now classify the irreducible representations of S_3. By Example 2.13.1, there are exactly two one-dimensional representations: the trivial and the sign representation. We also have, by Proposition 2.17.3, the two-dimensional reflection representation, which is also irreducible. Since $1^2 + 1^2 + 2^2 = 6 = |S_3|$, these are all of the representations.

Example 2.17.6. We next classify the irreducible representations of S_4. Again, there are two one-dimensional representations. If $d_1 = 1, d_2 = 1, d_3, \ldots, d_r$ are the dimensional of the irreducible representations, then $1^2 + 1^2 + d_3^2 + \cdots + d_r^2 = 24$, so $d_3^2 + \cdots + d_r^2 = 22$ with all squares appearing being $4, 9$, or 16. We can't have 16 appear since 6 is not a sum of these squares. With 4 and 9 we can't have an odd number of 9's appearing, and also clearly can't have only 4's appearing, so the only possibility is $4 + 9 + 9$. Thus, up to reordering, $d_3 = 2, d_4 = 3$, and $d_5 = 3$.

We can realise these representations. One of the 3-dimensional representations is the reflection representation, call it (V, ρ), with $V = \{(a_1, \ldots, a_4) \mid a_1 + \cdots + a_4 = 0\} \subseteq \mathbf{C}^4$. Another one can be given by multiplying by the sign: (V, ρ') with $\rho'(\sigma) = \mathrm{sign}(\sigma)\rho(\sigma)$ (it is either a permutation matrix or just like a permutation matrix but with -1's replacing 1 throughout). This is actually an example of tensor product, as we will see later; in the next exercise we will define this and show it gives an irreducible representation. Finally, we need a two-dimensional irreducible representation. We can get it from the reflection representation of S_3, since there is a surjective homomorphism $q : S_4 \to S_3$, given by the action of S_4 by permuting the elements of the conjugacy class $\{(14)(23), (13)(24), (12)(34)\}$ (or we can explicitly write the formula, $q((12)) = (12), q((23)) = (23), q((34)) = (12)$ and the elements $(12), (23), (34)$ generate S_4). Let (W, ρ_W) be the reflection representation of S_3, $\rho_W : S_3 \to \mathrm{GL}(W)$ with $\dim W = 2$. Then we can take $(W, \rho_W \circ q)$, and since (W, ρ_W) is irreducible, so is this one by the next exercise.

Exercise 2.17.7. The above example motivates the following constructions: (i) Let (V, ρ) be a representation of a group G and $\theta : G \to \mathbf{C}^\times$ a one-dimensional representation. Show that (V, ρ') given by $\rho'(g) = \theta(g)\rho(g)$ is also a representation, and that it is irreducible if and only if (V, ρ) is irreducible. (ii) Let (V, ρ) be a representation of G and $\varphi : H \to G$ a homomorphism of groups. Show that $(V, \rho \circ \varphi)$ is a representation of H. In the case that (V, ρ) is irreducible and φ is surjective, show that $(V, \rho \circ \varphi)$ is also irreducible.

Example 2.17.8. Let $G = C_4 = \{1, g, g^2, g^3\}$ and $\rho(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the rotation matrix. Then for (i) we can let $\theta(g) = i \in \mathbf{C}$, and then $\rho'(g) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. For (ii) we can consider the automorphism $\varphi : G \to G$ given by $\varphi(g) = g^{-1} = g^3$, and then $\rho \circ \varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark 2.17.9 (Omit from lecture, non-examinable)**.** As a special useful case of part (ii) of Exercise 2.17.7, we can compose any representation of G by an automorphism to get another representation, and this preserves irreducibility. Recall from group theory: An automorphism is *inner* if it is of the form $\mathrm{Ad}_g : G \to G, \mathrm{Ad}_g(h) = ghg^{-1}$ for some $g \in G$, and otherwise it is called outer. The notation "Ad" stands for "adjoint", referring to the conjugation action (see also Definition 2.18.10 below). As an exercise, you can show that, for an inner automorphism Ad_g, we have $(V, \rho) \cong (V, \rho \circ \mathrm{Ad}_g)$. So the operation of composing (irreducible) representations by automorphisms to get new (irreducible) representations is mostly interesting when the automorphism is outer.

Example 2.17.10. We can classify the representations of D_n for low n. For $n = 3$ we have $D_6 = S_3$ so that is already done. Let's try D_8. By Example 2.13.2, there are four one-dimensional representations, so $8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ shows that there can be exactly one two-dimensional irreducible representation, which is then the one of Example 2.2.6.

Next let's look at D_{10}. By Example 2.13.2, there are two one-dimensional representations. So the sum of squares of the other representations is $10-2 = 8$, which means there are exactly two irreducible two-dimensional representations. One is Example 2.13.2, call it $(\mathbf{C}^2, \rho)$, and the other is given by the construction of Exercise 2.17.7.(ii) using the automorphism $\varphi : D_{10} \to D_{10}$, given by $\varphi(x) = x^2, \varphi(y) = y$, where x, y are the generators of Example 2.13.2) [i.e., φ doubles the rotation angles and doubles the angles that the reflection axes make with the x-axis]. Thus the other one is $(\mathbf{C}^2, \rho \circ \varphi)$.

Let us see that these two-dimensional representations are not isomorphic. Indeed, $\rho(x)$ has eigenvalues $e^{\pm 2\pi i/5}$, whereas $\rho \circ \varphi(x) = \rho(x^2)$ has eigenvalues $e^{2\pm 4\pi i/5}$, and these are not equal. Hence $\rho(x)$ and $\rho \circ \varphi(x)$ are not conjugate in $\mathrm{GL}(\mathbf{C}^2) = \mathrm{GL}_2(\mathbf{C})$, so the two representations cannot be isomorphic.

[Omit from lecture, non-examinable: on the other hand, if we had used the automorphism ψ given by $\psi(x) = x^{-1}, \psi(y) = y$, then the representations would have been isomorphic, since now $\rho \circ \psi(x) = \rho(x)^{-1} = \rho(x^{-1}) = \rho(yxy^{-1})$. Since it is also true that $\rho \circ \psi(y) = \rho(y) = \rho(yyy^{-1})$, we obtain that $\rho \circ \psi(g) = \rho(y) \circ \rho(g) \circ \rho(y)^{-1}$ for all g, hence $\rho(y) : \mathbf{C}^2 \to \mathbf{C}^2$ is an isomorphism $(\mathbf{C}^2, \rho) \to (\mathbf{C}^2, \rho \circ \psi)$.]

Note that Exercise 2.17.7.(i) does *not* work to construct the second irreducible two-dimensional representation. The nontrivial one-dimensional representation is $\chi : D_{10} \to \mathbf{C}^\times$, χ sends rotations to 1 and reflections to -1 ($\chi(x^k) = 1$ and $\chi(x^k y) = -1$ for all k). Now unlike $\rho \circ \varphi$ above, $\chi \cdot \rho(x)$ *does* have the same eigenvalues as $\rho(x)$, so actually the two will have to be isomorphic (as there are only two irreducible representations). To prove it explicitly we can use matrices, and it is valid for any D_n: Suppose g_θ is the rotation by angle

θ and h_θ the reflection about the axis making angle θ with the x-axis. Then:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho(g_\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$= \rho(g_\theta) = \chi(g_\theta)\rho(g_\theta), \quad (2.17.11)$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho(h_\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$
$$= -\rho(h_\theta) = \chi(h_\theta)\rho(h_\theta). \quad (2.17.12)$$

[Omit from lecture, non-examinable: In vectors, we explain the above as follows: the map $T : \mathbf{C}^2 \to \mathbf{C}^2$ given by the matrix $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ defines an isomorphism $(\mathbf{C}^2, \rho) \to (\mathbf{C}^2, \rho')$ for $\rho'(g) = \chi(g)\rho(g)$. That is, $T \circ \rho = \chi \cdot \rho \circ T$. To see this we note that $T \circ \rho(x) = \rho(x) \circ T$, since T is a rotation and commutes with the rotation $\rho(x)$, whereas $T \circ \rho(y) = \rho(y) \circ T^{-1}$ since $\rho(y)$ is a reflection and T a rotation, but $T^{-1} = -I$, and we conclude $T \circ \rho(y) = -\rho(y) \circ T = \chi(y) \cdot \rho(y) \circ T$. Since x and y generate (or by applying the same argument for arbitrary rotations and reflections), this proves the statement.]

For general D_n, you will complete the classification of irreducible representations for general n in the following exercise.

Exercise 2.17.13. Find all irreducible representations of D_n. Other than the one-dimensional representations (see Example 2.13.2), show that they are all obtained from the one of Example 2.2.6, $(\mathbf{C}^2, \rho)$, by the construction of Exercise 2.17.7.(ii). In more detail, consider $(\mathbf{C}^2, \rho \circ \varphi_j)$ for the endomorphisms $\varphi_j : D_n \to D_n$, $\varphi_j(x^a y^b) = x^{ja} y^b$ (caution: these are not automorphisms)! Show that they are irreducible for $1 \leq j < n/2$. Moreover show that they are nonisomorphic for these values of j, by showing that $\mathrm{tr}(\rho(x^j)) \neq \mathrm{tr}(\rho(x^k))$ for $1 leq j < k < n/2$.

Here is a useful observation: Since the number of one-dimensional representations of a finite group G equals the size of G_{ab} (Corollary 2.13.13), it is in particular a factor of $|G|$. If G is not itself abelian, this is a proper factor.

Exercise 2.17.14 (For problems class)**.** Similarly classify the irreducible representations of A_4. First using the previous observation show the possibilities are (a) $1^2+1^2+1^2+3^2$ or (b) $1^2+1^2+1^2+1^2+2^2+2^2$. Then use the surjective homomorphism $q : S_4 \to S_3$, restricted to A_4, to get $q|_{A_4} : A_4 \to A_3$, a surjective homomorphism with abelian image. Using Corollary 2.13.9 show that we are in case (a). Then find the one-dimensional representations. Finally prove that the restriction of the three-dimensional reflection representation of S_4 to A_4 is irreducible, completing the classification.

2.18 The number of irreducible representations

Now we are ready to prove a fundamental formula. First some terminology:

Definition 2.18.1. A full set of nonisomorphic irreducible representations $(V_1, \rho_1), \ldots, (V_m, \rho_m)$ is one such that every irreducible representation is isomorphic to exactly one of the (V_i, ρ_i).

Theorem 2.18.2. Let G be a finite group and $(V_1, \rho_1), \ldots, (V_m, \rho_m)$ a full set of irreducible representations. Then m equals the number of conjugacy classes of G.

We will prove this theorem as a consequence of a strengthened form of Corollary 2.16.2. Let us motivate this a bit. The formula gives a suggestive identity of the size of G with the sum of the squares of the dimensions of the irreducible representations $V_1, \ldots, V_m$. We can reinterpret $(\dim V_i)^2$ as $\dim \mathrm{End}(V_i)$. With this in mind the vector spaces $\mathbf{C}[G]$ and $\bigoplus_{i=1}^m \mathrm{End}(V_i)$ have the same dimension, hence are isomorphic. But what we really want is a "canonical" isomorphism between them, i.e., an explicit isomorphism of the form

$$\mathbf{C}[G] \xrightarrow{\sim} \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_m), \tag{2.18.3}$$

which does not depend on bases (it should be given by a "nice" formula). Having such a canonical isomorphism, could say that $\mathbf{C}[G]$ and $\bigoplus_{i=1}^m \mathrm{End}(V_i)$ have the same *personality*.

There is indeed an obvious candidate for the map in (2.18.3): for each i we have $\rho_i : G \to \mathrm{GL}(V_i)$, and we can linearly extend this to $\rho_i : \mathbf{C}[G] \to \mathrm{End}(V_i)$, just by

$$\rho_i(\sum_g a_g g) = \sum_g a_g \rho_i(g) \in \mathrm{End}(V_i). \tag{2.18.4}$$

Putting these together yields a candidate for the desired isomorphism (2.18.3).

Remark 2.18.5. What we have just proposed to do is called *categorification* in mathematics: namely, replacing an equality of numbers (here, Corollary 2.16.4) by a "canonical" isomorphism of vector spaces, such that the original equality is recovered by taking dimension of both sides. (Corollary 2.16.2 didn't do this, since it depended on choosing bases, and anyway the RHS were not literally the same.)

How will this imply Theorem 2.18.2? From the RHS of (2.18.3) we will be able to recover m by taking G-invariants, since Schur's Lemma says that $\mathrm{End}(V_i)^G = \mathrm{End}_G(V_i)$ is one-dimensional for all i! That is, the dimension of the G-invariants of the RHS is m. **Caution:** this doesn't immediately solve our problem since the map (2.18.3) is actually *not* G-linear, in spite of being canonical. But since it is canonical we will be able to put a new G-action on $\mathbf{C}[G]$ that fixes this, and we will conclude Theorem 2.18.2.

Thus Theorem 2.18.2 will be a consequence of the following strengthening of Corollary 2.16.2 (or "categorification" of Corollary 2.16.4):

Theorem 2.18.6. The map $\Phi = (\rho_1, \ldots, \rho_m) : \mathbf{C}[G] \to \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_m)$ is a linear isomorphism.

Caution: as before, this is not G-linear (yet).

Proof. The map is linear by definition. To check it is an isomorphism, by the dimension equality Corollary 2.16.4, we only need to check it is injective. Suppose that $f \in \mathbf{C}[G]$ had the property that $\rho_i(f) = 0$ for all i. Since the (V_i, ρ_i) are a full set of irreducible representations, this implies that for every irreducible representation (W, ρ_W), we have $\rho_W(f) = 0$. But by Maschke's Theorem, every finite-dimensional representation of G is a direct sum of irreducible representations. Hence for *every* finite-dimensional representation (W, ρ_W) we have $\rho_W(f) = 0$. Now set $W = \mathbf{C}[G]$. Then $\rho_{\mathbf{C}[G]}(f) = 0$. Applying this to $1 \in G$, we get $\rho_{\mathbf{C}[G]}(f)(1) = f$, by definition. Thus $f = 0$, so Φ is indeed injective. $\square$

We now have to fix the problem that Φ is not G-linear. Let's see what property it does satisfy:

Lemma 2.18.7. The map Φ has the property

$$\Phi(ghg^{-1}) = \bigoplus_{i=1}^{m} \rho_{\mathrm{End}(V_i)}(g)(\Phi(h)). \tag{2.18.8}$$

Proof. Equivalently, we need to show that $\rho_i(ghg^{-1}) = \rho_{\mathrm{End}(V_i)}(g)(\rho_i(h))$. By Definition 2.14.3, we have:

$$\rho_{\mathrm{End}(V_i)}(g)(f) = \rho_i(g) \circ f \circ \rho_i(g)^{-1}. \tag{2.18.9}$$

Now plugging in $f = \rho_i(h)$ and using the fact that ρ_i is a homomorphism yields the statement. $\square$

This motivates the following new representation on the vector space $\mathbf{C}[G]$:

Definition 2.18.10. The *adjoint representation* $(\mathbf{C}[G], \rho_{\mathrm{ad}})$ is given by

$$\rho_{\mathrm{ad}}(g)(\sum_{h \in G} a_h h) = \sum_{h \in H} a_h ghg^{-1}. \tag{2.18.11}$$

The word "adjoint" refers to the conjugation action, as in Remark 2.17.9: $g \cdot h = \mathrm{Ad}_g(h)$.

Exercise 2.18.12. (i) Verify that the adjoint representation is a representation. Similarly show that the *right regular* representation, $\rho_{\mathrm{Rreg}} : G \to \mathrm{Aut}(\mathbf{C}[G]), \rho_{\mathrm{Rreg}}(g)(h) := hg^{-1}$, is a representation. Thus there are *three* different structures of a representation on the vector space $\mathbf{C}[G]$: the (left) regular one, the right regular one, and the ajoint one.

Then Lemma 2.18.7 and Theorem 2.18.6 immediately imply:

Corollary 2.18.13. The isomorphism Φ is G-linear as a map

$$(\mathbf{C}[G], \rho_{\mathrm{ad}}) \to \bigoplus_{i=1}^{m} \mathrm{End}(V_i). \tag{2.18.14}$$

Taking G-invariants yields:

Corollary 2.18.15. We have an isomorphism of vector spaces,

$$(\mathbf{C}[G], \rho_{\mathrm{ad}})^G \to \bigoplus_{i=1}^m \mathrm{End}_G(V_i) \cong \mathbf{C}^m, \tag{2.18.16}$$

To prove Theorem 2.18.2, we only need one more lemma. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{m'}$ be the conjugacy classes of G.

Lemma 2.18.17. The G-invariants $(\mathbf{C}[G], \rho_{\mathrm{ad}})^G$ have a basis $f_1, \ldots, f_{m'}$ given by:

$$f_i = \sum_{g \in \mathcal{C}_i} g. \tag{2.18.18}$$

Proof. Note that $f \in (\mathbf{C}[G], \rho_{\mathrm{ad}})^G$ if and only if $f = gfg^{-1}$ for every $g \in G$. Now, take an element $f = \sum_{h \in G} a_h h$. Then for a given $G \in G$, we have

$$gfg^{-1} = \sum_{h \in G} a_h ghg^{-1} = \sum_{h \in G} a_{g^{-1}hg} h.$$

So $f = gfg^{-1}$ for all $g \in G$ if and only if $a_h = a_{g^{-1}hg}$ for all $g, h \in G$. Equivalently, $a_h = a_{h'}$ if h, h' are in the same conjugacy class. Now for each i let $g_i \in \mathcal{C}_i$ be a representative. We obtain that

$$f = \sum_{i=1}^m a_{g_i} f_i.$$

Thus the elements f_i span $(\mathbf{C}[G], \rho_{\mathrm{ad}})^G$. They are obviously linearly independent. $\square$

Now taking dimensions in Corollary 2.18.15, we obtain $m' = m$, i.e., the number of conjugacy classes in G equals the number of isomorphism classes of irreducible representations. This completes the proof of Theorem 2.18.2.

Remark 2.18.19 (Non-examinable)**.** In fact, both sides of the isomorphim in Theorem 2.18.6 have a multiplication as well, coming from the group multiplication on the LHS and the composition of endomorphisms on the RHS. The map ψ respects these (more formally, this says that ψ is *ring isomorphism*). This will be a central observation of the last unit of the course, which we will use to give another proof of the formula for the number of irreducible representations of G.

2.19 Duals and tensor products

Recall that, given a vector space V, the dual vector space is $V^* := \mathrm{Hom}(V, \mathbf{C})$. By Definition 2.14.3, when (V, ρ_V) is a representation of G, we also obtain a representation (V^*, ρ_{V^*}). Let us write explicitly the formula: for $f \in V^*$, $g \in G$, and $v \in V$:

$$(\rho_{V^*}(g)(f))(v) := f(\rho_V(g)^{-1}(v)) = f(g^{-1} \cdot v). \tag{2.19.1}$$

Example 2.19.2. Let $\rho : G \to \mathbf{C}^\times$ be a one-dimensional representation. Let us compute its dual. Let $V = \mathbf{C}$. We obtain (V, ρ_V) where $\rho_V(g)(v) = \rho(g)v$ for all $g \in G$ and $v \in V$. Then the dual (V^*, ρ_{V^*}) satisfies $\rho_{V^*}(g)(f) = f \circ \rho_V(g)^{-1} = \rho(g)^{-1}f$. That is, dualising one-dimensional representations *inverts* $\rho(g)$ *for all* g.

Remark 2.19.3 (Important)**.** Let us work out the formula in terms of matrices for $\rho_{V^*}(g)$. Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of V, and let $\mathcal{C} = \{f_1, \ldots, f_n\}$ be the dual basis, given by

$$f_i(v_j) = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.19.4}$$

Here δ_{ij} is called the Kronecker delta function. We now compute:

$$\rho_{V^*}(g)(f_i)(v_j) = f_i(\rho_V(g^{-1})(v_j)). \tag{2.19.5}$$

The LHS is the (j, i)-th entry of $[\rho_{V^*}(g)]_{\mathcal{C}}$. The RHS is, on the other hand, the (i, j)-th entry of $[\rho_V(g^{-1})]_{\mathcal{B}} = ([\rho_V(g)]_{\mathcal{B}})^{-1}$. Putting this together we get:

$$[\rho_{V^*}(g)]_{\mathcal{C}} = (([\rho_V(g)]_{\mathcal{B}})^{-1})^t, \tag{2.19.6}$$

where the superscript of t denotes transpose.

Proposition 2.19.7. We have a canonical (and G-linear) inclusion $V \hookrightarrow (V^*)^*$ of vector spaces, $v \mapsto \varphi_v, \varphi_v(f) := f(v)$. This is an isomorphism if V is finite-dimensional.

The key point here in the proposition is not merely that we have isomorphisms but *that there is a nice formula for them not depending on bases*. That means that the two isomorphic quantities "have the same personalities" and we can interchange them, particularly when we want to be able to keep track of what a representation is on both sides.

Proof of Proposition 2.19.7. The given map is obviously linear. To see it is injective, suppose that $v \neq 0$. Then we can extend v to a basis of V, and therefore define a linear map $f \in V^*$ such that $f(v) = 1$ (and say $f(v') = 0$ for other elements of the basis). Then $\varphi_v(f) = f(v) \neq 0$, so $\varphi_v \neq 0$. To check G-linearity, we trace through the definitions:

$$\rho_{(V^*)^*}(g)(\varphi_v)(f) = \varphi_v(\rho_{V^*}(g)^{-1}(f)) = \varphi_v(f \circ \rho_V(g)) = f(\rho_V(g)(v)) = \varphi_{\rho_V(g)(v)}(f). \qquad \square$$

2.19.1 Definition of the tensor product of vector spaces

Next, given representations (V, ρ_V) and (W, ρ_W) of G, we would like to define the representation $(V \otimes W, \rho_{V \otimes W})$. To do this we first have to define the tensor product.

Definition 2.19.8. The tensor product $V \otimes W$ of vector spaces V and W is the quotient of the vector space $\mathbf{C}[V \times W] := \{\sum_{i=1}^n a_i(v_i, w_i) \mid a_i \in \mathbf{C}, v_i \in V, w_i \in W\}$ (requiring the pairs

(v_i, w_i) appearing be distinct), with basis $V \times W$, by the subspace spanned by the elements (called *relations*):

$$a(v,w)-(av,w), \quad a(v,w)-(v,aw), \quad (v+v',w)-(v,w)-(v',w), \quad (v,w+w')-(v,w)-(v,w'). \tag{2.19.9}$$

The image of $(v, w) \in \mathbf{C}[V \times W]$ under the quotient map $\mathbf{C}[V \times W] \twoheadrightarrow V \otimes W$ is denoted $v \otimes w$.

Note that the elements $v \otimes w$ span $V \otimes W$, since $a(v \otimes w) = (av) \otimes w$.

Example 2.19.10. Let $W = \mathbf{C}$. Then we have an isomorphism $\varphi : V \otimes \mathbf{C} \to V$ given by $(v \otimes a) \mapsto av$. First let us check that this is well-defined: obviously we have a map $\widetilde{\varphi} : \mathbf{C}[V \times \mathbf{C}] \to V$ given by $(v, a) \mapsto av$, and we just need to observe that the relations are satisfied. This is true because $(v, a) \mapsto av$ is bilinear; for instance, $\widetilde{\varphi}((a(v, b) + a'(v', b)) = abv + a'bv' = \widetilde{\varphi}(av + a'v, b)$.

To see that φ is an isomorphism, note first that it is obviously surjective and linear. Let us show it is injective. Suppose some element $x = \sum_i a_i(v_i \otimes b_i)$ is in the kernel. Applying φ we get that $\sum_i a_i b_i v_i = 0$. But also $x = \sum_i (a_i b_i v_i) \otimes 1$, applying linearity. So $x = 0 \otimes 1 = 0$.

This definition is rather abstract, but we will give two other definitions in the following proposition to make things easier to understand.

Proposition 2.19.11. (i) Suppose that $(v_i)_{i \in I}$ is a basis of V and $(w_j)_{j \in J}$ is a basis of W (for some sets I and J which index the bases: for this course you are welcome to assume they are finite). Then $(v_i \otimes w_j)_{i \in I, j \in J}$ is a basis for $V \otimes W$. In particular, $\dim(V \otimes W) = (\dim V)(\dim W)$.

(ii) There is a homomorphism

$$\iota : V^* \otimes W \to Hom(V, W), \quad \iota(f \otimes w)(v) = f(v)w. \tag{2.19.12}$$

If V is finite-dimensional, this homomorphism is an isomorphism.

In this course we will restrict to the case that V and W are finite-dimensional. Therefore, we can think alternatively of $V \otimes W$ as a vector space with basis $(v_i \otimes w_j)$ where (v_i) and (w_j) are (finite) bases of V and W respectively, or we can think of it as $\mathrm{Hom}(V^*, W)$.

Proof of Proposition 2.19.11. [non-examinable]

(i) [Prove only finite-dimensional case in lecture:] First we show that $(v_i \otimes w_j)$ span $V \otimes W$. We already observed that $V \otimes W$ is spanned by elements $v \otimes w$ for arbitrary $v \in V, w \in W$. But we can replace v by a linear combination of the v_i and w by a linear combination of the w_j, and apply the definition of tensor product: $\sum_{i,j}(a_i v_i) \otimes (b_j w_j) = \sum_{i,j} a_i b_j (v_i \otimes w_j)$, which proves the statement.

Next we prove linear independence. Suppose that $z := \sum_{k=1}^{n} a_{i_k, j_k} v_{i_k} \otimes w_{j_k}$ is zero, with all of the a_{i_k, j_k} nonzero and none of the pairs (i_k, j_k) identical, and assume n is minimal to get such a relation. If any of the i_k are equal we can combine terms and reduce n, and the same is true for the j_k. So assume all the i_k are unequal. Let $f \in V^*$ be an element such that $f(v_{i_1}) = 1$ and $f(v_{i_k}) = 0$ for $k > 1$. Then $\sum_{k=1}^{n} a_{i_k,j_k} f(v_{i_k}) \otimes w_{j_k} = a_{i_1,j_1} w_{j,k} \neq 0$. But the map $F : V \otimes W \to W, F(v \otimes w) = f(v)w$ is a well-defined linear map, since it is well-defined and linear on $\mathbf{C}[V \times W]$, and sends all of the relations to zero. Then $F(z) \neq 0$, but this contradicts $z = 0$ since F is a linear map.

(ii) To check this map is linear and well-defined, note first that we have a well-defined linear map $\widetilde{\iota} : \mathbf{C}[V^* \times W] \to \mathrm{Hom}(V, W)$, given by $\widetilde{\iota}(f, w)(v) = f(v)w$ for all $v \in V, w \in W$, extended linearly to $\mathbf{C}[V^* \times W]$. We only have to check that the relations defining $V^* \otimes W$ are in the kernel. This is an explicit check:

$$\widetilde{\iota}((af, w))(v) = f(av)w = af(v)w = a\widetilde{\iota}(f, w)(v) = \widetilde{\iota}(a(f, w))(v), \tag{2.19.13}$$

$$\widetilde{\iota}((f, w) + (f', w))(v) = f(v)w + f'(v)w = (f + f')(v)w = \widetilde{\iota}(f + f', w)(v). \tag{2.19.14}$$

We can similarly handle the other relations.

To prove it is an isomorphism if V is finite-dimensional, we give an explicit inverse in this case. Let (v_i) be a basis of V and (f_i) the dual basis of V^*. Then we can define $\mathrm{Hom}(V, W) \to V^* \otimes W$ by $T \mapsto \sum_i f_i \otimes T(v_i)$. It is straightforward to verify that this is an inverse. □

The main thing is that we need to think of $V \otimes W$ as a vector space whose elements are linear combinations (or sums) of elements of the form $v \otimes w, v \in V, w \in W$, subject to the defining relations (2.19.9) which in terms of tensor products can be written as the following equalities for all $v, v' \in V$, $w, w' \in W$, and $a, a' \in \mathbf{C}$:

$$(av + a'v') \otimes w = a(v \otimes w) + a'(v' \otimes w); \quad v \otimes (aw + a'w') = a(v \otimes w) + a'(v \otimes w'). \tag{2.19.15}$$

Some of you might like the following more abstract way of thinking about tensor product:

Proposition 2.19.16 (The universal property of the tensor product)**.** Given vector spaces U, V, W, for every bilinear map $F : V \times W \to U$, i.e., one satisfying

$$F(av, w) = aF(v, w) = F(v, aw), \tag{2.19.17}$$

$$F(v + v', w) = F(v, w) + F(v', w), \quad F(v, w + w') = F(v, w) + F(v, w'), \tag{2.19.18}$$

there is a unique homomorphism $\widetilde{F} : V \otimes W \to U$ with the property $\widetilde{F}(v \otimes w) = F(v, w)$. Conversely, all homomorphisms $V \otimes W \to U$ are of this form.

Proof. Construct the linear map $\widetilde{F} : \mathbf{C}[V \times W] \to U$ by linear extension of F. Because F is bilinear, the kernel includes the elements (2.19.9), as in the preceding proof. □

2.19.2 Natural isomorphisms involving tensor products

Proposition 2.19.19. (i) There are the following explicit isomorphisms, for U, V, and W arbitrary vector spaces:

$$V \otimes W \xrightarrow{\sim} W \otimes V, \quad (v \otimes w) \mapsto (w \otimes v); \tag{2.19.20}$$

$$(U \otimes V) \otimes W) \xrightarrow{\sim} U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w); \tag{2.19.21}$$

$$U \otimes (V \oplus W) \xrightarrow{\sim} (U \otimes V) \oplus (U \otimes W), \quad u \otimes (v, w) \mapsto (u \otimes v, u \otimes w), \tag{2.19.22}$$

$$\Phi : \mathrm{Hom}(U \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}(U, \mathrm{Hom}(V, W)), \quad \Phi(f)(u)(v) = f(u \otimes v). \tag{2.19.23}$$

(ii) For arbitrary V and W, there is an explicit linear injection,

$$\Phi : V^* \otimes W^* \to (V \otimes W)^*, \quad \Phi(f \otimes g)(v \otimes w) = f(v)g(w). \tag{2.19.24}$$

This is an isomorphism if V is finite-dimensional.

Proof. (i) The maps are obviously linear. It is easy to construct an explicit formula for their inverses. This is left as an exercise. For example, the inverse of the third one is given by the sum of the injective map $U \otimes V \cong U \otimes (V \oplus \{0\}) \hookrightarrow U \otimes (V \oplus W)$ and the same map with V replaced by W.

(ii) This map is obviously linear. To see it is injective, by Proposition 2.19.11.(ii), we have an injection $V^* \otimes W^* \to \mathrm{Hom}(V, W^*)$, and by (i) we can compose this with $\mathrm{Hom}(V, W^*) = \mathrm{Hom}(V, \mathrm{Hom}(W, \mathbf{C})) \cong \mathrm{Hom}(V \otimes W, \mathbf{C}) = (V \otimes W)^*$ to get a linear injection. It is easy to check that the result is the same as the given map Φ. Finally, if V is finite-dimensional, by Proposition 2.19.11.(ii) again, the injection is an isomorphism. $\square$

If we take dimensions of vector spaces, the first three isomorphisms in (ii) become the commutativity, associativity, and distributivity rules on $\mathbb{N}$. Namely, for $a = \dim U, b = \dim V, c = \dim W$, we get:

$$b \cdot c = c \cdot b, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c), \quad a \cdot (b + c) = a \cdot b + a \cdot c. \tag{2.19.25}$$

There is a term for this in mathematics: we say that these isomorphisms "categorify" the identities above.

2.19.3 Tensor products of representations

By the preceding, one way to think about tensor products of representations V and W, if V is finite-dimensional, is simply as the representation $\mathrm{Hom}(V^*, W)$. However it is useful to write a formula in terms of tensor product symbols:

Definition 2.19.26. The tensor product representation $(V \otimes W, \rho_{V \otimes W})$ is given by

$$\rho_{V \otimes W} : G \to \mathrm{GL}(V \otimes W), \quad \rho_{V \otimes W}(g)(v \otimes w) = \rho_V(g)(v) \otimes \rho_V(g)(w). \tag{2.19.27}$$

(This definition also has the advantage of working even if V is not finite-dimensional, although we won't need this.)

Example 2.19.28. Let $(V, \rho_V), (V', \rho_{V'})$ be one-dimensional representations. Let $\rho, \rho' : G \to \mathbf{C}^\times$ be defined by the properties $\rho_V(g)(v) = \rho(g)v$ and $\rho_{V'}(g)(v') = \rho'(g)v'$ for all $g \in G, v \in V$, and $v' \in V'$. Then $(V \otimes V', \rho_{V \otimes V'})$ has the property $\rho_{V \otimes V'}(v \otimes v') = \rho(g)\rho'(g)(v \otimes v')$. That is, the tensor product of the one-dimensional representations $\rho, \rho' : G \to \mathbf{C}^\times$ in terms of matrices is the product,

$$(\rho \otimes \rho')(g) = \rho(g)\rho'(g). \tag{2.19.29}$$

Remark 2.19.30. Combining Examples 2.19.28 and 2.19.2, we conclude that the set of homomorphisms $\rho : G \to \mathbf{C}^\times$ (matrix one-dimensional representations) form a group under the tensor product operation (2.19.29) with inversion given by the matrix corresponding to dualisation: the inverse ρ' to ρ satisfies $\rho'(g) = \rho(g)^{-1}$ for all $g \in G$. Of course we didn't need tensor products to define this. However the tensor product and dualisation constructions work on vector spaces, not just on matrices, and provide the conceptual explanation for this operation.

Remark 2.19.31 (Important)**.** It is useful to give a formula for the tensor product representation in terms of matrices. Let $\mathcal{B} = \{v_1, \ldots, v_m\}$ be a basis of V and $\mathcal{C} = \{w_1, \ldots, w_n\}$ be a basis of W. Let $\mathcal{D} := \{v_i \otimes w_j\}$ be the resulting basis of $V \otimes W$, indexed by *pairs* (i, j). Take $g \in G$ and let $M = [\rho_V(g)]_{\mathcal{B}}$ and $N = [\rho_W(g)]_{\mathcal{C}}$ be the matrices of the representation. Then

$$\rho_{V\otimes W}(g)(v_k \otimes w_\ell) = \rho_V(g)(v_k) \otimes \rho_W(g)(w_\ell) = \sum_{i,j} M_{ik} v_i \otimes N_{j\ell} w_j. \tag{2.19.32}$$

Thus, as $mn \times mn$ matrices $A_{ij,k\ell}$ for $1 \leq i, k \leq m$ and $1 \leq j \leq n$, we have:

$$([\rho_{V\otimes W}(g)]_{\mathcal{D}})_{ij,k\ell} = (M_{ik} N_{j\ell}). \tag{2.19.33}$$

We can define $M \otimes N$ as the LHS matrix (i.e., thinking of matrices as linear transformations $M : \mathbf{C}^m \to \mathbf{C}^m, N : \mathbf{C}^n \to \mathbf{C}^n$ by multiplication, then $M \otimes N : (\mathbf{C}^m \otimes \mathbf{C}^n) \to (\mathbf{C}^m \otimes \mathbf{C}^n)$ is the resulting $mn \times mn$ matrix with entries labeled by pairs (a, b) with $1 \leq a \leq m$ and $1 \leq b \leq n$). It is also called the *Kronecker product* matrix of M and N. Then the equation above gives the standard formula for this product:

$$(M \otimes N)_{ij,k\ell} = M_{ik} N_{j\ell}. \tag{2.19.34}$$

Exercise 2.19.35. Show that, under the isomorphism $\iota : V \otimes W \to \mathrm{Hom}(V^*, W)$, that Definition 2.19.26 agrees with Definition 2.14.3.

In fact, (2.19.27) is easier to deal with than the formula you get from taking two hom spaces. So much so that it is useful to turn the situation around, using Proposition 2.19.11.(ii). Now instead of using the formula for the representation $\mathrm{Hom}(V, W)$ we can feel free to begin with V^* and once we understood its formula, to use $V^* \otimes W$. Here is an example which shows why this might be useful:

Example 2.19.36. Let $(\mathbf{C}_\theta, \theta)$ be a one-dimensional representation, given by $\mathbf{C}_\theta = \mathbf{C}$ and $\theta : G \to \mathbf{C}^\times = \mathrm{GL}_1(\mathbf{C})$. Let (W, ρ_W) be an arbitrary representation. To compute the representation on $\mathbf{C}_\theta \otimes W$, first note that $\mathbf{C}_\theta \otimes W \cong W$ as vector spaces by Example 2.19.10. In terms of this, we have

$$\rho_{\mathbf{C}_\theta\otimes W}(g)(w) = \rho_{\mathbf{C}_\theta\otimes W}(g)(1 \otimes w) = \theta(g) 1 \otimes \rho_W(g)(w) = \theta(g)\rho_W(g)(w). \tag{2.19.37}$$

So we just multiply ρ_W by θ: it's just the construction of Exercise 2.17.7.(i)!

Note, on the other hand, if we want to compute the representation $\mathrm{Hom}(\mathbf{C}_\theta, W)$, this vector space is also isomorphic to W, but the correct representation is given by $\mathrm{Hom}(\mathbf{C}_\theta, W) \cong \mathbf{C}_\theta^* \otimes W$ so that we have to invert the $\theta(g)$: we get

$$\rho_{\mathrm{Hom}(\mathbf{C}_\theta, W)}(g)(w) = \theta(g)^{-1} \rho_W(g)(w). \tag{2.19.38}$$

This, I think, is easier to think about than computing directly from the formula of Definition 2.14.3 for $\mathrm{Hom}(\mathbf{C}_\theta, W)$ in terms of linear maps $\mathbf{C}_\theta \to W$.

2.20 External tensor product

An illustration of the power of tensor product is the following:

Definition 2.20.1. Let G and H be groups, (V, ρ_V) be a representation of G, and (W, ρ_W) be a representation of H. Form the representation, called the *external tensor product* and often denoted $V \boxtimes W$ of $G \times H$ as follows: as a vector space, $V \boxtimes W = V \otimes W$ is the ordinary tensor product, but with homomorphism given by

$$\rho_{V \boxtimes W} : G \times H \to \mathrm{GL}(V \otimes W), \quad \rho_{V \boxtimes W}(g, h) = \rho_V(g) \otimes \rho_W(h). \tag{2.20.2}$$

Exercise 2.20.3. (i) Show that this construction recovers the tensor product of representations in the following sense: for $G = H$, we have

$$\rho_{V \otimes W}(g) = \rho_{V \boxtimes W}(g, g). \tag{2.20.4}$$

(ii) More conceptually, define the *diagonal map*, $\Delta : G \to G \times G$ by $g \mapsto (g, g)$, which is a homomorphism. Then show that $\rho_{V \otimes W} = \rho_{V \boxtimes W} \circ \Delta$. In other words, $V \otimes W$ is obtained from $V \boxtimes W$ via the map Δ and the construction of Exercise 2.17.7.(ii).

Proposition 2.20.5. Let G and H be two finite groups. Then every irreducible representation of $G \times H$ is of the form $V \boxtimes W$ for (V, ρ_V) and (W, ρ_W) irreducible representations of G and H, respectively. For V, V', W, W' irreducible, we have $V' \boxtimes W' \cong V \boxtimes W$ if and only if $V' \cong V$ and $W' \cong W$.

In fact the statement is true more generally if G and H are not finite, under the assumption only that V and W are finite-dimensional (see Remark 2.20.13 below).

Thus we can list the irreducible representations of $G \times H$ from those of G and H by taking external tensor products. This gives a generalisation of Exercise 2.12.7.

Proof of Proposition 2.20.5. (Maybe omit from the lectures and make non-examinable.) Suppose that (V, ρ_V) and (W, ρ_W) are irreducible representations of G and H, respectively. They are finite-dimensional by Proposition 2.8.6, and hence so is $V \boxtimes W$. Now, we compute $\mathrm{End}_{G \times H}(V \boxtimes W)$. Note that

$$\mathrm{End}_{G \times H}(V \boxtimes W) = \mathrm{End}(V \boxtimes W)^{G \times H} = (\mathrm{End}(V \boxtimes W)^G)^H = \mathrm{End}_G(V \boxtimes W)^H, \tag{2.20.6}$$

so we first compute $\mathrm{End}_G(V \boxtimes W)$. There is a map

$$\mathrm{End}(W) \to \mathrm{End}_G(V \boxtimes W), \quad T \mapsto (I \otimes T), \quad (I \otimes T)(v \otimes w) = v \otimes T(w). \qquad (2.20.7)$$

We claim it is an isomorphism of H-representations. Given the claim we have:

$$\mathrm{End}_{G\times H}(V \boxtimes W) = \mathrm{End}_G(V \boxtimes W)^H = \mathrm{End}(W)^H = \mathrm{End}_H(W) = \mathbf{C}, \qquad (2.20.8)$$

and by Proposition 2.15.12, this implies that $V \boxtimes W$ is indeed irreducible.

Let us now prove the claim that (2.20.7) is an isomorphism of H-representations. It is clear that the map is H-linear and injective, so we just have to show it is surjective. The easiest (although not the best) way to do this is by dimension counting (see the remark below for a better proof). Note that, as G-representations, $V \boxtimes W \cong V^{\dim W}$, since if we take any basis $w_1, \ldots, w_m$ of W, we get $V \otimes W = \bigoplus_{i=1}^m V \otimes w_i$. Then $\dim \mathrm{End}_G(V \boxtimes W) = \dim \mathrm{End}_G(V^{\dim W}) = (\dim W)^2$ by Proposition 2.15.12 again. As this equals $\dim \mathrm{End}(W)$ and the map (2.20.7) is injective and linear, the map is surjective.

Now for the uniqueness, suppose that $(V' \boxtimes W') \cong (V \boxtimes W)$. By the above, as G-representations, we get $(V')^{\dim W'} \cong V^{\dim W}$, and by Corollary 2.15.10, this implies $V' \cong V$. Similarly $W' \cong W$. (For a better proof that doesn't use bases of W and W', see the last paragraph of the following remark.)

Finally, we need to show that all irreducible representations are of the form $V \boxtimes W$ for V and W irreducible. We can deduce this by counting (for a better proof that doesn't count and therefore doesn't require G to be finite, see Remark 2.20.13). Namely, if $V_1, \ldots, V_m$ and $W_1, \ldots, W_n$ are the irreducible representations of G and H up to isomorphism (i.e., every irreducible representation is isomorphic to exactly one of these), by Corollary 2.16.4, $|G| = \sum_{i=1}^m (\dim V_i)^2$ and $|H| = \sum_{i=1}^n (\dim W_i)^2$. Also, $\dim(V_i \otimes W_j) = (\dim V_i)(\dim W_j)$ by Proposition 2.19.11.(i). Thus we get

$$|G \times H| = |G||H| = \sum_{i=1}^m (\dim V_i)^2 \sum_{j=1}^n (\dim W_j)^2 = \sum_{i,j} \dim(V_i \otimes W_j)^2. \qquad (2.20.9)$$

This implies that the $V_i \boxtimes W_j$ must be all the irreducible representations of $G \times H$ up to isomorphism. $\square$

Remark 2.20.10 (Non-examinable)**.** Here is a better proof of the claim that $\mathrm{End}(W) \to \mathrm{End}_G(V \boxtimes W)$ is surjective, that doesn't use bases or dimensions. Let $T \in \mathrm{End}_G(V \boxtimes W)$. For every $w \in W$ and $f \in W^*$, we can consider the composition

$$V \xrightarrow{I\otimes w} V \otimes W \xrightarrow{T} V \otimes W \xrightarrow{I\otimes f} V, \qquad (2.20.11)$$

where the first map has the form $(I \otimes w)(v) = v \otimes w$ and the last map has the form $(I \otimes f)(v \otimes w) = f(w)v$. This map is a G-homomorphism and hence is a multiple of the identity for all f. So we see that $T(v \otimes w) = v \otimes w'$ for some $w' \in W$. Let $v' \in V$ be another vector. We claim also that $T(v' \otimes w) = v' \otimes w'$. Indeed, since V is irreducible, v' is in the span of $\rho(g)(v)$, so we can write $v' = \sum_{g \in G} a_g \rho(g)(v)$ for some $a_g \in \mathbf{C}$. Then

$$T \circ \sum_{g \in G} (a_g \rho(g) \otimes I) = \sum_{g \in G} (a_g \rho(g) \otimes I) \circ T, \tag{2.20.12}$$

and applying (2.20.12) to $v \otimes w$, we get $T(v' \otimes w) = v' \otimes w'$, as desired.

This also gives a better proof that $V' \boxtimes W' \cong V \boxtimes W$ implies $V' \cong V$ and $W' \cong W$. Indeed, the above can be generalised to show that $\mathrm{Hom}_G(V' \boxtimes W', V \boxtimes W) = \mathrm{Hom}_G(V', V) \otimes \mathrm{Hom}(W', W)$, which by Schur's Lemma is nonzero if and only if $V' \cong V$ as G-representations. Thus if $V' \boxtimes W' \cong V \boxtimes W$ we get $V' \cong V$. Similarly, $W' \cong W$.

Remark 2.20.13 (Non-examinable)**.** Actually, Proposition 2.20.5 is valid more generally: we don't need G and H to be finite, and only need to assume that V and W are finite-dimensional. Together with Remark 2.20.10, the proof of the proposition shows that $\mathrm{End}_{G \times H}(V \boxtimes W) = \mathbf{C}$ when V and W are finite-dimensional and irreducible. More generally (using the last paragraph of Remark 2.20.10), it shows that $\mathrm{Hom}_{G \times H}(V' \boxtimes W', V \boxtimes W)$ is nonzero if and only if $V' \cong V$ and $W' \cong W$, in which case it is one-dimensional. So we have the final statement, and only have to prove that $V \boxtimes W$ is irreducible when V and W are. We just can't use Proposition 2.15.12 to finish the proof because that required $G \times H$ to be finite. So let's find another proof of irreducibility.

Given any irreducible representation U of $G \times H$, we claim that there is a surjective homomorphism of representations $(V \boxtimes W) \to U$ for some irreducible representation V of G and some representation W of H. Indeed, let V be any irreducible representation isomorphic to a G-subrepresentation of U. Then we have a canonical map, $\varphi : V \boxtimes \mathrm{Hom}_G(V, U) \to U, (v \boxtimes \varphi) \mapsto \varphi(v)$. Let $W' := \mathrm{Hom}_G(V, U)$. It is a representation of H via the action of H on U (with trivial action on V). So we get a homomorphism of $G \times H$ representations, $V \boxtimes W' \to U$. By choice of V, this is nonzero, and since U is irreducible, it must be surjective. We now show how to replace W' by an irreducible representation. Let $W'' \subseteq W'$ be the largest subspace such that $V \boxtimes W''$ is in the kernel of φ (we can take the sum of all subspaces whose tensor product with V are in the kernel). Then we get a surjection $V \boxtimes (W'/W'') \to U$. Now let $W \subseteq W'/W''$ be any irreducible subrepresentation (which exists by finite-dimensionality of W'/W''). Then $V \boxtimes W \to U$ is a nonzero homomorphism, which is therefore surjective by irreducibility of U.

Now let us return to the situation where V and W are irreducible. Suppose that $U \subseteq V \boxtimes W$ is an irreducible $G \times H$-subrepresentation. Let $V' \boxtimes W' \to U$ be a surjective homomorphism with V' and W' irreducible. Taking the composition we obtain a nonzero homomorphism of $G \times H$-representations, $V' \boxtimes W' \to V \boxtimes W$. By the last paragraph of Remark 2.20.10, this implies $V' \cong V$ and $W' \cong W$, so we can set $V' = V$ and $W' = W$ to begin with. Now the composition $T : V \boxtimes W \to V \boxtimes W$ is a nonzero homomorphism of representations. Since $\mathrm{End}_G(V \boxtimes W) = \mathbf{C}$, we conclude that T is a multiple of the identity. But then T is surjective. Since the image of T is U, this implies $U = V \boxtimes W$, so that $V \boxtimes W$ is irreducible as desired.

To prove that all of the irreducible finite-dimensional representations of $G \times H$ are of the form $V \boxtimes W$ for V, W irreducible, note that in the preceding paragraph we showed that for every finite-dimensional irreducible representation U, there is a surjective homomorphism of representations $V \boxtimes W \to U$ for some irreducible V and W. But since $V \boxtimes W$ is itself irreducible, we conclude that $U \cong V \boxtimes W$.

Remark 2.20.14 (Non-examinable)**.** Parallel to Remark 2.12.5, we can get a counterexample to the proposition if we allow V and W to be infinite-dimensional. As in Remark 2.12.5, let $V = \mathbf{C}(x)$ and $G = \mathbf{C}(x)^\times$, and let also $W = V$. Then $V \otimes W = \mathbf{C}(x) \otimes \mathbf{C}(x)$ is not irreducible, since there is a nonzero non-injective homomorphism of representations $\psi : \mathbf{C}(x) \otimes \mathbf{C}(x) \to \mathbf{C}(x)$ given by multiplying fractions. This is clearly surjective and hence nonzero, but $f \otimes 1 - 1 \otimes f$ is in the kernel for all $f \in \mathbf{C}(x)$. This element $f \otimes 1 - 1 \otimes f$ is nonzero if f is nonconstant, since we can also write an injective homomorphism $\mathbf{C} \otimes \mathbf{C}(x) \to \mathbf{C}(x, y)$, the field of fractions of polynomials in two variables x and y, by $f(x) \otimes g(x) \mapsto f(x)g(y)$, and then $f \otimes 1 - 1 \otimes f$ maps to $f(x) - f(y)$ which is nonzero if f is nonconstant. In fact this injective homomorphism realises $\mathbf{C}(x) \otimes \mathbf{C}(x)$ as the subring of $\mathbf{C}(x, y)$ consisting of elements of the form $\sum_{i=1}^n f_i(x)g_i(y)$ for rational fractions f_i, g_i of one variable. (For an alternative proof that $f \otimes 1 - 1 \otimes f$ is nonzero for some f, take the map $\mathbf{C}(x) \otimes \mathbf{C}(x) \to \mathbf{C}(x)$, $f \otimes g \mapsto f(x)g(-x)$, in which case again $f \otimes 1 - 1 \otimes f \mapsto f(x) - f(-x)$ which is nonzero whenever f is not even; for instance when $f = x$ itself.)

2.21 Summary of section

Here we recap the highlights of this section, in order.

We began by presenting many of the important examples of representations (Section 2.2), for cyclic, symmetric, and dihedral groups.

We defined representations of a group G in two ways: (a) *abstractly*, as a pair (V, ρ_V) of a vector space and a homomorphism $\rho_V : G \to \mathrm{GL}(V)$ (Definition 2.4.1); and (b) via *matrices*, as a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ for some dimension n (Definition 2.4.7). We showed how to go between these two definitions (Definition 2.4.8 and Example 2.5.1) and proved that they give the same isomorphism classes of representations (Proposition 2.4.19). Both definitions give the same notion of dimension, (ir)reducibility, (in)decomposability, etc.

We defined the key notion of representations from group actions (Definition 2.6.1), and the important special case of the (left) regular representation (Example 2.7.1).

We proceeded to define subrepresentations (Section 2.8) and direct sums (Section 2.9) of representations and defined a representation to be reducible or decomposable if it has a nonzero proper subrepresentation or a nontrivial decomposition as a sum of subrepresentations, respectively. Otherwise the representation is called irreducible or indecomposable, respectively (Exception: the zero representation is neither reducible nor irreducible, and neither decomposable nor indecomposable: this is like the number one being neither prime nore composite). We defined a representation to be *semisimple* if it is a direct sum of irreducible subrepresentations (Definition 2.9.6). We observed that an indecomposable representation is semisimple if and only if it is actually irreducible (Example 2.9.7).

We then proved the first and most important theorem on (complex) representations of finite groups: Maschke's Theorem (Theorem 2.10.2), that finite-dimensional such representations are always semisimple. In fact the proof of the theorem constructs, for every subrepresentation of a finite-dimensional representation of a finite group, a complementary subrepresentation.

Next we proved the second most important result in the subject (but which is also valid for infinite groups): Schur's Lemma (Lemma 2.11.1). Part (i) shows that nonzero homomorphisms of irreducible representations are isomorphisms (which requires no assumptions on the field or the dimension and is relatively easy and quick to prove). Part (ii), assumes that the representations are finite-dimensional and, using that the field is complex, gives us that there is at most one such isomorphism up to scaling: $\mathrm{End}_G(V) = \mathbf{C} \cdot I$ for V a finite-dimensional irreducible representation. This was also not difficult to prove, using mainly the fact that square matrices over the complex numbers always have an eigenvalue (or better, that an endomorphism of a finite-dimensional complex vector space has an eigenvalue), by taking any root of the characteristic polynomial.

As an application of Schur's Lemma, we proved that every finite-dimensional representation of an abelian group is one-dimensional (Proposition 2.12.1). As a special case we classified these for cyclic (Corollary 2.12.6) and all finite abelian groups (Exercise 2.12.7), seeing that the number of irreducible representations equals the size of the group.

We next studied one-dimensional representations, showing that one-dimensional representations of a group G are in explicit bijection with those of the *abelianisation*, $G_{\mathrm{ab}} := G/[G,G]$ (Definition 2.13.4 and Corollary 2.13.9). We classified one-dimensional representations for symmetric and dihedral groups (Examples 2.13.1 and 2.13.2). Namely, these are the trivial and sign representations of the symmetric groups, and two or four examples in the dihedral cases, depending on whether n is odd or even, respectively (the generating rotation and reflection can get sent to ± 1, except in the odd case where the rotation must get sent to 1). We observed that perfect (which includes simple) groups can have no nontrivial one-dimensional representations (Examples 2.13.14 and 2.13.15).

We then proceeded to define the representation $\mathrm{Hom}(V,W)$ where V and W are themselves representations of a group G (Section 2.14). We defined, for every representation V, the G-invariant subrepresentation V^G (Definition 2.14.7), satisfying the property $\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$. We presented an averaging formula for a G-linear projection $V \to V^G$, and explained that the proof of Maschke's theorem crucially involves the projection $\mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$, in order to turn a linear projection $V \to W$ into a G-linear projection, whose kernel produces the desired complementary subrepresentation.

We then considered decompositions. Assume that the group G is finite. By Maschke's theorem, if V is a finite-dimensional representation, there is an isomorphism $V \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}$ with the V_i nonisomorphic irreducible representations. We proved that this decomposition is unique up to replacing each V_i by an isomorphic representation (Corollary 2.15.10). Namely, $r_i = \dim \mathrm{Hom}_G(V_i, V)$ for all i (Proposition 2.15.2). Moreover we proved the formula $\dim \mathrm{End}_G(V) = r_1^2 + \cdots + r_m^2$. This includes a converse of Schur's lemma: $\mathrm{End}_G(V)$ is one-dimensional if and only if V is irreducible.

Applying this to the regular representation, we proved the fundamental formula $|G| = \sum_{i=1}^m (\dim V_i)^2$, where $V_1, \ldots, V_m$ are all of the irreducible representations of G up to isomorphism (with $V_i \ncong V_j$ for $i \neq j$), and in particular that m is finite (Corollary 2.16.4). We can use to classify irreducible representations for S_3, S_4, and the dihedral groups (see Examples 2.17.5, 2.17.6, and 2.17.10). Along the way we proved that the reflection representation of S_n is always irreducible (Proposition 2.17.3) and gave some general constructions of representations (Exercise 2.17.7).

Next we proved that the number of irreducible representations of a finite group equals its number of conjugacy classes (Theorem 2.18.2). The key idea here was to enhance the fundamental formula $|G| = \sum_{i=1}^m (\dim V_i)^2$ to an explicit isomorphism, $\mathbf{C}[G] \cong \bigoplus_{i=1}^m \mathrm{End}(V_i)$. This is not G-linear at first, but it is if we equip $\mathbf{C}[G]$ with the *adjoint representation* (Definition 2.18.10). Then taking G-invariants and taking dimension yields the desired formula.

We then defined duals and tensor products of representations (Definition 2.14.3 and Example 2.19.1; Definitions 2.19.8 and 2.19.26). The tensor product generalizes the construction of Exercise 2.17.7.(i) in the case one of the representations is one-dimensional. We showed that $V^* \otimes W \cong \mathrm{Hom}(V, W)$ as representations when V is finite-dimensional.

Finally, generalising this, we briefly introduced the *external tensor product* of representations (V, ρ_V) and (W, ρ_W) of two (distinct) groups G and H: the representation $(V \otimes W, \rho_{V \boxtimes W})$, also denoted $V \boxtimes W$, of the product $G \times H$ (Definition 2.20.1). This is useful because, if we know the irreducible representations of G and of H, then the irreducible representations of $G \times H$ are precisely the external tensor products of irreducible representations. Thus the number of irreducible representations of $G \times H$ equals the product of the number for G and H (which also follows from the formula in terms of conjugacy classes). This generalizes the aforementioned classification (Exercise 2.12.7) of irreducible representations of finite abelian groups (which are all one-dimensional).

3 Character theory

3.1 Traces and definitions

Character theory is a way to get rid of the vector spaces, and replace them by *numerical information depending only on isomorphism class.* The main idea is as follows: Suppose that two square matrices A and B are conjugate, $B = PAP^{-1}$. Then we have the fundamental identity:

$$\mathrm{tr}(B) = \mathrm{tr}(PAP^{-1}) = \mathrm{tr}(P^{-1}PA) = \mathrm{tr}(A). \tag{3.1.1}$$

Here, recall from linear algebra that $\mathrm{tr}(A) := \sum_i A_{ii}$. Also, you should have seen that $\mathrm{tr}(CD) = \mathrm{tr}(DC)$ for all matrices C and D such that CD (equivalently DC) is square. Indeed $\mathrm{tr}(CD) = \sum_{i,j} C_{ij} D_{ji} = \mathrm{tr}(DC)$. This verifies (3.1.1). We can then define:

Definition 3.1.2. The character of a matrix representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is the function $\chi_\rho : G \to \mathbf{C}$, $\chi_\rho(g) = \mathrm{tr}\, \rho(g)$.

We then see that two equivalent representations give the same character (we will record this observation in Proposition 3.1.9 below).

Here is another useful formula for the trace. Since trace is invariant under conjugation, and there is some choice B of conjugate matrix which is upper-triangular with the eigenvalues (with multiplicity) on the diagonal, we also get that, if A has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicity,

$$\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n. \tag{3.1.3}$$

To extend this to, abstract representations let's recall how to define the trace of linear endomorphisms:

Definition 3.1.4. If $T : V \to V$ is a linear endomorphism of a finite-dimensional vector space V, then for any basis $\mathcal{B}$ of V, we set $\mathrm{tr}(T) := \mathrm{tr}[T]_{\mathcal{B}}$.

For this to make sense we have to verify that it is actually independent of the choice of $\mathcal{B}$: for any other basis $\mathcal{B}'$, letting $P = [I]_{\mathcal{B}',\mathcal{B}}$ be the change of basis matrix (see (2.4.11) and (2.4.12)), we have

$$\mathrm{tr}[T]_{\mathcal{B}'} = \mathrm{tr}([I]_{\mathcal{B},\mathcal{B}'}[T]_{\mathcal{B},\mathcal{B}}[I]_{\mathcal{B}',\mathcal{B}}) = \mathrm{tr}(P^{-1}[T]_{\mathcal{B}}P) = \mathrm{tr}[T]_{\mathcal{B}}. \tag{3.1.5}$$

As before we can verify that, if $T : V \to W$ and $S : W \to V$ are linear maps, then

$$\mathrm{tr}(S \circ T) = \mathrm{tr}(T \circ S), \tag{3.1.6}$$

by taking matrices and reducing to the aforementioned identity $\mathrm{tr}(CD) = \mathrm{tr}(DC)$. Similarly we deduce that, if T has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicity, then $\mathrm{tr}(T) = \lambda_1 + \cdots + \lambda_n$. The latter definition is, in a sense, better, since it doesn't really require using a basis at all.

Remark 3.1.7 (Non-examinable)**.** It is also possible to define trace of endomorphisms in a more direct way without using bases. For this recall that $\mathrm{End}(V) = \mathrm{Hom}(V, V) \cong V^* \otimes V$, since V is finite-dimensional (by Proposition 2.19.11.(ii)). It suffices therefore to define trace on $V^* \otimes V$. Here we can define it by $\mathrm{tr}(f \otimes v) = f(v)$ for $f \in V^*$ and $v \in V$, extended linearly to all of $V^* \otimes V$.

This motivates the following:

Definition 3.1.8. Given a finite-dimensional representation (V, ρ_V) of a group G, the *character* is the function $\chi_{(V,\rho_V)} : G \to \mathbf{C}$, given by $\chi_{(V,\rho_V)}(g) = \mathrm{tr}\,\rho_V(g)$ for all g. To ease notation this is also called χ_V or χ_{ρ_V}.

Note that the definition of character *only makes sense for finite-dimensional representations*, so whenever we write χ_V we can assume V is finite-dimensional.

Proposition 3.1.9. Isomorphic abstract representations, or equivalent matrix representations, have the same character.

Proof. For matrices, if $\rho, \rho' : G \to \mathrm{GL}_n(\mathbf{C})$ are equivalent, then $\rho(g)$ and $\rho'(g)$ are conjugate for all g (in fact, by the same invertible matrix P, but we won't need this). So the result follows from (3.1.1). For abstract representations, the result follows from the correspondence between matrix and abstract representations (alternatively, if $T : (V, \rho_V) \to (W, \rho_W)$ is an isomorphism, we can directly verify by (3.1.6) that $\operatorname{tr} \rho_W(g) = \operatorname{tr}(T \circ \rho_V(g) \circ T^{-1}) = \operatorname{tr}(T^{-1} \circ T \circ \rho_V(g)) = \operatorname{tr} \rho_V(g)$.) $\square$

The first main result we will prove is that *the converse is also true for finite groups:*

Theorem 3.1.10. If G is finite, and (V, ρ_V) and (W, ρ_W) two finite-dimensional representations, then $(V, \rho_V) \cong (W, \rho_W)$ if and only if $\chi_V = \chi_W$.

Shortly we will reduce this statement, by Maschke's theorem, to the case of irreducible representations, and then we will prove it using Proposition 2.15.12 together with an explicit formula for $\dim \operatorname{End}_G(V)$ in terms of the character χ_V. But first, we will explain the fundamental beautiful properties of the character, some of which we will need.

We conclude with some examples.

Example 3.1.11. If $(\mathbf{C}, \rho_{\mathrm{triv}})$ is the trivial representation, then $\chi_{\mathbf{C}}(g) = 1$ for all g. That is, $\chi_{\mathbf{C}} = 1$.

Example 3.1.12. Suppose that (V, ρ_V) is a one-dimensional representation. Then $\rho_V(g) = \chi_V(g) I$ for every g. This shows that, given V itself, the data of ρ_V and χ_V are equivalent. If we take a matrix one-dimensional representation, $\rho : G \to \mathbf{C}^\times$, then we see that literally $\rho(g) = \chi_\rho(g)$. In particular χ_ρ is multiplicative: $\chi_\rho(gh) = \chi_\rho(g)\chi_\rho(h)$, and similarly χ_V is multiplicative for every one-dimensional representation (V, ρ_V).

Example 3.1.13. For any (V, ρ_V), note that $\rho_V(1) = I$ so $\chi_V(1) = \dim V$. Hence if V is at least two-dimensional, then χ_V is not multiplicative: $1^2 = 1$ whereas $(\dim V)^2 = \dim V$ only if $\dim V \leq 1$.

Example 3.1.14. We return to Example 2.12.8: $G = \{\pm 1\} \times \{\pm 1\} \cong C_2 \times C_2$, and ρ is given by $\rho(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then $\chi(a, b) = a + b$, so $\chi(1, 1) = 2$, $\chi(-1, -1) = -2$, and $\chi(1, -1) = 0 = \chi(-1, 1)$. Here it is clear that χ is not multiplicative.

Example 3.1.15. Let G act on a finite set X and take the associated representation $(\mathbf{C}[X], \rho)$. Using the basis X of $\mathbf{C}[X]$ (put in any ordering), $[\rho(g)]_X$ is a permutation matrix. The trace of a permutation matrix P_σ is the number of diagonal entries, i.e., the number of basis vectors e_i such that $Pe_i = e_i$, or equivalently the number of i such that $\sigma(i) = i$. Thus here we have $\chi_\rho(g) = \operatorname{tr}(\rho(g)) = |\{x \in X \mid g \cdot x = x\}|$. This can also be written as $|X^g|$, by the following definition.

Definition 3.1.16. For $g : X \to X$, the fixed point set X^g is defined as $X^g := \{x \in X \mid g(x) = x\}$. If $G \times X \to X$ is an action then we use the same notation for $g \in G$, via $g(x) := g \cdot x$.

Remark 3.1.17 (Non-examinable, philosophical)**.** The theorem shows that passing from representations to characters is essentially the same as identifying isomorphic representations, at least in the case of finite groups. Of course, as I have said several times, identifying isomorphic objects is "wrong": isomorphic objects are still different and "can have different personalities." However, as we will see, character theory is very powerful and produces numbers with a great deal of structure. It is very useful not merely for classifying representations, but also for applications to combinatorics and number theory. So while we shouldn't just throw away the representations and replace them by characters, it is very useful to understand what information is left, and how it is organized, after passing to characters.

Remark 3.1.18 (Non-examinable)**.** The trace is not the only number you can get from a matrix invariant under conjugation: one can also take the determinant. More generally, the trace and the determinant appear as coefficients of the *characteristic polynomials* of A and B, which are equal when A and B are conjugate. Alternatively, the eigenvalues of A and B are equal (with multiplicity); this is the same information though, since the eigenvalues are the roots of the characteristic polynomial.

Surprisingly, in the case of groups, these other numbers don't provide any additional information. That is, for a group G, if we know the traces $\chi_V(g)$ of *all* elements $g \in G$, we actually can recover the characteristic polynomials of every element as well (this is obviously false if we only know the trace of a single element). In the case of finite groups it is a consequence of Theorem 3.1.10, but here is a direct proof for arbitrary G. Suppose that (V, ρ_V) is m-dimensional, and that the eigenvalues of $\rho_V(g)$ are $\lambda_1, \ldots, \lambda_m$ with multiplicity. Then the eigenvalues of $\rho_V(g^k)$ are $\lambda_1^k, \ldots, \lambda_m^k$ with multiplicity. So $\chi_V(g^k) = \lambda_1^k + \cdots + \lambda_m^k$ for all k. But knowing these sums for all $k \geq 0$ actually determines the coefficients of the characteristic polynomial $(x - \lambda_1) \cdots (x - \lambda_m)$, which are known to be polynomials in these power sums $\lambda_1^k + \cdots + \lambda_m^k$ (we only need $1 \leq k \leq m$). So the character of ρ_V determines the characteristic polynomial of $\rho_V(g)$.

3.2 Characters are class functions and other basic properties

Here we observe that characters are not arbitrary functions, but have a highly constrained structure:

Definition 3.2.1. A function $f : G \to \mathbf{C}$ is a *class function* if $f(g) = f(h^{-1}gh)$ for all $h \in G$.

In other words, class functions are functions that are constant on conjugacy classes.

Definition 3.2.2. Let $\mathrm{Fun}(G, \mathbf{C})$ denote the vector space of all complex-valued functions on G, and let $\mathrm{Fun}_{\mathrm{cl}}(G, \mathbf{C}) \subseteq \mathrm{Fun}(G, \mathbf{C})$ denote the vector space of all class functions.

Remark 3.2.3. Note that this is also denoted $\mathbf{C}^G$ and $\mathbf{C}^G_{\mathrm{cl}}$, but I think that these notations could be confusing since it *clashes* with the superscript of G to denote G-invariants.

Exercise 3.2.4. Show that the set of class functions $f : G \to \mathbf{C}$ form a vector space of dimension equal to the number of conjugacy classes of G. In particular, every function is a class function if and only if G is abelian (this is also clear from the definition). (Hint: a basis is given in (3.5.7) below.)

Exercise 3.2.5. (i) Show that $\mathrm{Fun}(G, \mathbf{C})$ is a representation of G under the adjoint action $(g \cdot f)(h) := f(g^{-1}hg)$. (ii) Show that using this action, $\mathrm{Fun}_{\mathrm{cl}}(G, \mathbf{C}) = \mathrm{Fun}(G, \mathbf{C})^G$. (iii) Show that there also two other actions on $\mathrm{Fun}(G, \mathbf{C})$, analogous to the right and left regular representations, for which this isomorphism does *not* hold in general. (iv) In fact, show that these are all related: these three representations on $\mathrm{Fun}(G, \mathbf{C})$ are the duals of the adjoint, left, and right regular representations.

Proposition 3.2.6. Every character χ_V is a class function.

Proof. $\chi_V(h^{-1}gh) = \mathrm{tr}(\rho_V(h^{-1}gh)) = \mathrm{tr}(\rho_V(h)^{-1}\rho_V(g)\rho_V(h)) = \mathrm{tr}(\rho_V(g)) = \chi_V(g)$, using (3.1.6). $\square$

To proceed, we need the following lemma:

Lemma 3.2.7. Suppose that $T \in \mathrm{GL}(V)$ has finite order: $T^m = I$ for some $m \geq 1$. Then T is diagonalisable with eigenvalues which are m-th roots of unity, i.e., in some basis $\mathcal{B}$ of V, we have $[T]_{\mathcal{B}}$ is diagonal whose entries are in $\{\zeta \in \mathbf{C} \mid \zeta^m = 1\}$.

Proof. This is linear algebra. Namely, T satisfies the polynomial $x^m - 1 = 0$, which has distinct roots (the m-th roots of unity). The minimal polynomial $p(x)$ of T is a factor of $x^m - 1$, so also has roots of multiplicity one, which are m-th roots of unity. Therefore T is diagonalisable to a diagonal matrix whose entries are m-th roots of unity. (In other words, the Jordan normal form of T has diagonal entries which are m-th roots of unity, and nothing above the diagonal since $p(T) = 0$ and p has distinct roots.) $\square$

Remark 3.2.8 (Sketch in lecture)**.** The lemma can also be proved using representation theory. As in Exercise 2.4.22, we can define a representation $\rho : C_m = \{1, g, \ldots, g^{m-1}\} \to \mathrm{GL}(V)$ given by $\rho(g^k) = T^k$. Then Maschke's Theorem together with Corollary 2.12.6 show that $V = V_1 \oplus \cdots \oplus V_n$ for some one-dimensional representations $V_1, \ldots, V_n$. Picking a basis $\mathcal{B} := (v_1, \ldots, v_n)$ such that $v_i \in V_i$ for all i, we get that $[\rho(g)]_{\mathcal{B}}$ is diagonal and the entries are obviously m-th roots of unity.

Proposition 3.2.9. Let (V, ρ_V) be a finite-dimensional representation of G and $g \in G$ an element of finite order.

(i) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

(ii) $|\chi_V(g)| \leq \dim V$, with equality holding if and only if g is a scalar matrix (=a scalar multiple of the identity).

(iii) $\chi_V(1) = \dim V$.

Proof. (i) Let the eigenvalues of $\rho_V(g)$ be, with multiplicity, $\zeta_1, \ldots, \zeta_n$. Then, by (3.1.3), $\chi_V(g) = \zeta_1 + \cdots + \zeta_n$. Also, the eigenvalues of $\rho_V(g^{-1}) = \rho_V(g)^{-1}$ are $\zeta_1^{-1}, \ldots, \zeta_n^{-1}$. Since each ζ_j is a root of unity, it follows that $\zeta_j^{-1} = \overline{\zeta_j}$: after all, $\zeta_j \overline{\zeta_j} = |\zeta_j| = 1$. Thus

$$\chi_V(g^{-1}) = \zeta_1^{-1} + \cdots + \zeta_n^{-1} = \overline{\zeta_1} + \cdots + \overline{\zeta_n} = \overline{\chi_V(g)}. \qquad (3.2.10)$$

(ii) Applying the triangle inequality,

$$|\chi_V(g)| = |\zeta_1 + \cdots + \zeta_n| \leq |\zeta_1| + \cdots + |\zeta_n| = n = \dim V, \qquad (3.2.11)$$

with equality holding if and only if all of the ζ_j are positive multiples of each other. But to have this, since $|\zeta_j| = 1$, all the ζ_j have to be equal. (iii) This was already observed in Example 3.1.13. $\square$

Exercise 3.2.12. (i) Find the functions $f : C_2 = \{1, g\} \to \mathbf{C}$ that satisfy the conditions of the above lemmas. (ii) Give an example to show that they cannot all be characters. (iii) Compute all functions which actually are characters. (Hint: $T = \rho(g)$ will have to be diagonalisable with entries ± 1, since $T^2 = I$ hence $(T - I)(T + I) = 0$.)

3.3 Characters of direct sums and tensor products

Next we explain how characters turn direct sums and tensor products into ordinary sums and products, and we will also discuss duals and hom spaces.

As an introduction, recall the formulas

$$\dim(V \oplus W) = \dim V + \dim W, \quad \dim(V \otimes W) = \dim V \dim W. \qquad (3.3.1)$$

Since $\chi_V(1) = \dim V$, we obtain that, for $g = 1$, with V and W finite-dimensional,

$$\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g), \quad \chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g). \qquad (3.3.2)$$

Below we will explain that in fact *this holds for all* g!

Remark 3.3.3 (Non-examinable)**.** This feature says that direct sums and tensor products of representations "categorify" ordinary sums and products of characters, in that they replace numbers by vector spaces in such a way that they reverse the operation of taking character. This was already true for vector spaces under $\oplus$ and $\otimes$ by (3.3.1).

As a matter of notation, if we have two functions f_1, f_2, then $f_1 f_2$ is the product function $f_1 f_2(g) := f_1(g) f_2(g)$. Similarly $\overline{f}$ is the function $\overline{f}(g) = \overline{f(g)}$.

Proposition 3.3.4. Let (V, ρ_V) and (W, ρ_W) be finite-dimensional representations of a group G. Then:

(i) $\chi_{V \oplus W} = \chi_V + \chi_W$;

(ii) $\chi_{V \otimes W} = \chi_V \chi_W$.

Next suppose that G is finite. Then also:

(iii) $\chi_{V^*} = \overline{\chi_V}$;

(iv) $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$.

Here $\overline{f}(g) := \overline{f(g)}$ is the complex conjugation operation.

Proof. Throughout let $\mathcal{B} = \{v_1, \ldots, v_m\}$ and $\mathcal{C} = \{w_1, \ldots, w_n\}$ be bases of V and W, respectively. Fix $g \in G$ and let $M := [\rho_V(g)]_{\mathcal{B}}$ and $N := [\rho_W(g)]_{\mathcal{C}}$. Thus, $\chi_V(g) = \operatorname{tr} M$ and $\chi_W(g) = \operatorname{tr} N$. Then the computations reduce to finding the matrices for the representations on the LHS and computing their traces.

(i) Let $\mathcal{D} := \mathcal{B} \sqcup \mathcal{C}$ be a basis of $V \oplus W$ (to be precise, for $v \in \mathcal{B}$, then $(v, 0) \in (V \oplus W)$ is the corresponding basis element in $\mathcal{D}$). Then we get the block-diagonal matrix for $\rho_{V\oplus W}(g)$:

$$[\rho_{V\oplus W}(g)]_{\mathcal{D}} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \tag{3.3.5}$$

since $\rho_{V\oplus W}(g)$ sends elements of $\mathcal{B}$ to linear combinations of $\mathcal{B}$ according to $\rho_V(g)$, and similarly for $\mathcal{C}$ and W. Then

$$\chi_{V\oplus W}(g) = \operatorname{tr}[\rho_{V\oplus W}(g)]_{\mathcal{D}} = \operatorname{tr} M + \operatorname{tr} N = \chi_V(g) + \chi_W(g). \tag{3.3.6}$$

(ii) We take the basis $\mathcal{D} = \{v_i \otimes w_j\}$ of $V \otimes W$. By (2.19.33) we have $[\rho_{V\otimes W}(g)]_{\mathcal{D}} = M \otimes N$ where $(M \otimes N)_{ij,k\ell} = M_{ik}N_{j\ell}$ for $i, k \in \{1, \ldots, m\}$ and $k, \ell \in \{1, \ldots, n\}$. Thus

$$\begin{aligned}\chi_{V\otimes W}(g) = \operatorname{tr}(M \otimes N) = \sum_{i,j} (M \otimes N)_{ij,ij} &= \sum_{i,j} M_{ii}N_{jj} \\ &= \sum_i M_{ii} \sum_j N_{jj} = \operatorname{tr}(M)\operatorname{tr}(N) = \chi_V(g)\chi_W(g). \end{aligned}\tag{3.3.7}$$

(iii) Here we take the dual basis $\mathcal{D} = \{f_1, \ldots, f_m\}$ of V^*, given by $f_i(v_j) = \delta_{ij}$. By Remark 2.19.3, we get

$$\chi_{V^*}(g) = \operatorname{tr}[\rho_{V^*}(g)]_{\mathcal{D}} = \operatorname{tr}(M^{-1})^t = \operatorname{tr} M^{-1} = \operatorname{tr}[\rho_V(g^{-1})]_{\mathcal{B}} = \chi_V(g^{-1}), \tag{3.3.8}$$

since $M^{-1} = \rho_V(g^{-1})_{\mathcal{B}}$. The result now follows from Proposition 3.2.9.(i).

(iv) By Proposition 2.19.11.(ii), $\mathrm{Hom}(V, W) \cong V^* \otimes W$, so by Proposition 3.1.9, the result follows from (ii) and (iii). $\square$

Example 3.3.9. Here's an example to show why (iii) and (iv) require G to be finite. Let $G = \mathbf{Z}$ and $a \in \mathbf{C}^\times$. Then we can take the one-dimensional representation $\rho : G \to \mathbf{C}^\times, \rho(m) = a^m$. Then as in Example 3.1.12, $\chi_\rho = \rho$, but $\chi_\rho(-1) = a^{-1} \neq \overline{a}$ in general. [Note that (iii) and (iv) hold in this case if and only if $|a| = 1$, which is weaker than the condition that a is a root of unity.]

Example 3.3.10. Let's take $G = S_n$ and let $(\mathbf{C}^n, \rho_{\mathrm{perm}})$ be the permutation representation and (V, ρ_V) be the reflection representation ($V \subset \mathbf{C}^n$). Let $(\mathbf{C}, \rho_{\mathrm{triv}})$ denote the trivial representation. Then $\mathbf{C}^n = V \oplus \mathbf{C} \cdot (1, 1, \ldots, 1)$, with $\mathbf{C} \cdot (1, 1, \ldots, 1) = \{(a, a, \ldots, a) \mid a \in \mathbf{C}\}$ isomorphic to the trivial representation. Hence $\chi_{\mathbf{C}^n} = \chi_V + \chi_{\mathbf{C}}$. By Examples 3.1.11 and 3.1.15, we get

$$\chi_V(\sigma) = |\{1, \ldots, n\}^\sigma| - 1. \tag{3.3.11}$$

Example 3.3.12. Now let (V, ρ_V) be an arbitrary representation of a group, and $(\mathbf{C}_\theta, \theta)$ a one-dimensional representation, for $\theta : G \to \mathbf{C}^\times$ a homomorphism. Then taking the tensor product, we get $\chi_{\mathbf{C}_\theta \otimes V} = \theta \cdot \chi_V$. This equals χ_V if and only if $\chi_V(g) = 0$ whenever $\theta(g) \neq 1$, i.e., whenever $g \notin \ker \theta$. In the case that G is finite, by Theorem 3.1.10 (still to be proved), this gives an explicit criterion when the construction of Exercise 2.17.7.(i) produces a nonisomorphic representation to the first one.

Example 3.3.13. As a special case of the preceding, by Theorem 3.1.10, for $(\mathbf{C}_-, \rho_{\mathrm{sign}})$ the sign representation of S_n and (V, ρ_V) the reflection representation, we see that $\mathbf{C}_- \otimes V \cong V$ if and only if $|\{1, \ldots, n\}^\sigma| = 1$ whenever σ is an odd permutation. We see that this is true if and only if $n = 3$. (Even for $n = 2$ it is not true, and indeed in this case the reflection and sign representations are isomorphic, and their tensor product is the trivial representation which is not isomorphic to the reflection representation!) Thus for $n \geq 4$, the reflection representation and the reflection tensor sign are two nonisomorphic irreducible representations of dimension $n - 1$.

Example 3.3.14. Let $(\mathbf{C}[G], \rho_{\mathbf{C}[G]})$ be the left regular representation of a finite group G, $\rho_{\mathbf{C}[G]}(g)(\sum_{h \in G} a_h h) = \sum_{h \in G} a_h gh$. This representation is induced by the group action $G \times G \to G$, so $\chi_{\mathbf{C}[G]}(g) = |G^g|$ where $G^g := \{h \in G \mid gh = h\}$. But applying cancellation, $gh = h$ if and only if $g = 1$. Therefore $G^g = \emptyset$ for $g \neq 1$ and $G^1 = G$. We get

$$\chi_{\mathbf{C}[G]}(g) = \begin{cases} |G|, & g = 1, \\ 0, & g \neq 1. \end{cases} \tag{3.3.15}$$

Example 3.3.16. The same applies to the right regular representation, $g \cdot (\sum_{h \in G} a_h h) = \sum_{h \in G} a_h h g^{-1}$. So these two have the same character; so by Theorem 3.1.10 we see that the left and right regular representations are isomorphic (we can also write an explicit isomorphism).

Example 3.3.17. On the other hand, the *adjoint* representation $\rho_{\mathrm{ad}}(g)(h) = ghg^{-1}$ does *not* have the same character: $\chi_{\rho_{\mathrm{ad}}}(g) = |Z_G(g)|$ where $Z_G(g) := \{h \in G \mid gh = hg\}$, which is a subgroup of G called the *centraliser* of g. Note that, since it is a subgroup, $|Z_G(g)| \geq 1$ (i.e., $1 \in Z_G(g)$ for all g), so $\chi_{\rho_{\mathrm{ad}}} \neq \chi_{\mathbf{C}[G]}$ unless $G = \{1\}$. So the adjoint representation is not isomorphic to the left and right regular representations.

3.4 Inner product of functions and characters

Suppose $f_1, f_2 : G \to \mathbf{C}$ are functions and G is a finite group. Then we consider the (complex) inner product

$$\langle f_1, f_2 \rangle := |G|^{-1} \sum_{g \in G} f_1(g)\overline{f_2(g)}. \tag{3.4.1}$$

Remark 3.4.2 (Non-examinable)**.** This makes sense if G is replaced by any finite set. There is also an analogue for functions on more general spaces, where replace the finite sum by an infinite sum or even by an integral. For example, there is the complex vector space $L^2(\mathbf{R})$ of square-integrable functions on $\mathbf{R}$ (functions $f : \mathbf{R} \to \mathbf{C}$ such that $\int_{\mathbf{R}} |f(x)|^2 dx < \infty$), with the inner product $\langle f, g \rangle = \int_{\mathbf{R}} f(x)\overline{g(x)} dx$.

Now it is easy to verify that $\langle -, - \rangle$ forms an inner product:

Definition 3.4.3. Let V be a complex vector space. A (Hermitian) inner product is a pairing $\langle -, - \rangle : V \times V \to \mathbf{C}$ satisfying, For all $\lambda \in \mathbf{C}$ and vectors $u, v, w \in V$:

(i) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and similarly $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$;

(ii) $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle = \langle u, \overline{\lambda} v \rangle$;

(iii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$;

(iv) $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$.

Properties (i) and (ii) are called *sequilinearity*. Property (iii) is called *conjugate-symmetry*. And property (iv) is called *positive-definiteness*.

Lemma 3.4.4. The pairing $\langle -, - \rangle$ on $\mathrm{Fun}(G, \mathbf{C})$ is an inner product.

Proof. Note that (i), (ii), and (iii) are immediate from the definition (3.4.1). For (iv), observe that $\langle f, f \rangle = |G|^{-1} \sum_{g \in G} |f(g)|^2$, which implies the statement. $\square$

This is essentially just the standard inner product $\langle v, w \rangle = v \cdot \overline{w}$ on the vector space $\mathbf{C}^G$, with components indexed by G instead of by $\{1, \ldots, n\}$.

Recall from linear algebra the following:

Definition 3.4.5. Let V be a vector space with an inner product $\langle -, - \rangle$. Then $v, w \in V$ are called *orthogonal*, written $v \perp w$, if $\langle v, w \rangle = 0$. A set $v_1, \ldots, v_n$ is *orthonormal* if $\langle v_i, v_j \rangle = \delta_{ij}$, i.e., $v_i \perp v_j$ for $i \neq j$ and $\langle v_i, v_i \rangle = 1$ for all i.

We have the basic fact:

Lemma 3.4.6. If $(v_1, \ldots, v_n)$ is orthonormal, then it is linearly independent. Moreover, if $v = a_1 v_1 + \cdots + a_n v_n$, then $a_i = \langle v, v_i \rangle$.

Proof. The second statement is immediate from the linearity of $\langle -, - \rangle$ in the first component. Then $v = 0$ implies $a_i = 0$ for all i, hence the first statement also follows. $\square$

3.5 Dimension of homomorphisms via characters

We are now ready to prove a fundamental result:

Theorem 3.5.1. Let (V, ρ_V) and (W, ρ_W) be finite-dimensional representations of a finite group. Then:

$$\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_G(V, W). \tag{3.5.2}$$

3.5.1 Applications of Theorem 3.5.1

Before we prove the theorem, let's look at some motivating consequences:

Corollary 3.5.3. Let G be a finite group and $V_1, \ldots, V_m$ a full set of irreducible representations.

(i) Let V and W be irreducible representations. Then

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1, & \text{if } V \cong W; \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.4}$$

(ii) The characters χ_{V_i} form an orthonormal basis of $\mathrm{Fun}_{\mathrm{cl}}(G, \mathbf{C})$.

(iii) Let V be a finite-dimensional representation. Then $V \cong \bigoplus_{i=1}^m V_i^{\langle \chi_{V_i}, \chi_V \rangle}$. Moreover, $\chi_V = \sum_{i=1}^m \langle \chi_{V_i}, \chi_V \rangle \chi_{V_i}$.

(iv) Under the same assumptions as in (iii),

$$\langle \chi_V, \chi_V \rangle = \sum_i r_i^2 = \sum_i \langle \chi_{V_i}, \chi_V \rangle^2. \tag{3.5.5}$$

(v) A finite-dimensional representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Proof. (i) This is (2.14.2) together with Theorem 3.5.1.

(ii) The fact that the χ_{V_i} form an orthonormal set is an immediate consequence of (i). By Lemma 3.4.6, the characters are linearly independent. By Theorem 2.18.2, m equals the number of conjugacy classes of G. By Exercise 3.2.4, this is the dimension of the vector space of class functions (since there is another basis $\delta_{\mathcal{C}}$ indexed by the conjugacy classes: see also (3.5.7)). Thus this must form a basis.

(iii) The first statement is an immediate consequence of Proposition 2.15.2 together with Theorem 3.5.1. The second statement then follows from Proposition 3.3.4.(i).

(iv) This is an immediate consequence of (ii) and (iii). Alternatively this is Proposition 2.15.12 together with Theorem 3.5.1.

(v) This is an immediate consequence of (iv). □

Note that part (iii) immediately implies Theorem 3.1.10.

This motivates the following shorthand:

Definition 3.5.6. An *irreducible character* is a character of an irreducible representation.

Note that, in Exercise 3.2.4, we used an obvious basis of the vector space of class functions: for each conjugacy class $\mathcal{C} \subseteq G$, we can consider the function $\delta_{\mathcal{C}}$ given by

$$\delta_{\mathcal{C}}(g) = \begin{cases} 1, & \text{if } g \in \mathcal{C}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.7}$$

So the content of character theory is **there are two natural bases of the vector space of class functions**, given by the Kronecker delta functions of conjugacy classes (3.5.7), or by the irreducible characters (Corollary 3.5.3.(iii)).

Remark 3.5.8. The basis of Kronecker delta functions is not orthonormal: it is still true that $\langle \delta_{\mathcal{C}}, \delta_{\mathcal{C}'} \rangle = 0$ (i.e., $\delta_{\mathcal{C}} \perp \delta_{\mathcal{C}'}$), but $\langle \delta_{\mathcal{C}}, \delta_{\mathcal{C}} \rangle = |G|^{-1} \sum_{g \in \mathcal{C}} 1 = \frac{|\mathcal{C}|}{|G|}$. So an orthonormal basis is given by the functions $\sqrt{\frac{|G|}{|\mathcal{C}|}} \delta_{\mathcal{C}}$.

Remark 3.5.9. There is an interpretation of the normalisation coefficients in terms of the orbit-stabiliser theorem: note that $\mathcal{C}$ is an orbit of G under the conjugation action $G \times G, (g, h) \mapsto ghg^{-1}$. Let $h \in \mathcal{C}$; then $\mathcal{C} = G \cdot h$ under this action. Hence $\frac{|G|}{|\mathcal{C}|}$ is the size of the stabiliser of h. But the stabiliser is just the centraliser, $Z_G(h) = \{g \in G \mid gh = hg\}$. Thus the orthonormal basis consists of functions $\sqrt{|Z_G(h)|}\delta_{\mathcal{C}}$ (for $h \in \mathcal{C}$).

Aside from the normalisation issue, the bases $\delta_{\mathcal{C}}$ are still quite different:

Example 3.5.10. Let $G = C_n = \{1, g, \ldots, g^{n-1}\}$ be a cyclic group. Then $\mathrm{Fun}_{\mathrm{cl}}(G, \mathbf{C}) = \mathrm{Fun}(G, \mathbf{C})$, with dimension equal to $|G|$. The basis $\delta_g, g \in G$ are just the functions which are one on a single element $g \in G$ and zero elsewhere (the "Kronecker delta functions"). On the other hand, the irreducible representations are of the form $(\mathbf{C}_\zeta, \rho_\zeta)$ where $\mathbf{C}_\zeta := \mathbf{C}$ and $\rho_\zeta(g^k) = \zeta^k \in \mathbf{C}^\times$, for $\zeta^n = 1$. This is also the same as the basis $\chi_{\mathbf{C}_\zeta} = \rho_\zeta$ of characters since taking the character of a one-dimensional matrix representation doesn't do anything.

If we rather use the isomorphism $G \cong \mathbf{Z}/n\mathbf{Z} = \{\overline{0}, \ldots, \overline{n-1}\}$, then for $\zeta = e^{i\theta}$ (for $\theta = 2\pi i\ell/n$ for some $\ell \in \mathbf{Z}$), we get $\chi_{\mathbf{C}_\zeta}(\overline{k}) = e^{i\theta k}$, which is a basis of exponential functions. The change of basis matrix between the Kronecker delta functions and these exponential functions is the *discrete Fourier transform*, computed efficiently by the famous "fast Fourier transform".

The formula for this transform $f = (f(k))_{0 \leq k \leq n} \mapsto (r_\zeta)_{\zeta^n = 1}$, with $f = \sum_\zeta r_\zeta \chi_{\mathbf{C}_\zeta}$, is given by Corollary 3.5.3.(iii):

$$r_\zeta = \langle f, \chi_{\mathbf{C}_\zeta} \rangle = \frac{1}{n} \sum_{k=0}^{n-1} f(k) \zeta^{-k}. \tag{3.5.11}$$

Example 3.5.12. If G is a finite abelian group, almost the same thing as in the preceding example holds, except now we have to take a product of cyclic groups and exponential functions on a product of groups $\mathbf{Z}/n_i\mathbf{Z}$. The change of basis is a product of discrete Fourier transforms.

With the preceding example(s) in mind, the general change-of-basis matrix between Kronecker delta functions and irreducible characters can be thought of as a *nonabelian Fourier transform.*

Example 3.5.13. Let's consider the reflection representation (V, ρ_V) of S_n. Since this is irreducible, the formula (3.3.11) yields the identity

$$n! = n!\langle \chi_V, \chi_V \rangle = \sum_{\sigma \in S_n} (|\{1, \ldots, n\}^\sigma| - 1)^2. \tag{3.5.14}$$

This is a purely combinatorial identity but does not seem very obvious from combinatorics alone! For example, when $n = 3$ we get $6 = (3-1)^2 + 3 \cdot (1-1)^2 + 2 \cdot (0-1)^2$, considering first the identity, then the two-cycles, then the three-cycles. When $n = 4$ we get

$$24 = (4-1)^2 + 6 \cdot (2-1)^2 + 8 \cdot (1-1)^2 + 3 \cdot (0-1)^2 + 6 \cdot (0-1)^2,$$

considering the identity, the two-cycles, the three-cycles, the products of two disjoint two-cycles, and finally the four-cycles.

Example 3.5.15. Let's return to $G = C_n = \{1, g, \ldots, g^{n-1}\}$ in the notation of Example 3.5.10. Corollary 3.5.3.(i) says that $\langle \chi_{\mathbf{C}_\zeta}, \chi_{\mathbf{C}_\xi} \rangle = \delta_{\zeta,\xi}$. We can verify this explicitly:

$$\langle \chi_{\mathbf{C}_\zeta}, \chi_{\mathbf{C}_\xi} \rangle = \frac{1}{n} \sum_{\ell=0}^{n-1} \zeta^\ell (\xi)^{-\ell} = \frac{1}{n} \sum_{\ell=0}^{n-1} (\zeta/\xi)^\ell, \tag{3.5.16}$$

which is indeed $\delta_{\zeta,\xi}$ since ζ/ξ is an n-th root of unity.

Example 3.5.17. Let us keep the notation from the previous example. By Exercise 2.4.22, an m-dimensional representation $(\mathbf{C}^m, \rho)$ is the same as a matrix $T \in \mathrm{GL}_m(\mathbf{C})$ with $T^n = I$, by the correspondence $\rho(g^k) = T^k$. Let us decompose this as

$$(\mathbf{C}^m, \rho) \cong \bigoplus_{\zeta^n = 1} (\mathbf{C}, \rho_\zeta)^{r_\zeta}. \tag{3.5.18}$$

By Corollary 3.5.3.(iii) (cf. (3.5.11)),

$$r_\zeta = \overline{r_\zeta} = \langle \chi_\rho, \chi_{\mathbf{C}_\zeta} \rangle = n^{-1} \sum_{\ell=0}^{m-1} \mathrm{tr}(T^\ell) \zeta^{-\ell}, \tag{3.5.19}$$

which is applying the discrete Fourier transform to χ_ρ.

Of course, we can also compute the r_ζ with linear algebra: it is the multiplicity of ζ as an eigenvalue of T, i.e., the multiplicity of ζ as a root of the characteristic polynomial $\det(xI - T)$ of T. This is equivalent to the above answer though: since T is diagonalisable with r_ξ entries of ξ on the diagonal for all ξ with $\xi^n = 1$, we have

$$\mathrm{tr}(T^\ell) = \sum_{\xi^n = 1} r_\xi \xi^\ell. \tag{3.5.20}$$

Therefore (3.5.19) becomes

$$r_\zeta = n^{-1} \sum_{\ell=0}^{n-1} \zeta^{-\ell} \sum_{\xi^n=1} r_\xi \xi^\ell = n^{-1} \sum_{\xi^n=1} r_\xi \sum_{\ell=0}^{n-1} (\xi/\zeta)^\ell, \tag{3.5.21}$$

which is again true since ξ/ζ is an n-th root of unity. (This is just reproving that $\chi_{\mathbf{C}_\zeta}$ and $\chi_{\mathbf{C}_\xi}$ are orthogonal for $\zeta \neq \xi$, as in Example 3.5.15.)

Remark 3.5.22 (Non-examinable)**.** Actually, (3.5.19) and (3.5.20) are almost the same transformations: this is saying (which we just proved) that the discrete Fourier transform is almost inverse to itself, up to negating the coefficient in the sum, as well as the extra normalisation of n^{-1}. To fix the second issue we can consider the *normalised* discrete Fourier transform, call it F, by multiplying the RHS of (3.5.11) by $n^{-\frac{1}{2}}$; then $F^2(f)(x) = f(n-x)$, i.e., F^2 reverses the order of a sequence. Thus F^4 is the identity. (This is actually the same as the normalisation of Remark 3.5.8: it is what is needed to make F unitary.)

3.5.2 Proof of Theorem 3.5.1

The main idea of the proof is to count the dimension of $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$ via the projection $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)^G$, onto G-invariants. The count is done using the following linear algebra result:

Lemma 3.5.23. Let $S : V \to V$ be a linear projection onto $W := \mathrm{im}(S)$. Then $\dim W = \mathrm{tr}\, S$.

Proof. By Lemma 2.8.16 we can write $V = W \oplus U$ for $U := \ker(S)$. Pick bases $w_1, \ldots, w_m$ and $u_1, \ldots, u_n$ of W and U, so that $\mathcal{B} := (w_1, \ldots, w_m, u_1, \ldots, u_n)$ is a basis of V. Then in this basis we have the block matrix

$$[S]_{\mathcal{B}} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \tag{3.5.24}$$

with $\mathrm{tr}[S]_{\mathcal{B}} = m$. Hence $\mathrm{tr}\, S = m = \dim W$. $\square$

In order to make the proof of the theorem more transparent, it is worth proving first the special case where $V = \mathbf{C}_{\mathrm{triv}}$ is the trivial representation (this is not needed: see Remark 3.5.33 for a direct and simple proof). In this case $\mathrm{Hom}_G(\mathbf{C}_{\mathrm{triv}}, W)$ becomes just W^G. Let us record this statement, which is an analogue of (2.16.1), replacing the regular by the trivial representation:

Lemma 3.5.25. For every representation (V, ρ_V) of a group G, we have an isomorphism of vector spaces

$$\mathrm{Hom}_G(\mathbf{C}_{\mathrm{triv}}, V) \cong V^G, \quad \varphi \mapsto \varphi(1). \tag{3.5.26}$$

Proof. First, $\mathrm{Hom}(\mathbf{C}_{\mathrm{triv}}, V) \cong V$, under the linear map $\varphi \mapsto \varphi(1)$ (since $\varphi(z) = z\varphi(1)$ for all $z \in \mathbf{C}$). This map is compatible with the G action since G acts trivially on $\mathbf{C}_{\mathrm{triv}}$. Applying Exercise 2.14.9, we get $\mathrm{Hom}_G(\mathbf{C}_{\mathrm{triv}}, V) = \mathrm{Hom}(\mathbf{C}_{\mathrm{triv}}, V)^G = V^G$. (Alternatively, $\varphi : \mathbf{C}_{\mathrm{triv}} \to V$ is G-linear if and only if $\varphi(1)$ is G-invariant, since $1 \in \mathbf{C}_{\mathrm{triv}}$ is G-invariant.) $\square$

Proof of Theorem 3.5.1. The main idea of the proof is contained in the case $V = \mathbf{C}_{\mathrm{triv}}$; let us assume this. By Lemma 3.5.25, we need to show

$$\langle \chi_{\mathbf{C}_{\mathrm{triv}}}, \chi_W \rangle = \dim W^G. \tag{3.5.27}$$

Recall the G-linear projection $S : W \twoheadrightarrow W^G$ from (2.14.11), given by $S = |G|^{-1} \sum_{g \in G} \rho(g)$. By Lemma 2.8.16 we have $\operatorname{tr} S = \dim W^G$. We can now complete the proof with a quick computation:

$$\begin{aligned}\langle \chi_{\mathbf{C}_{\mathrm{triv}}}, \chi_W \rangle &= |G|^{-1} \sum_{g \in G} \overline{\chi_W(g)} = |G|^{-1} \sum_{g \in G} \operatorname{tr}(\overline{\rho_W(g)}) \\ &= \operatorname{tr}(|G|^{-1} \sum_{g \in G} \overline{\rho_W(g)}) = \operatorname{tr} \overline{S} = \overline{\operatorname{tr} S} = \overline{\dim W^G} = \dim W^G. \end{aligned} \tag{3.5.28}$$

Let us deduce the general case from this (see Remark 3.5.33 for a direct proof). First note that

$$\langle f_1, f_2 \rangle = \langle 1, \overline{f_1} f_2 \rangle, \tag{3.5.29}$$

where 1 is the constant function, $1(g) = 1$ for all $g \in G$. Now by Example 3.1.11 and Proposition 3.3.4.(iv), setting $f_1 = \chi_V$ and $f_2 = \chi_W$, this becomes

$$\langle \chi_V, \chi_W \rangle = \langle \chi_{\mathbf{C}_{\mathrm{triv}}}, \chi_{\mathrm{Hom}(V,W)} \rangle. \tag{3.5.30}$$

On the other hand, by Lemma 3.5.25, we have

$$\mathrm{Hom}_G(\mathbf{C}_{\mathrm{triv}}, \mathrm{Hom}(V, W)) = \mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W), \tag{3.5.31}$$

using Exercise 2.14.9 for the last equality.

Therefore $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$ follows from

$$\langle \chi_{\mathbf{C}_{\mathrm{triv}}}, \chi_{\mathrm{Hom}(V,W)} \rangle = \mathrm{Hom}_G(\mathbf{C}_{\mathrm{triv}}, \mathrm{Hom}(V, W)), \tag{3.5.32}$$

which is the case of the theorem with $V = \mathbf{C}_{\mathrm{triv}}$ and W set to $\mathrm{Hom}(V, W)$. $\square$

Remark 3.5.33. The reduction to $V = \mathbf{C}_{\mathrm{triv}}$ was not required; we did this to explain what is going on (in the spirit of the explanation of part of the proof of Maschke's theorem given by Proposition 2.14.10 and the surrounding text). Without making this reduction the proof is actually shorter: the point is that $\langle \chi_V, \chi_W \rangle = \operatorname{tr} S$ where now

$$S(\varphi) = |G|^{-1} \sum_{g \in G} \rho_W(g) \circ \varphi \circ \rho_V(g)^{-1} \tag{3.5.34}$$

is the projection operator $\mathrm{Hom}(V, W) \to \mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$ used in the proof of Maschke's theorem (see also the discussion after Proposition 2.14.10). Then on the one hand, Lemma 2.8.16 implies that $\operatorname{tr} S = \dim \mathrm{Hom}_G(V, W)$. On the other hand, $S = |G|^{-1} \sum_{g \in G} \rho_{\mathrm{Hom}(V,W)}$, which implies that

$$\operatorname{tr} S = |G|^{-1} \sum_{G \in G} \chi_{\mathrm{Hom}(V,W)}(g)) = \langle \chi_V, \chi_W \rangle, \tag{3.5.35}$$

applying Proposition 3.3.4.(iv) for the second equality.

3.6 Character tables

3.6.1 Definition and examples

The idea of a character table is very simple: list the irreducible characters. For finite groups, this determines all of the characters by Maschke's theorem.

Let G be a finite group and let $V_1, \ldots, V_m$ be a full set of irreducible representations (Definition 2.18.1). Recall that the number m also equals the number of conjugacy classes (Theorem 2.18.2): label these $\mathcal{C}_1, \ldots, \mathcal{C}_m \subseteq G$, and let $g_i \in \mathcal{C}_i$ be representatives.

Definition 3.6.1. The character table of G is the table whose i, j-th entry is $\chi_{V_i}(g_j)$.

Note that the definition does *not* depend on the choice of V_i up to isomorphism (by Proposition 3.1.9) nor on the choice of representatives g_j (by Proposition 3.2.6). However, it *does* depend on the choice of ordering of the V_i and of the $\mathcal{C}_j$. So really the table itself only is well-defined up to reordering the rows and the columns, although in practice we will always indicate with the table the representations corresponding to each row and the conjugacy classes corresponding to each column.

Another way to interpret the character table is that it gives the (transpose of the) change-of-basis matrix between the basis of Kronecker delta functions $\delta_{\mathcal{C}_i}$ (see (3.5.7)) and the basis of irreducible characters χ_{V_i} (which up to reordering does not depend on the choice of the irreducible representations V_i).

Example 3.6.2. The character table for C_n is given as follows. Let $\xi := e^{2\pi i/n}$ (one could equally use any primitive n-th root of unity). For each row we list $\zeta = 1, \xi, \xi^2, \ldots$, then $\chi_{\mathbf{C}_\zeta}(h)$ for the group element $h = 1, g, g^2, \ldots, g^{n-1}$ listed above each column.

	$h=1$	$h=g$	$h=g^2$	$\ldots$	$h=g^{n-1}$
$\chi_{\mathbf{C}_1}(h)$	1	1	1	$\ldots$	1
$\chi_{\mathbf{C}_\xi}$	1	ξ	ξ^2	$\ldots$	ξ^{n-1}
$\chi_{\mathbf{C}_{\xi^2}}$	1	ξ^2	ξ^4	$\ldots$	$\xi^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\chi_{\mathbf{C}_{\xi^{n-1}}}$	1	ξ^{n-1}	$\xi^{2(n-1)}$	$\ldots$	$\xi^{(n-1)^2}$

The $n \times n$ matrix here is also the Vandermonde matrix for the n-th roots of unity $(1, \xi, \xi^2, \ldots, \xi^{n-1})$: see, e.g., `https://en.wikipedia.org/wiki/Vandermonde_matrix`. It represents the discrete Fourier transform, and as you will see in Exercise 3.6.3, it is essentially unitary up to normalisation.

Exercise 3.6.3. Let A be the matrix appearing in Example 3.6.2, i.e., $A_{kl} = \xi^{kl}$ for ξ a primitive n-th root of unity. Verify directly that $A\overline{A}^t = nI$, i.e., $\overline{A}^t = nA^{-1}$. Deduce that the matrix $U := n^{-1/2}A$ is unitary: $\overline{U}^t = U^{-1}$. (Note that this is a consequence of Corollary 3.5.3.(i), but we will use this argument to generalise to all finite groups soon.)

Remark 3.6.4. Note that the matrix above is symmetric, but this is an artifact of the ordering we chose of the rows and the columns. If we had picked a different ordering of the rows (irreducible characters) without correspondingly changing the ordering of the columns, then the matrix would not have been symmetric. For a non-abelian group, the character table cannot be symmetric: see Exercise 3.6.10.

Example 3.6.5. It is worth giving an example of a non-cyclic abelian group. Let $G = C_2 \times C_2 = \{(1,1),(a,1),(1,b),(a,b)\}$. Then by Exercise 2.12.7 the irreducible representations are of the form $\rho_{\zeta,\xi}$ for $\zeta, \xi \in \{\pm 1\}$, defined by $\rho_{\zeta,\xi}(a^p, b^q) = \zeta^p \xi^q$ for $p, q \in \{0,1\}$. (In other words, $\rho_{\zeta,\xi} = \rho_\zeta \otimes \rho_\xi$.) The character table is then:

	$h=1$	$h=(a,1)$	$h=(1,b)$	$h=(a,b)$
$\chi_{1,1}(h)$	1	1	1	1
$\chi_{-1,1}(h)$	1	-1	1	-1
$\chi_{1,-1}(h)$	1	1	-1	-1
$\chi_{-1,-1}(h)$	1	-1	-1	1

In fact, this 4×4 matrix is just the Kronecker product of the two 2×2 matrices for the factors C_2. See Exercise 3.6.9.

Example 3.6.6. We compute the character table for S_3. The irreducible representations, by Example 2.17.5, are the trivial, sign, and reflection representations. The character of the one-dimensional representations (the trivial and sign representations) are just the values of the representation (as one-by-one matrices). For the reflection representation the character is $\chi_V(\sigma) = |\{1,2,3\}^\sigma| - 1$ by Example 3.3.10. Let $\mathbf{C} = \mathbf{C}_{\text{triv}}$ be the trivial representation and $\mathbf{C}_-$ be the sign representation. Let V denote the reflection representation. We get:

	$h=1$	$h=(12)$	$h=(123)$
$\chi_{\mathbf{C}}(h)$	1	1	1
$\chi_{\mathbf{C}_-}(h)$	1	-1	1
$\chi_V(h)$	2	0	-1

Note that, as required by Corollary 3.5.3.(i), the rows are orthogonal by the inner product $\langle (a,b,c),(a',b',c')\rangle = \frac{1}{6}(aa' + 3bb' + 2cc')$, where $\frac{1}{6} = |G|^{-1}$ and $1, 3, 2$ are the sizes of the conjugacy classes of $1, (12)$, and (123), respectively. (In fact, this relation allows you to deduce any one of the rows from the other two.)

Exercise 3.6.7. Let G and H be finite groups and V and W representations of G and H. Show, by the proof of Proposition 3.3.4.(i), that

$$\chi_{V \boxtimes W}(g,h) = \chi_V(g)\chi_W(h). \tag{3.6.8}$$

This specialises to Proposition 3.3.4.(ii) when we recall that $\rho_{V\otimes W}(g) = \rho_{V\boxtimes W}(g,g)$ (2.20.4).

Exercise 3.6.9. Maintain the notation of Exercise 3.6.7. Show that, choosing the representations and conjugacy classes appropriately, the character table of $G \times H$ is the Kronecker product (2.19.34) of the character tables for G and H. (Hints: if $(V_i), (W_j)$ are full sets of irreducible representations for G and H, use Proposition 2.20.5 to see that $(V_i \boxtimes W_j)$ is a full set of irreducible representations for $G \times H$. Then apply Exercise 3.6.7. Finally for $(\mathcal{C}_i)$ and $(\mathcal{D}_j)$ the conjugacy classes of G and H, then $(\mathcal{C}_i \times \mathcal{D}_j)$ forms the conjugacy classes of $G \times H$.)

Exercise 3.6.10. In this exercise we will show that, if G is non-abelian, then there is no ordering of the rows and columns to get a symmetric matrix.

(i) If G is nonabelian, prove that there is an irreducible representation of some dimension $n > 1$. (Hint: the number of one-dimensional representations is $|G_{\text{ab}}| < |G|$).

(ii) Show that the only irreducible character with all positive values is the trivial character. (Hint: use the orthogonality, Corollary 3.5.3.(i), with $V = \mathbf{C}_{\text{triv}}$).

(iii) Now prove that there is no ordering of the rows and columns to give a symmetric character table. (Hint: without loss of generality, put the trivial representation in the first row; show using (ii) that $\{1\}$ is the first column but by (i) that the result is still not symmetric.)

3.6.2 Row and column orthogonality; unitarity

Let's maintain the notation of the previous subsection. Observe that the orthogonality relation $\langle \chi_{V_i}, \chi_{V_j} \rangle$ implies that the rows of the character table must be orthogonal, in the following sense:

$$\delta_{ij} = \langle \chi_{V_i}, \chi_{V_j} \rangle = |G|^{-1} \sum_{g \in G} \chi_{V_i}(g) \overline{\chi_{V_j}}(g) = |G|^{-1} \sum_{k=1}^{m} |\mathcal{C}_k| \chi_{V_i}(g_k) \overline{\chi_{V_j}}(g_k), \tag{3.6.11}$$

which just means we need to take into account the sizes of the conjugacy classes. When G is abelian, then $|\mathcal{C}_k| = 1$ for all k, and we get that $\langle \chi_{V_i}, \chi_{V_j} \rangle$ is just $|G|^{-1}$ times the dot product of the i-th row with the complex conjugate of the j-th row, which explains Exercise 3.6.3.

Let A be the matrix given by the character table: $A_{ij} = \chi_{V_i}(g_j)$. Rewriting (3.6.11), we have

$$|G|^{-1} \sum_{k=1}^{m} |\mathcal{C}_k| A_{ik} \overline{A_{jk}} = |G|^{-1} \sum_{k=1}^{m} |\mathcal{C}_k| \chi_{V_i}(g_k) \overline{\chi_{V_j}}(g_k) = \delta_{ij}. \tag{3.6.12}$$

To turn this into a genuine dot product with complex conjugation, define the renormalised matrix U by:

$$U_{ij} := \sqrt{\frac{|\mathcal{C}_j|}{|G|}} A_{ij}. \tag{3.6.13}$$

Then we obtain

$$\sum_{k=1}^{m} U_{ik}\overline{U_{jk}} = \delta_{ij}. \tag{3.6.14}$$

In other words, (3.6.14) states that $U\overline{U^t} = I$, i.e., U is invertible with $U^{-1} = \overline{U^t}$. So U is a *unitary matrix*:

Definition 3.6.15. A unitary matrix is $U \in \mathrm{GL}_n(\mathbf{C})$ such that $U^{-1} = \overline{U^t}$.

Lemma 3.6.16. The following are equivalent: (a) U is unitary; (b) the rows of U are orthonormal with respect to $\langle v, w\rangle = v \cdot \overline{w}$ (3.6.14); (c) the columns are orthonormal with respect to this inner product, i.e.,

$$\sum_{k=1}^{m} U_{ki}\overline{U_{kj}} = \delta_{ij}. \tag{3.6.17}$$

Proof. Part (b) says that $U\overline{U^t} = I$ and part (c) says that $\overline{U^t}U = I$. So the equivalence is the result from linear algebra that $AB = I$ if and only if $BA = I$, for square matrices A and B of the same size (which is implicit in the definition of the inverse matrix). $\square$

From (3.6.17) we deduce the column orthogonality relations:

$$\begin{aligned}\delta_{ij} = \sum_{k=1}^{m} U_{ki}\overline{U_{kj}} = \frac{\sqrt{|\mathcal{C}_i||\mathcal{C}_j|}}{|G|}\sum_{k=1}^{m} A_{ki}\overline{A_{kj}} &= \frac{\sqrt{|\mathcal{C}_i||\mathcal{C}_j|}}{|G|}\sum_{k=1}^{m} A_{ki}\overline{A_{kj}} \\ &= \frac{\sqrt{|\mathcal{C}_i||\mathcal{C}_j|}}{|G|}\sum_{k=1}^{m} \chi_{V_k}(g_i)\overline{\chi_{V_k}}(g_j).\end{aligned} \tag{3.6.18}$$

Example 3.6.19. As a special case, let $g_i = g_j = 1$. Then we obtain

$$1 = |G|^{-1}\sum_{k=1}^{m} \chi_{V_k}(1)\overline{\chi_{V_k}}(1) = |G|^{-1}\sum_{k=1}^{m}(\dim V_k)^2, \tag{3.6.20}$$

i.e., the sum of squares formula (Corollary 2.16.4), proved in yet another way!

Example 3.6.21. More generally, letting $g_i = g_j$, we obtain

$$\frac{|G|}{|\mathcal{C}_i|} = \sum_{k=1}^{m} |\chi_{V_k}(g_i)|^2. \tag{3.6.22}$$

This allows one to obtain $|\mathcal{C}_i|$, the size of the conjugacy class, directly from the character table. Note that the LHS can also be interpreted as the size of the centraliser $|Z_G(g_i)|$, see also Remark 3.5.9. In particular, g_i is central if and only if $|\mathcal{C}_i| = 1$, i.e., $\sum_{k=1}^{m} |\chi_{V_k}(g_i)|^2 = |G|$. By Proposition 3.2.9.(ii), this is equivalent to the statement that $|\chi_{V_k}(g_i)| = \dim V_k$ for all k, or that $\rho_{V_k}(g_i)$ is a *scalar matrix* for all k. This can also be proved without using character theory: see Remark 2.12.4.

Remark 3.6.23. Using the row and column orthogonality, we can fill in a single missing row or column. Indeed, in a unitary matrix, the condition that the last column v_n is orthogonal to the span of the preceding columns $v_1, \ldots, v_{n-1}$ shows that $v_n \in \mathrm{Span}(v_1, \ldots, v_{n-1})^\perp$ which is one-dimensional; this fixes v_n up to scaling (by a number of absolute value one). In the case of the character table we know the first entry of the column is one (if we put the trivial representation first) and this fixes the scaling. Similarly for a missing row, the same argument applies except now the first entry is equal to the dimension, which we can compute from the sum of squares formula.

Concretely, the way to compute v_n up to scaling, for a unitary matrix, is to take any vector v not in the span of $v_1, \ldots, v_{n-1}$; then by Lemma 3.4.6, $v = \sum_{i=1}^n a_i v_i$ with $a_i = v \cdot \overline{v_i}$. Then v_n is a scalar multiple of $v - \sum_{i=1}^{n-1} a_i v_i$. The same technique works for a missing row. Applying this to the unitary matrix U obtained from the character table A by (3.6.13), we can get a missing row or column up to scaling, and then we simply rescale as indicated in the preceding paragraph.

In the case of a missing row, we can also use an easier technique: we know that

$$\chi_{\mathbf{C}[G]} = |G|\delta_1 = \sum_{i=1}^m \dim V_i \chi_{V_i}, \tag{3.6.24}$$

so if we know all of the χ_{V_i} except one, we can get the remaining one (first computing the remaining dimension by the sum of squares formula). This yields concretely for $g_j \neq 1$:

$$\chi_{V_i}(g_k) = -(\dim V_i)^{-1} \sum_{j \neq i} \dim V_j \cdot \chi_{V_j}(g_k). \tag{3.6.25}$$

3.6.3 More examples

You do not need to memorise any of this information for the exam.

Example 3.6.26. Let us compute the character table of D_8. Recall the description $D_8 = \{x^a y^b \mid 0 \leq a \leq 3, 0 \leq b \leq 1\}$ with x the counterclockwise 90° rotation and y the reflection about the x-axis. The conjugacy classes of rotations are $\{1\}, \{x, x^3\}$, and $\{x^2\}$ (since $yxy^{-1} = x^{-1} = x^3$ but x^2 is central), and the conjugacy classes of reflections are $\{y, x^2y\}$ and $\{xy, x^3y\}$ (since $xyx^{-1} = x^2y$ shows that in general $x^a y$ and $x^{a'}y$ are conjugate if and only if $2 \mid (a - a')$). This makes five conjugacy classes. As explained in Example 2.17.10, there are four one-dimensional representations (by Example 2.13.2) and one two-dimensional irreducible representation up to isomorphism (the one of Example 2.2.6). Call the one-dimensional representations $(\mathbf{C}_{\pm,\pm}, \rho_{\pm,\pm})$, where $\rho_{+,+}$ is the trivial representation, $\rho_{-,+}$ is the representation sending reflections to -1 and rotations to 1, $\rho_{+,-}$ sends the generating rotation x to -1 and the generating reflection y to 1, and $\rho_{-,-}$ sends both x and y to -1. Call the two-dimensional representation $\mathbf{C}^2$. The trace of a rotation matrix by angle θ is $2\cos\theta$, so $\chi_{\mathbf{C}^2}(x) = 0$ whereas $\chi_{\mathbf{C}^2}(x^2) = -2$. Also the trace of a reflection matrix is always zero since it has one and minus one as eigenvalues (or from the explicit description). Putting this together, we get:

	$\{1\}$	$\{x, x^3\}$	$\{x^2\}$	$\{y, x^2y\}$	$\{xy, x^3y\}$
Size of class:	1	2	1	2	2
$\rho_{+,+}$	1	1	1	1	1
$\rho_{-,+}$	1	1	1	-1	-1
$\rho_{+,-}$	1	-1	1	1	-1
$\rho_{-,-}$	1	-1	1	-1	1
$\chi_{\mathbf{C}^2}$	2	0	-2	0	0

Example 3.6.27. We can compute the character table also of $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ (with multiplication given by $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$, and $(-1)^2 = 1$). The conjugacy classes are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}$, and $\{\pm k\}$. By coursework, the one-dimensional representations are $\rho_{\pm,\pm}$ determined by the values on i and j which are ± 1 (so the two subscripts of ρ give the signs of the images of i and j, respectively). There is one two-dimensional representation, $\rho(\pm 1) = \pm I$, $\rho(\pm i) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\rho(\pm j) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\rho(\pm k) = \pm \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. We get:

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
Size of class:	1	1	2	2	2
$\rho_{+,+}$	1	1	1	1	1
$\rho_{-,+}$	1	1	-1	1	-1
$\rho_{+,-}$	1	1	1	-1	-1
$\rho_{-,-}$	1	1	-1	-1	1
$\chi_{\mathbf{C}^2}$	2	-2	0	0	0

Remark 3.6.28. Notice this is *the same table as for* D_8 if we swap the columns accordingly: in the D_8 case we can instead order the columns as $\{1\}, \{x^2\}, \{y, x^2y\}, \{x, x^3\}, \{xy, x^3y\}$. This is remarkable since $Q_8 \not\cong D_8$: for instance, in Q_8 there are six elements of order four, whereas in D_8 there are only two. It shows that *the character table does not determine the group up to isomorphism.*

Actually in this case, the two character tables have to coincide since 8 is a small number for a nonabelian group (e.g., the only possible sum of squares of dimensions is $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$ since 1 has to occur for the trivial representation but we can't have only 1's since the group is nonabelian). See Exercise 3.6.35.

Example 3.6.29. Let us compute the character table for A_4. As explained in Exercise 2.17.14, there are three irreducible one-dimensional representations, $(\mathbf{C}_\zeta, \rho_\zeta \circ q|_{A_4})$, for $\zeta^3 = 1$, defined as follows. Let $\mathbf{C}_\zeta = \mathbf{C}$. There is a surjection $q : S_4 \to S_3, q(12) = (12), q(23) = (23), q(34) = (12)$, which restricts to a surjection $q|_{A_4} : A_4 \to A_3$. Then $A_3 \cong C_3$ so its irreducible representations are ρ_ζ for $\zeta^3 = 1$ (Corollary 2.12.6). Then as outlined in Exercise 2.17.14 it follows that $A_3 \cong (A_4)_{\mathrm{ab}}$ and there is one three-dimensional irreducible representation. The three-dimensional irreducible representation can be given by restricting the reflection representation (V, ρ_V) of S_4 to the group A_4, i.e., $(V, \rho_V \circ i)$ for $i : A_4 \to S_4$

the inclusion. Now there are four conjugacy classes in A_4, given by the classes $[1] = \{1\}$, $[(12)(34)] = \{(12)(34), (13)(24), (14)(23)\}$, and the three-cycles must split into two conjugacy classes: $[(123)] = \{(123), (134), (142), (243)\}$, $[(132)] = \{(132), (124), (143), (234)\}$. For another way to compute the latter two conjugacy classes, see the following remark.

Taking traces we get, for $\omega := e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (using (3.3.11)):

	$[1]$	$[(12)(34)]$	$[(123)]$	$[(132)]$
Size of class:	1	3	4	4
$\mathbf{C}_1$	1	1	1	1
$\mathbf{C}_\omega$	1	1	ω	ω^2
$\mathbf{C}_{\omega^2}$	1	1	ω^2	ω
V	3	-1	0	0

Remark 3.6.30 (Non-examinable)**.** Here is another way to compute the conjugacy classes of A_4. In general, every conjugacy class of S_n can split into at most two conjugacy classes in A_n, since a conjugacy class is the orbit of the group under the conjugation (adjoint) action, and $[S_n : A_n] = 2$. In the case the conjugacy class splits into two conjugacy classes, the two classes must have the same size (since the two are conjugate in S_n). Now, since characters have the same value on every element of a conjugacy class, the image of each conjugacy class uner q in A_3 must be the same element (every element of A_3 takes a different value under a nontrivial one-dimensional representation). Since $q((123)) = (123)$ and $q((132)) = (132)$, this implies $[(123)] \subseteq q^{-1}((123))$ and $[(132)] \subseteq q^{-1}((132))$. Since there can be only two conjugacy classes of three-cycles, there are exactly two. Thus $[(123)]$ consists of the three-cycles in $q^{-1}((123))$ and similarly for $[(132)]$. In fact all elements of $q^{-1}((123))$ and $q^{-1}((132))$ are three-cycles (which is also implied by the fact that the sizes of these fibers are $[A_4 : A_3] = 4$ each).

Example 3.6.31. The character table of S_4: let $\mathbf{C}, \mathbf{C}_-$ be the trivial and sign representations, V the reflection representation, $V_- = V \otimes \mathbf{C}_-$ the other three-dimensional irreducible representation, and U the two-dimensional irreducible representation (the reflection representation of S_3 composed with the map $S_4 \twoheadrightarrow S_3$), see Example 2.17.6.

	$[1]$	$[(12)]$	$[(123)]$	$[(12)(34)]$	$[(1234)]$
Size of class	1	6	8	3	6
$\chi_{\mathbf{C}}$	1	1	1	1	1
$\chi_{\mathbf{C}_-}$	1	-1	1	1	-1
χ_U	2	0	-1	2	0
χ_V	3	1	0	-1	-1
χ_{V_-}	3	-1	0	-1	1

Remark 3.6.32. In order to compute the character table for the groups A_n, we need to analyse the conjugacy classes. If $\sigma \in A_n$, then $[\sigma]_{A_n}$ is either all of $[\sigma]_{S_n}$ or half of it, since $[\sigma]_{S_n} = [\sigma]_{A_n} \cup [(12)\sigma(12)]$ (as $S_n = A_n \cup (12)A_n$). The only question is whether σ is conjugate to $(12)\sigma(12)$ or not. Now, $[\sigma]_{A_n}$ is an orbit in A_n under the action $A_n \times A_n \to A_n$

by conjugation. The stabiliser of σ is the centraliser $Z_{A_n}(\sigma) := \{\tau \in A_n \mid \tau\sigma = \sigma\tau\}$. By the orbit-stabiliser theorem, $|[\sigma]_{A_n}| = \frac{|A_n|}{|Z_{A_n}(\sigma)|}$. Similarly $|[\sigma]_{S_n}| = \frac{|S_n|}{|Z_{S_n}(\sigma)|}$. Therefore we conclude that $[\sigma]_{A_n} = [\sigma]_{S_n}$ if and only if $[Z_{S_n}(\sigma) : Z_{A_n}(\sigma)] = 2$; otherwise $[\sigma]_{A_n}$ is half of $[\sigma]_{S_n}$ and $Z_{S_n}(\sigma) = Z_{A_n}(\sigma)$. In other words, $[\sigma]_{S_n}$ splits into two conjugacy classes if and only if it does not commute with an odd permutation.

Example 3.6.33. Consider $G = A_5$. The even conjugacy classes in S_5 are $[1]$, $[(123)]$, $[(12)(34)]$, and $[(12345)]$. The first conjugacy class cannot split into two for A_5. The second and third do not commute either since (123) commutes with the odd permutation (45) and $(12)(34)$ commutes with (12). But $Z_{S_5}(12345) = \langle(12345)\rangle$ which does not contain any odd permutations. Therefore (12345) does split into two conjugacy classes. We get a total of five conjugacy classes of sizes $1, 20, 15, 12$, and 12. Next as shown in problems class, the dimensions of the irreducible representations are $1, 3, 3, 4, 5$. Let us recall why. Since A_5 is simple, it has only one one-dimensional representation. For the other dimensions, we seek $a, b, c, d \geq 1$ such that $a^2 + b^2 + c^2 + d^2 = 59$. Taking this modulo 4, three of these must be odd (since squares are 1 or 0 modulo 4), and the remaining one even. Suppose $a \leq b \leq c \leq d$. We have $d \leq 7$. In fact $d \neq 7$, since $a, b, c \geq 2$. If $d = 6$ we have $a^2 + b^2 + c^2 = 23$, obviously impossible since the only possible values of a, b, c are three. So $d \leq 5$. We cannot have $d = 3$ since $4 + 9 + 9 + 9 < 59$. We also cannot have $d = 4$ since $9 + 9 + 9 + 16 < 59$. So $d = 5$. Then $a^2 + b^2 + c^2 = 34$. Again $c \neq 5$ since $a^2 + b^2 = 9$ is impossible. Also $c > 3$ since $4 + 9 + 9 < 34$. So $c = 4$. Then we get $a = b = 3$. So the dimensions of the nontrivial representations are: $3, 3, 4, 5$.

Now we can construct some of these irreducible representations. One is the restriction of the reflection representation of S_5 (either check it using the character, or better you can prove that the reflection representation of S_n restricts to an irreducible representation of A_n for all $n \geq 4$, by a similar proof to how we showed the reflection representation was irreducible in the first place.) This gives the four-dimensional irreducible representation, and the character is given by the formula (3.3.11).

We can use geometry to construct a three-dimensional irreducible representation since we know that A_5 acts as the group of rotations of the icosahedron or dodedecahedron (see the third problems class sheet and `https://en.wikipedia.org/wiki/Icosahedral_symmetry#Group_structure`). We actually don't need to know much about this representation, only that an element of order n must act as a rotation about some axis by some angle $2\pi m/n$ with $\gcd(m, n) = 1$. The trace of such a rotation is $2\cos(2\pi m/n) + 1 = \zeta + \zeta^{-1} + 1$ for $\zeta = e^{2\pi i m/n}$ (the 1 is for the axial direction). In the case $n = 2, 3$, we get -1 and 0, respectively. In the case of 5 we get two possibilities: for $\xi = e^{2\pi i/5}$, we get either $1 + 2\cos(2\pi/5) = 1 + \xi + \xi^{-1}$ or $1 + 2\cos(4\pi/5) = 1 + \xi^2 + \xi^{-2}$. These values are well-known but let us give a quick derivation (it is the only way I can remember the values). Let $z := \xi + \xi^{-1}$. Then $1 + \xi + \xi^2 + \xi^3 + \xi^4 = 0$ implies $\xi^2 + \xi^{-2} = -1 - z$. But $z^2 = 2 + \xi^2 + \xi^{-2} = 1 - z$ implies $z^2 + z - 1 = 0$, hence $z = \frac{-1 \pm \sqrt{5}}{2}$. In fact $z = \frac{-1+\sqrt{5}}{2}$ since it is easy to see from geometry that z is positive. Then $\xi^2 + \xi^2 = \frac{-1-\sqrt{5}}{2}$. Thus the possible trace values for the order five elements are $\frac{1 \pm \sqrt{5}}{2}$. One of these are assigned to each of the conjugacy classes of elements of order five. In fact, since these conjugacy classes are obtained from each other by conjugating by an odd permutation (e.g., (12)), we see that the only difference between the two is relabeling the five compounds being permuted by an odd permutation: thus both possibilities occur and there are *two* nonisomorphic rotation representations, depending on how we label the five compounds. Another way to say this is that the two resulting representations are $(\mathbf{C}^3, \rho)$ and $(\mathbf{C}^3, \rho \circ \mathrm{Ad}_{(12)})$, for $(\mathbf{C}^3, \rho)$ one rotation representation. For lack of a better notation, call these two representations $\mathbf{C}_1^3$ and $\mathbf{C}_2^3$.

This gives all of the irreducible representations except for the dimension five one. We get the partial table (we choose the second five-cycle to be $(12345)^2 = (13524)$):

	[1]	[(12)(34)]	[(123)]	[(12345)]	[(13524)]
Size of class	1	15	20	12	12
$\chi_{\mathbf{C}}$	1	1	1	1	1
$\chi_{\mathbf{C}_1^3}$	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\chi_{\mathbf{C}_2^3}$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\chi_{\mathbf{C}^4}$	4	0	1	-1	-1
$\chi_{\mathbf{C}^5}$	5	?	?	?	?

To fill in the last row, the easiest technique is to use (3.6.25), for $k > 1$:

$$\chi_{\mathbf{C}^5}(g_k) = -\frac{1}{5}(1 + 3\chi_{\mathbf{C}_1^3}(g_k) + 3\chi_{\mathbf{C}_2^3}(g_k) + 4\chi_{\mathbf{C}^4}(g_k)). \tag{3.6.34}$$

This yields:

	[1]	[(12)(34)]	[(123)]	[(12345)]	[(13524)]
Size of class	1	15	20	12	12
$\chi_{\mathbf{C}}$	1	1	1	1	1
$\chi_{\mathbf{C}_1^3}$	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\chi_{\mathbf{C}_2^3}$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\chi_{\mathbf{C}^4}$	4	0	1	-1	-1
$\chi_{\mathbf{C}^5}$	5	1	-1	0	0

The remaining entry has all integer entries, which makes it seem likely it is related to the action of A_5 on a set: the character of such a representation $\mathbf{C}[X]$ has all nonnegative integer values by Example 3.1.15, and it always has the trivial representation as the subrepresentation where all coefficients are equal (just like for the reflection representation), so $\chi_{\mathbf{C}[X]} - \chi_{\mathbf{C}}$ is also a character. In order to have $\chi_{\mathbf{C}^5} = \chi_{\mathbf{C}[X]} - \chi_{\mathbf{C}}$, X should have size six, and A_5 acts with (12)(34) fixing no elements, (123) fixing two, and five-cycles fixing one element. *Such a set X exists!* It is given by the set of pairs of opposite vertices of the icosahedron (there are twelve vertices and hence six pairs of opposite vertices). This is acted on by A_5 as the group of rotational symmetries. Thus, $\chi_{\mathbf{C}^5}$ is the character of the subrepresentation of $\mathbf{C}[X]$ with coefficients summing to zero, analogously to the reflection representation.

Exercise 3.6.35. Show that, up to reordering rows and columns, the character table of Q_8 is the only one that can occur for a nonabelian group of size eight. Here is an outline of one way to do this:

Step 1: Show that the dimensions of the irreducible representations are $1, 1, 1, 1, 2$.

Step 2: Conclude that $|G_{\mathrm{ab}}| = 4$.

Step 3: Prove the lemma: if $N \lhd G$ is a normal subgroup of size two, then it is central (i.e., for all $n \in N$ and $g \in G$, $gn = ng$, that is, N is a subgroup of the center of G.). Using the lemma, prove that $[G, G]$ is central.

Step 4: Prove that $G_{\mathrm{ab}} \cong C_2 \times C_2$. For this, observe first that either this is true or $G_{\mathrm{ab}} \cong C_4$, and to rule out the latter possibility, prove the lemma: if $Z < G$ is a central subgroup with G/Z cyclic, then G is itself abelian.

Step 5: Using the lemma in Step 4, show that $[G, G]$ is the entire center of G. Conclude that there are two conjugacy classes of size one in G. Now prove the other conjugacy classes all have size 2 (show the sizes are factors of $|G$ and use Theorem 2.18.2).

Step 6: Now using Steps 4 and 5, show that the rows for the four one-dimensional representations are uniquely determined up to ordering the columns as well as these rows.

Step 7: Finally, use orthogonality to show that this determines the whole table.

3.7 Kernels of representations and normal subgroups

The next proposition shows that the kernel of a representation can be read off from the character.

Proposition 3.7.1. Let (V, ρ_V) be a representation of a finite group. Then $\ker \rho_V = \{g \in G \mid \chi_V(g) = \dim V\}$.

Proof. It is clear that $g \in \ker \rho_V$ implies $\chi_V(g) = \dim V$. We only need to show the converse. Suppose $\chi_V(g) = \dim V$. By Proposition 3.2.9.(ii), $|\chi_V(g)| = \dim V$ if and only if $\chi_V(g)$ is a scalar matrix; so $\rho_V(g) = \lambda I$ for some $\lambda \in \mathbf{C}$. But then $\chi_V = \lambda \dim V$, so that $\lambda = 1$. Therefore $g \in \ker \rho_V$. $\square$

Remark 3.7.2. This is one of the few places we needed Proposition 3.2.9.(ii)! So maybe you should review it: the result followed by applying the triangle inequality (viewing $\mathbf{C}$ as $\mathbf{R}^2$) to $\chi_V(g) = \zeta_1 + \cdots + \zeta_n$ where ζ_i are the eigenvalues of $\rho_V(g)$ with multiplicity (which are roots of unity, hence of absolute value one).

Now we show that every normal subgroup is an intersection of kernels of irreducible representations. Together with Proposition 3.7.1, this shows that we can read off *all* normal subgroups (in terms of unions of conjugacy classes) directly from the character table. Let G be a finite group and $V_1, \ldots, V_m$ be a full set of irreducible representations, and let $K_i := \ker \rho_{V_i}$.

Proposition 3.7.3. Let $N \lhd G$ be any normal subgroup. Let $J := \{j \in \{1, \ldots, m\} \mid N \leq K_i\}$. Then $N = \bigcap_{j \in J} K_j$.

To prove the proposition we need two lemmas, interesting in themselves.

Lemma 3.7.4. Suppose $V \cong \bigoplus_{i=1}^m V_i^{r_i}$. Let $J := \{j \in \{1, \ldots, m\} \mid r_j \geq 1\}$. Then $\ker V = \bigcap_{j \in J} K_j$.

Proof. This follows because $\rho_V(g) = I$ if and only if $\rho_{V_j}(g) = I$ for every $j \in J$. $\square$

The next lemma is a generalisation of part of the argument of Theorem 2.18.6 (there we used the statement below for the case $H = \{1\}$). For $H \leq G$ a subgroup, recall that the *left translation action* $G \times G/H \to G/H$ is given by $g \cdot g'H := gg'H$.

Lemma 3.7.5. Let $N \lhd G$ be a normal subgroup. Let $(\mathbf{C}[G/N], \rho_{\mathbf{C}[G/N]})$ be the representation associated to the left translation action. Then $\ker \rho_{\mathbf{C}[G/N]} = N$.

Caution that the hypothesis that H is normal is necessary (obviously, kernels are always normal).

Proof of Lemma 3.7.5. Given $n \in N$ and $g \in G$, normality implies $ngN = gN$. Hence $n \in \ker \rho_{\mathbf{C}[G/N]}$. On the other hand, if $g \notin H$, then $gN \neq N$, so $\rho_{\mathbf{C}[G/N]}(g) \neq I$. $\square$

Proof of Proposition 3.7.3. By Lemma 3.7.5, $\ker \rho_{\mathbf{C}[G/N]} = N$. By Lemma 3.7.4, it follows that $N = \bigcap_{j \in J'} K_j$ where J' is as defined there. Note that $K_j \leq N$ for $j \in J$, hence $J' \subseteq J$. Therefore $N = \bigcap_{j \in J'} K_j \supseteq \bigcap_{j \in J} K_j$. Since $N \leq K_j$ for all $j \in J$, the reverse inclusion is clear. $\square$

As a result of all of the above, we can give an explicit way to read off the normal subgroups from the character table:

Corollary 3.7.6. The normal subgroups of G are precisely the subgroups N_J of the form

$$N_J := \{n \in G \mid \chi_{V_j}(n) = \chi_{V_j}(1), \forall j \in J\}, \quad J \subseteq \{1, \ldots, m\}. \tag{3.7.7}$$

Proof. By Proposition 3.7.1, since $\dim V_j = \chi_{V_j}(1)$, we have $N_J = \bigcap_{j \in J} K_j$. Since K_j is normal, so is N_J. Conversely, by Proposition 3.7.3, every normal subgroup is obtained in this way. $\square$

Example 3.7.8. Let's take the example of D_8 (note that Q_8 has the same character table!). By Example 3.6.26, the proper nontrivial normal subgroups of the form K_i are $\{1\} \cup \{x^2\} \cup \mathcal{C}$ where $\mathcal{C} \in \{\{x, x^3\}, \{y, x^2y\}, \{xy, x^3y\}\}$. There are three of these, all of size four. The intersection of any two of them is $\{1, x^2\} = Z(D_8)$.

Example 3.7.9. Next consider Q_8. Of course, the character table is the same as that of D_8 so the answer has to be the same, but the elements are written by different labels. By Example 3.6.27, the proper nontrivial normal subgroups K_i are $\{1\} \cup \{-1\} \cup \mathcal{C}$ for $\mathcal{C} \in \{\{\pm i\}, \{\pm j\}, \{\pm k\}\}$. There are again three of these, of size four. The intersection of any two of them is $\{\pm 1\} = Z(Q_8)$.

Example 3.7.10. Consider now A_4. Looking at Example 3.6.29, the only proper nontrivial K_i are both equal to $\{1, (12)(34), (13)(24), (14)(23)\}$, which is the commutator subgroup $[A_4, A_4]$ (i.e., the kernel of $q|_{A_4} : A_4 \to A_3 \cong (A_4)_{\mathrm{ab}}$).

Example 3.7.11. Let's take the example of S_4. By Example 3.6.31, we can get two nontrivial proper subgroups N_j: the subgroup $K_2 = A_4$ for $V_2 = \mathbf{C}_-$ and the subgroup $K_3 = \{1, (12)(34), (13)(24), (14)(23)\}$ for $V_3 = U$. Their intersection is again K_3, so these are all of the proper normal subgroups.

Example 3.7.12. Finally, let's look at A_5. Here we already know this group is simple so has no proper nontrivial normal subgroups, and indeed from Example 3.6.33 we see that all the K_j are trivial (except $K_1 = A_5$ of course)!

3.8 Automorphisms [mostly non-examinable]

The material of this subsection will only be sketched in lecture and is non-examinable, in the sense that you do not need to know any of it to solve the exam questions. However, exam problems could still be related to automorphisms and their action on the character table. Indeed, we even discussed similar results already (particularly in Exercise 2.17.7.(ii) and on CW 1).

The character table also includes partial information about automorphisms of a group G. Indeed, it is obvious that, if we have an automorphism $\varphi : G \to G$, then it will induce a symmetry of the character table, by relabeling the irreducible representations and the conjugacy classes. Let us make this explicit.

Recall that, if $\varphi : G \to G$ is an automorphism and (V, ρ_V) an irreducible representation, then $(V, \rho_V \circ \varphi)$ is also an irreducible representation (by Exercise 2.17.7.(ii)). Therefore, we get a permutation of the rows of the character table:

$$\sigma_\varphi \in S_m, \quad (V_i, \rho_{V_i} \circ \varphi^{-1}) \cong (V_{\sigma_\varphi(i)}, \rho_{V_{\sigma_\varphi(i)}}). \tag{3.8.1}$$

On characters, this says that

$$\chi_{V_i}(\varphi^{-1}(g)) = \chi_{V_{\sigma_\varphi(i)}}(g). \tag{3.8.2}$$

Of course, φ also induces a permutation of the columns:

$$\tau_\varphi \in S_m, \varphi(\mathcal{C}_i) = \mathcal{C}_{\tau_\varphi(i)}. \tag{3.8.3}$$

Putting the previous equations together and replacing g by $\varphi(g_j)$, we get

$$\chi_{V_i}(g_j) = \chi_{V_{\sigma_{\varphi(i)}}}(g_{\tau_\varphi(i)}). \tag{3.8.4}$$

When are these permutations trivial? Recall the following from group theory (also mentioned in Remark 2.17.9):

Definition 3.8.5. An inner automorphism is one of the form $\mathrm{Ad}_g : G \to G, \mathrm{Ad}_g(h) = ghg^{-1}$.

Lemma 3.8.6. The permutations σ_φ and τ_φ are trivial if $\varphi : G \to G$ is an inner automorphism, i.e., $\varphi = \mathrm{Ad}_g, \mathrm{Ad}_g(h) = ghg^{-1}$ for some $g \in G$.

Proof. It is clear that $\varphi = \mathrm{Ad}_g$ preserves all of the conjugacy classes, so $\tau_\varphi = 1$. But then by (3.8.4), $\chi_{V_i} = \chi_{V_{\sigma_\varphi(i)}}$ for all i, which implies σ is also trivial. □

Remark 3.8.7. The proof actually shows that σ_φ and τ_φ are trivial if and only if $\varphi(\mathcal{C}_i) = \mathcal{C}_i$ for all i. Such an automorphism is called a *class-preserving automorphism*. It is a nontrivial fact that there exist finite groups with class-preserving automorphisms that are not inner: see, e.g., `https://groupprops.subwiki.org/wiki/Class-preserving_not_implies_inner#A_finite_group_example`.

Definition 3.8.8. The group $\mathrm{Aut}(G)$ is the group of all automorphisms of G. The subgroup $\mathrm{Inn}(G)$ of inner automorphisms is the group $\{\mathrm{Ad}_g \mid g \in G\}$.

Note that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. Moreover, there is a surjective homomorphism $G \to \mathrm{Inn}(G), g \mapsto \mathrm{Ad}_g$, with kernel $Z(G)$. Thus $G/Z(G) \cong \mathrm{Inn}(G)$ (by the first isomorphism theorem).

Definition 3.8.9. The group of outer automorphisms $\mathrm{Out}(G)$ is defined as $\mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Definition 3.8.10. Let the *character table symmetry group* be the subgroup $\mathrm{CTS}(G) \leq S_m \times S_m$ of all pairs (σ, τ) satisfying (3.8.4).

Remark 3.8.11. We could have used $S_{m-1} \times S_{m-1}$, since the trivial representation and trivial conjugacy class can't change by applying an automorphism.

Proposition 3.8.12. There is a homomorphism $\Psi : \mathrm{Out}(G) \to \mathrm{CTS}(G) \leq S_m \times S_m$, given by $[\varphi] \mapsto (\sigma_\varphi, \tau_\varphi)$.

Proof. It remains only to show that this map is multiplicative, but that follows from the definitions. $\square$

In general, computing all automorphisms of a group is an important and difficult problem. Since the inner ones are easy to describe, the problem reduces to computing the outer automorphisms. The proposition can be a useful tool, since the codomain $\mathrm{CTS}(G)$ is easy to compute from the table. However, Ψ need not be injective or surjective, although it is often injective and its kernel can be described (Remark 3.8.7). It is easy to produce an example where Ψ is not surjective (see Example 3.8.14), but difficult to produce one where it is not injective (see Remark 3.8.7).

Example 3.8.13. Consider $G = Q_8$ following Example 3.6.27. The symmetry group of this character table is S_3: we can't move the row corresponding to the trivial representation nor the one corresponding to the two-dimensional representation, but the three rows corresponding to nontrivial one-dimensional representations can all be permuted with a corresponding permutation of the columns, without changing the table.

Here the automorphism group is easy to compute: if φ is an automorphism, $\varphi(i) \in \{\pm i, \pm j, \pm k\}$, and $\varphi(j)$ can also be any of these other than $\pm\varphi(i)$. That makes a total of $6 \cdot 4 = 24$ automorphisms. The inner automorphism group is isomorphic to $G/Z(G) = G/\{\pm 1\} \cong C_2 \times C_2$, so $\mathrm{Out}(G)$ has size six. There is a surjective homomorphism $\mathrm{Out}(G) \to S_3$ given by permutations of the collection of conjugacy classes $\{\mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5\} = \{\{\pm i\}, \{\pm j\}, \{\pm k\}\}$, which is therefore an isomorphism.

So the homomorphism $\Psi : \mathrm{Out}(G) \to \mathrm{CTS}(G)$ is an isomorphism.

Example 3.8.14. Now consider $G = D_8$ following Example 3.6.26. The character table is the same as for Q_8 so $\mathrm{CTS}(G) \cong S_3$ again. But now the outer automorphism group is smaller: first note that since $x, x^3 = x^{-1}$ are the only elements of order 4, any automorphism φ satisfies $\varphi(x) = x^{\pm 1}$. Therefore $\varphi(y) = x^a y$ for some a. Conversely any such assignment extends to an automorphism. So $|\operatorname{Aut}(D_8)| = 2 \cdot 4 = 8$. The inner automorphism group is again of size four, so $|\operatorname{Out}(D_8)| = 2$. There is only one nontrivial outer automorphism, and one representative (modulo inner automorphisms) is the automorphism $\varphi(x) = x, \varphi(y) = xy$. This gives a nontrivial symmetry of the character table: the one swapping the last two columns and the third and fourth rows. So $\Psi : C_2 \cong \operatorname{Out}(D_8) \to \mathrm{CTS}(D_8) \cong S_3$ is injective but not surjective.

Example 3.8.15. Let's consider A_4, following Example 3.6.29. The second and third row can be permuted under an automorphism, provided we also permute the third and fourth columns. So $\mathrm{CTS}(A_4) \cong C_2$. Indeed there is an automorphism realising this symmetry: the map $\varphi(\sigma) = (12)(\sigma)(12)$, a non-inner automorphism which comes from conjugation inside the larger group S_4. So $\Psi : \operatorname{Out}(A_4) \to \mathrm{CTS}(A_4) \cong C_2$ is surjective. We claim that it is also injective, so that $\operatorname{Out}(A_4) \cong C_2$ as well, generated by conjugation by an odd element of S_4.

To prove the claim, by Remark 3.8.7, it is equivalent to show that every class-preserving automorphism is inner. Let φ be class-preserving. By composing with an inner automorphism we can assume that $\varphi((123)) = (123)$. Now $\mathrm{Ad}_{(123)}$ induces a cyclic permutation of the elements of $[(123)]$ other than (123) itself, so composing with this we can also assume $\varphi((134)) = (134)$. But (123) and (134) generate A_4 (taking conjugation they get the whole class $[(123)]$, and taking inverses gives $[(132)]$; the three-cycles generate A_4 by Remark 2.13.20).

Example 3.8.16. Let's consider S_4, following Example 3.6.31. There are no rows which can be permuted. So $\mathrm{CTS}(S_4) = \{1\}$ is trivial. We claim that Ψ is injective, and hence $\operatorname{Out}(S_4) = \{1\}$.

To prove this, we need to show again that any class-preserving automorphism is inner. By composing such an automorphism φ with an inner one, we can assume that $\varphi((12)) = (12)$. Then (23) maps to a transposition (ab) which does not commute with (12), so of the form $(1m)$ or $(2m)$ for $3 \leq m \leq 4$. Composing with $\mathrm{Ad}_{(12)}$ if necessary we can assume it is $(1m)$, and composing with a permutation of $\{3, 4\}$ we can assume $m = 3$, so $\varphi((23)) = (23)$ as well. Then $\varphi((34))$ has to be a transposition commuting with (12), which is not (12), so $\varphi((34)) = (34)$ as well. Therefore φ fixes $(12), (23)$, and (34), but these elements generate S_4, so we see actually that φ is the identity.

Example 3.8.17. Now consider $G = A_5$. Looking at Example 3.6.33, we see again that $\mathrm{CTS}(A_5) \cong C_2$, generated by the outer automorphism of conjugation by an odd permutation such as (12). So $\Psi : \operatorname{Out}(A_5) \to \mathrm{CTS}(A_5) \cong C_2$ is surjective. Similarly to Example 3.8.15, we can prove also that Ψ is injective, so that $\operatorname{Out}(A_5) \cong C_2$, generated by the adjoint action of an odd permutation in S_5.

Example 3.8.18. For general $G = A_n, S_n$, with $n \neq 2, 6$, it turns out that $\operatorname{Out}(A_n) \cong C_2$, generated again by the conjugation by an odd element of S_n, and $\mathrm{Out}(S_n) = \{1\}$. The map Ψ is always injective.

The interesting thing happens when $n = 6$. Let's look at the character table of S_6. This was generated by Magma. The conjugacy classes here are, in order,

$$([1], [(12)], [(12)(34)(56)], [(12)(34)], [(123)], [(123)(456)], \\ [(1234)(56)], [(1234)], [(12345)], [(123)(45)], [(123456)]). \quad (3.8.19)$$

Note below that the reflection representation is V_6.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$	$\mathcal{C}_{11}$
$\lvert\mathcal{C}_i\rvert$	1	15	15	45	40	40	90	90	144	120	120
V_1	1	1	1	1	1	1	1	1	1	1	1
V_2	1	-1	-1	1	1	1	1	-1	1	-1	-1
V_3	5	-1	3	1	-1	2	-1	1	0	-1	0
V_4	5	-3	1	1	2	-1	-1	-1	0	0	1
V_5	5	1	-3	1	-1	2	-1	-1	0	1	0
V_6	5	3	-1	1	2	-1	-1	1	0	0	-1
V_7	9	3	3	1	0	0	1	-1	-1	0	0
V_8	9	-3	-3	1	0	0	1	1	-1	0	0
V_9	10	2	-2	-2	1	1	0	0	0	-1	1
V_{10}	10	-2	2	-2	1	1	0	0	0	1	-1
V_{11}	16	0	0	0	-2	-2	0	0	1	0	0

It is easy to see that the symmetry group of this table is C_2: first, the symmetry group is determined by which column the second column (the class $[(12)]$) maps to, since the only permutation of rows fixing the second column is the swap of V_6 and V_7, which can't happen since the dimensions aren't equal. Then, visibly, the only column that we can swap this one with is the third column. If we do that we have to use the row permutation $\sigma = (36)(45)(9,10) \in S_{11}$. Doing this row permutation we indeed get a symmetry with column permutation $\tau = (23)(56)(10,11)$. [Observe that the even conjugacy classes are preserved, which one can see already by using row V_2 which cannot be swapped with any other row.]

Thus this leads us to suspect there is an automorphism of S_6 which swaps conjugacy classes $[(12)]$ with $[(12)(34)(56)]$, $[(123)]$ with $[(123)(456)]$, and $[(123)(45)]$ with $[(123456)]$. *Indeed this is the case:* it is called the *exotic automorphism* of S_6 (defined up to composing with an inner automorphism), and there are many beautiful constructions of it. *This is the only symmetric group with* $\mathrm{Out}(S_n) \neq \{1\}$*!*

Next let's look at the character table of A_6, which is more manageable. Again I generated this with Magma. The conjugacy classes here are, in the order appearing below,

$$[1], [(12)(34)], [(123)], [(123)(456)], [(1234)(56)], [(12345)], [(13245)]. \quad (3.8.20)$$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
$\lvert\mathcal{C}_i\rvert$	1	45	40	40	90	72	72
V_1	1	1	1	1	1	1	1
V_2	5	1	2	-1	-1	0	0
V_3	5	1	-1	2	-1	0	0
V_4	8	0	-1	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
V_5	8	0	-1	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
V_6	9	1	0	0	1	-1	-1
V_7	10	-2	1	1	0	0	0

Here it is easy to check that

$$\mathrm{CTS}(A_5) = \{[1,1], [((23),(34))], [((45),(67))], [((23)(45),(34)(67))]\} \cong C_2 \times C_2. \quad (3.8.21)$$

The element $((45),(67))$ is $\Psi(\mathrm{Ad}_{(12)})$ (coming from the adjoint action of S_6) whereas the other two nontrivial elements are Ψ applied to restrictions of exotic automorphisms of S_6. Note that the latter yield two distinct elements of $\mathrm{Out}(A_6)$, since we can compose any one exotic automorphism with $\mathrm{Ad}_{(12)}$ to get another exotic automorphism (and restricted to A_6 they are only the same modulo $\mathrm{Inn}(S_6)$, not modulo $\mathrm{Inn}(A_6)$).

4 Algebras and modules

4.1 Definitions and examples

Definition 4.1.1. A $\mathbf{C}$-algebra is a tuple $(A, m_A, 1_A)$ of a vector space A, a multiplication, $m_A : A \times A \to A$, and an element $1_A \in A$, satisfying the following for all $a, b, c \in A$ and $\lambda \in \mathbf{C}$:

- Bilinearity: $\lambda\, m_A(a,b) = m_A(\lambda a, b) = m_A(a, \lambda b)$ and:
  - Distributivity: $m_A(a, b+c) = m_A(a,b) + m_A(a,c)$.
- Associativity: $m_A(a, m_A(b,c)) = m_A(m_A(a,b),c)$ for $a, b, c \in A$;
- Identity: $m_A(1_A, a) = a = m_A(a, 1_A)$;

We will usually write $m_A(a,b)$ as $a \cdot b$ or simply ab. We omit the subscript A of 1_A and just write 1 whenever there is no confusion. The algebra itself is denoted by A rather than by $(A, m_A, 1_A)$ with the additional structure understood.

Observe that there are *three operations* on an algebra: multiplication, scalar multiplication, and addition (since the latter two come from being a vector space).

Remark 4.1.2. The multiplication and scalar multiplication are *distinct operations*: the first is a map $A \times A \to A$ and the second $\mathbf{C} \times A \to A$. Nonetheless there is a relation: for $\lambda \in \mathbf{C}$, we have the element $\lambda 1_A \in A$, and for all $a \in A$, $\lambda a = (\lambda 1_A)a$. We will state this more formally in Example 4.1.10.

Example 4.1.3. $\mathbf{C}$ itself is an algebra with its usual multiplication.

Example 4.1.4. Given two algebras A and B, the direct sum $A \oplus B := A \times B$ is an algebra, with componentwise multiplication:

$$(a, b) \cdot (a', b') = (aa', bb') \tag{4.1.5}$$

In particular, $\mathbf{C} \oplus \mathbf{C}$ is an algebra.

Example 4.1.6. The vector space $\mathbf{C}[x] := \{\sum_{i=0}^{m} a_i x^i \mid a_i \in \mathbf{C}, a_m \neq 0\} \cup \{0\}$ of polynomials with complex coefficients is an algebra with the usual polynomial multiplication, scalar multiplication, and addition.

Example 4.1.7. The vector space $A = \mathbf{C}[\varepsilon]/(\varepsilon^2) := \{a + b\varepsilon \mid a, b \in A\}$ with multiplication $(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon$ is an algebra. The multiplication is determined by the identities $1 + 0\varepsilon = 1_A$ and $\varepsilon^2 = 0$. This is just like the preceding example but cutting off (setting to zero) terms x^2 and higher. This algebra is important in differential calculus, since it expresses the idea of working only to first order (ε is so small that actually $\varepsilon^2 = 0$).

Definition 4.1.8. An algebra homomorphism $\varphi : A \to B$ is a linear map which is multiplicative and sends 1_A to 1_B. It is an isomorphism if there is an inverse algebra homomorphism $\varphi^{-1} : B \to A$.

Exercise 4.1.9. As for groups and vector spaces (and group representations), show that $\varphi : A \to B$ is an isomorphism if and only if it is bijective. The point is that, if φ is bijective, the inverse map $\varphi^{-1} : B \to A$ must be an algebra homomorphism.

Example 4.1.10. If A is any algebra, then there is always an algebra homomorphism $\mathbf{C} \to A$ given by $\lambda \mapsto \lambda 1_A$. This explains the observation in Remark 4.1.2.

All of the preceding examples have been *commutative*. Now let's consider some noncommutative examples.

Example 4.1.11. The vector space of $n \times n$ matrices, $\mathrm{Mat}_n(\mathbf{C})$, is an algebra under matrix multiplication.

Example 4.1.12. Let V be a vector space. Then $\mathrm{End}(V)$ is an algebra, with multiplication given by composition.

Example 4.1.13. Given a basis $\mathcal{B}$ of V with $\dim V = n$ finite, the map writing endomorphisms in bases produces an algebra isomorphism $\mathrm{End}(V) \xrightarrow{\sim} \mathrm{Mat}_n(\mathbf{C})$, $S \mapsto [S]_{\mathcal{B}}$. You already know this is a vector space isomorphism, and it is multiplicative since $[S]_{\mathcal{B}}[T]_{\mathcal{B}} = [S \circ T]_{\mathcal{B}}$ (2.4.10). So Examples 4.1.11 and 4.1.12 are isomorphic (for $\dim V = n$).

Example 4.1.14. Let G be a finite group. The vector space $\mathbf{C}[G]$ is an algebra with the multiplication linearly extended from the group multiplication. Explicitly:

$$\Big(\sum_{g \in G} a_g g\Big)\Big(\sum_{g \in G} b_g g\Big) = \Big(\sum_{g \in G}\Big(\sum_{h \in G} a_h b_{h^{-1}g}\Big) g\Big). \tag{4.1.15}$$

Remark 4.1.16 (Non-examinable)**.** As in Remark 2.6.6, the above definition also extends to the case G is infinite, provided we require that all sums have all but finitely many coefficients a_g and b_g nonzero ($\mathbf{C}[G] = \{\sum_{g \in G} a_g g \mid a_g \in \mathbf{C}, \text{all but finitely many } a_g = 0\}$).

Finally, let us give some geometric examples, of *commutative* algebras of functions.

Example 4.1.17. (Gelfand-Naimark duality) For $X \subseteq \mathbf{R}^n$, say for example the sphere $S^2 \subseteq \mathbf{R}^3$, we can consider the algebra A of continuous (or differentiable) complex-valued functions on X. It turns out that, in suitable cases (X is compact Hausdorff), the study of X is equivalent to that of A (Gelfand-Naimark duality)! *Unlike the group and matrix algebras, A is commutative.*

Example 4.1.18. (Algebraic Geometry) Similarly, given a collection of polynomial equations $f_1 = \cdots = f_m = 0$ in n variables $x_1, \ldots, x_n$, we can consider the zero locus $X \subseteq \mathbf{C}^n$. Again, it turns out that the geometry of X is completely captured by the algebra $A = \mathbf{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, defined as the set of polynomials in n variables $x_1, \ldots, x_n$ modulo the relation that all polynomials f_i (and their multiples) are zero. The addition, multiplication, and scalar multiplication are all as for polynomials without relations. This is a central idea of algebraic geometry. Note as in the preceding example that A is always commutative.

Definition 4.1.19. Given an algebra A, we define the opposite algebra by reversing the order of multiplication: $A^{\mathrm{op}} = A, 1_{A^{\mathrm{op}}} = 1_A$, but

$$m_{A^{\mathrm{op}}}(a, b) = m_A(b, a). \tag{4.1.20}$$

Remark 4.1.21. In the case of a group, we can also define the "opposite group" G^{op} by reversing the multiplication, but it isn't really interesting since inversion gives an isomorphism $G \to G^{\mathrm{op}}, g \mapsto g^{-1}$. For algebras we can't invert in general (e.g., not all matrices are invertible) so this is not an option.

Example 4.1.22. In the case of the group algebra, there is an isomorphism $\iota : \mathbf{C}[G] \xrightarrow{\sim} \mathbf{C}[G]^{\mathrm{op}}$ given by $\iota\big(\sum_{g \in G} a_g g\big) = \sum_{g \in G} a_g g^{-1}$, i.e., inverting the elements $g \in G$ of the group.

Remark 4.1.23. *Caution:* the map ι is *not* inverting all elements of $\mathbf{C}[G]$! As a trivial example, 0 has no inverse: $0 \cdot v = 0 \neq 1$ for all $v \in \mathbf{C}[G]$ (and indeed for any algebra). For a less trivial example, $e := |G|^{-1} \sum_{g \in G} g$ has the property $e\iota(e) = e^2 = e \neq 1$ if $G \neq \{1\}$.

Example 4.1.24. We can also produce an isomorphism $\mathrm{Mat}_n(\mathbf{C})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Mat}_n(\mathbf{C})$ by applying matrix transpose: $M \mapsto M^t$.

Example 4.1.25 (Non-examinable)**.** By the preceding example and Example 4.1.13, if V is any vector space, we can produce an isomorphism $\mathrm{End}(V)^{\mathrm{op}} \cong \mathrm{End}(V)$ from a choice of basis. This isomorphism depends on the choice of basis. However, one can produce a basis-independent isomorphism $\mathrm{End}(V)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}(V^*)$ by dualising endomorphisms, which as we have seen becomes the transpose operation on matrices after choosing a basis.

4.2 Modules

Modules are essentially the same thing as representations, but working more generally over algebras rather than groups (in the case of the group algebra, it will just reduce to the case of group representations).

Definition 4.2.1. A left module over an algebra A is a pair (V, ρ_V) where $\rho_V : A \to \text{End}(V)$ is an algebra homomorphism. A right module over A is a left module over A^{op}. A homomorphism of modules $T : V \to W$ is a linear map compatible with the actions: $T \circ \rho_V(a) = \rho_W(a) \circ T$. Such a homomorphism is also called an A-linear map. Let $\text{Hom}_A(V, W)$ denote the vector space of homomorphisms and $\text{End}_A(V)$ the vector space of endomorphisms.

This notation is close to that of group representations. Just as we can also think of a representation V of G as a map $G \times V \to V$, one traditionally defines left and right modules instead as bilinear, associative maps $A \times V \to V$ and $W \times A \to W$, respectively. The correspondence is again $(a, v) \mapsto \rho_V(a)(v)$ and $(w, a) \mapsto \rho_W(a)(w)$, for algebra homomorphisms $\rho_V : A \to \text{End}(V)$ and $\rho_W : A^{\text{op}} \to \text{End}(W)$.

Exercise 4.2.2. Show that the traditional definition of left and right modules is equivalent to Definition 4.2.1.

We won't really need to use right modules or opposite algebras much, but it would be a mistake not to see the definition from the start, due to their general importance and the symmetry of the definition (at least in the traditional sense).

Just as for representations, we have:

Definition 4.2.3. A submodule W of a module V is a linear subspace such that $\rho_V(a)(W) \subseteq W$ for all $a \in A$. A module is called simple, or irreducible, if it does not contain a proper nonzero submodule. A quotient module of V is a quotient V/W for $W \subseteq V$ a submodule.

It is clear that, if $W \subseteq V$ is a submodule, then $\rho_V : A \to \text{End}(V)$ determines a homomorphism $\rho_W : A \to \text{End}(W)$, by restricting each $\rho_V(a)$ to W. Similarly, we obtain $\rho_{V/W} : A \to \text{End}(V/W)$ by $\rho_{V/W}(a)(v + W) := \rho_V(a)(v) + W$.

As before we have:

Proposition 4.2.4. If $T : (V, \rho) \to (V', \rho')$ is a homomorphism of left modules, then $\ker(T)$ and $\text{im}(T)$ are submodules of V and V', respectively. The first isomorphism theorem holds: $\text{im}(T) \cong V/\ker(T)$.

The proof is the same as for groups, but we provide it for illustration:

Proof. If $T(v) = 0$ then $T(\rho_V(a)v) = \rho_{V'}(a)T(v) = 0$ for all $a \in A$, thus $\ker(T)$ is a submodule of V. Also, for all $v \in V$, $\rho_{V'}(a)T(v) = T(\rho_V(a)v) \in \text{im}(T)$, so $\text{im}(T)$ is a submodule of V'. $\square$

Then, Schur's Lemma 2.11.1 still holds, with the same proof:

Lemma 4.2.5 (Schur's Lemma)**.** Let (V, ρ_V) and (W, ρ_W) be simple left modules for an algebra A.

(i) If $T : V \to W$ is an A-linear map, then T is either an isomorphism or the zero map.

(ii) Suppose V is finite-dimensional. If $T : V \to V$ is A-linear then $T = \lambda I$ for some $\lambda \in \mathbf{C}$.

We omit the proof this time.

Definition 4.2.6. A direct sum of modules $\bigoplus_{i=1}^n V_i$ is the direct sum of vector spaces, equipped with the product action, $a(v_1, \ldots, v_n) = (av_1, \ldots, av_n)$. A module isomorphic to a direct sum of two nonzero modules is called decomposable, and otherwise it is called indecomposable. A module isomorphic to a direct sum of simple modules is called semisimple.

Example 4.2.7. Let $A = \mathrm{Mat}_n(\mathbf{C})$. Then $V = \mathbf{C}^n$ is a left module, with $\rho_{\mathbf{C}^n} : \mathrm{Mat}_n(\mathbf{C}) \to \mathrm{End}(\mathbf{C}^n)$ the map we have seen before (in Section 2.5): $\rho_{\mathbf{C}^n}(M)(v) = Mv$. In other words $A \times V \to V$ is the left multiplication map. This module is simple, since given any $v, v' \in V = \mathbf{C}^n$, there exists $M \in A = \mathrm{Mat}_n(\mathbf{C})$ such that $Mv = v'$.

Example 4.2.8. Similarly, let $A = \mathrm{End}(V)$. Then V is itself a module with $\rho_V : A \to \mathrm{End}(V)$ the identity map. Let $\dim V = n$ be finite and take a basis $\mathcal{B}$ of V. Then the isomorphism $\mathrm{End}(V) \cong \mathrm{Mat}_n(\mathbf{C})$ of Example 4.1.13 produces the previous example: we have $[S]_{\mathcal{B}}[v]_{\mathcal{B}} = [S(v)]_{\mathcal{B}}$. So the algebra isomorphism and the linear isomorphism $V \xrightarrow{\sim} \mathbf{C}^n$ are compatible with the module structures on either side. In particular, V is also an simple left module over $\mathrm{End}(V)$.

Example 4.2.9. Let $A = \mathbf{C}[G]$ be a group algebra and V a representation. Then we obtain a left module $(V, \widetilde{\rho}_V)$ by extending $\rho_V : G \to \mathrm{GL}(V)$ linearly to $\widetilde{\rho}_V : \mathbf{C}[G] \to \mathrm{End}(V)$. (A linear combination of elements of $\mathrm{GL}(V)$ is always still in the vector space $\mathrm{End}(V) \supset \mathrm{GL}(V)$ although it need not be invertible.)

Conversely, if $(V, \widetilde{\rho}_V)$ is a left module over $A = \mathbf{C}[G]$, then $(V, \widetilde{\rho}_V|_G)$ is a representation. This is because $\widetilde{\rho}_V(g)\widetilde{\rho}(g^{-1}) = I = \widetilde{\rho}_V(g^{-1})\widetilde{\rho}(g)$, which implies that $\widetilde{\rho}(g) \in \mathrm{GL}(V)$.

These operations are inverse to each other. Informally speaking, representations of groups are equivalent to left modules of their group algebras (this can be formalised and is true, but we won't explain it).

Example 4.2.10. Consider again the algebra $A = \mathbf{C}[\varepsilon]/(\varepsilon^2)$ of Example 4.1.7. Then A itself, as a left A-module, is indecomposable but not simple. Indeed, the only proper nonzero left submodule of A is $\{b\varepsilon \mid b \in \mathbf{C}\}$. This can be seen because every element of A is a multiple of any polynomial of the form $a + b\varepsilon$ with $a \neq 0$.

Exercise 4.2.11. Prove the claims of the previous example.

Example 4.2.12. Given left modules $(V, \rho_V), (W, \rho_W)$ over algebras A, B, we obtain a left module $(V \oplus W, \rho_{V \oplus W})$ over $A \oplus B$ with $\rho_{V \oplus W}(a, b) = \rho_V(a) + \rho_W(b)$. (This actually includes V and W themselves if we set the other one to be the zero module, the approach taken in CW2, #5).

Exercise 4.2.13. Expanding on CW2, #5.(a), show that every left module U over $A \oplus B$ is of the form Example 4.2.12, for unique submodules $V, W \subseteq U$. Namely, $V = (1,0)U$ and $W = (0,1)U$.

4.3 Group algebras as direct sums of matrix algebras

Let G be a finite group and $V_1, \ldots, V_m$ a full set of irreducible representations. Thanks to Theorem 2.18.6, we have a linear isomorphism

$$\Phi : \mathbf{C}[G] \cong \bigoplus_{i=1}^{m} \mathrm{End}(V_i), \quad \Phi(g) = (\rho_{V_1}(g), \ldots, \rho_{V_m}(g)) \forall g \in G, \tag{4.3.1}$$

extended linearly to all of $\mathbf{C}[G]$.

Theorem 4.3.2. This isomorphism Φ is an algebra isomorphism.

Proof. This follows from the fact that each $\rho_{V_i} : \mathbf{C}[G] \to \mathrm{End}(V_i)$ is an algebra homomorphism (Example 4.2.9), by the definition of the direct sum (Example 4.1.4). $\square$

4.4 Representations of matrix algebras

By Maschke's theorem, finite-dimensional representations of finite groups are direct sums of copies of the irreducible representations. On the other hand representations of (finite) groups are the same as of their group algebras. By Theorem 4.3.2, the group algebra of a finite group is a direct sum of matrix algebras, one for each irreducible representation. This leads us to suspect (by Exercise 4.2.13) the following theorem:

Theorem 4.4.1. Let $n \geq 1$. Then every finite-dimensional left module over $\mathrm{Mat}_n(\mathbf{C})$ is isomorphic to $(\mathbf{C}^n)^r$ for some $r \geq 0$.

By Example 4.2.8, this immediately implies the following:

Corollary 4.4.2. If V is any finite-dimensional vector space, then every left module over $\mathrm{End}(V)$ is isomorphic to V^r for some $r \geq 0$.

In terms of dimension, we can restate the result as follows:

Corollary 4.4.3. (a) If V is a finite-dimensional left module over $\mathrm{Mat}_n(\mathbf{C})$ then $U \cong (\mathbf{C}^n)^{\dim U/n}$. (b) If W is a finite-dimensional left module over $\mathrm{End}(V)$, with V finite-dimensional, then $W \cong V^{\dim W/\dim V}$.

We are going to give more general results that imply Theorem 4.4.1 later. One can also give a proof directly using Maschke's theorem together with Theorem 4.3.2: see CW2, #5. Therefore we will omit a direct proof in lecture.

Nonetheless, in these notes, we will provide a nice direct proof which has the advantage of being generalisable to other situations (replacing the element $e_{11} \in \mathrm{Mat}_n(\mathbf{C})$ in the proof by elements $e \in A$ satisfying $e^2 = 1$ and $AeA = A$).

Proof of Theorem 4.4.1. (Non-examinable) Let (V, ρ_V) be a representation of $A = \mathrm{Mat}_n(\mathbf{C})$. For ease of notation we write $av := \rho_V(a)v$. Consider the vector subspace $U := \mathrm{Span}(e_{11}v)_{v \in V} \subseteq V$. Let $(u_1, \ldots, u_m)$ be a basis of U. We claim that (a) $Au_i \cong \mathbf{C}^n$ for every i, and (b) $V = \bigoplus_{i=1}^m Au_i$. This clearly implies the theorem.

For part (a), fix i. Let u_i' be such that $u_i = e_{11}u_i'$. First of all, multiplying by e_{11}, we see that

$$e_{11}u_i = e_{11}^2 u_i' = e_{11}u_i' = u_i. \tag{4.4.4}$$

So we can actually assume $u_i' = u_i$. Next,

$$e_{ij}u_i = e_{ij}e_{11}u_i = \delta_{j1}e_{i1}u_i. \tag{4.4.5}$$

So Au_i is spanned by $v_{ji} := e_{j1}u_i$, for $1 \leq j \leq n$. Now,

$$e_{k\ell}v_{ji} = e_{k\ell}e_{j1}u_i = \delta_{j\ell}e_{k1}u_i = \delta_{j\ell}v_{ki}. \tag{4.4.6}$$

This is the same formula as the action of $A = \mathrm{Mat}_n(\mathbf{C})$ on $\mathbf{C}^n$, i.e., we have a surjective module homomorphism

$$\mathbf{C}^n \to Au_i, \quad e_j \mapsto v_{ji}. \tag{4.4.7}$$

Since $\mathbf{C}^n$ is simple (Example 4.2.7), Schur's Lemma (Lemma 4.2.5) implies that this map is either zero or an isomorphism. But it cannot be zero since $u_i \neq 0$. So it is an isomorphism. This proves (a).

For (b), since a basis of each Au_i is given by v_{ji}, it suffices to show that the v_{ji} together form a basis of V. Let us first show the spanning property. For general $v \in V$, we have

$$v = Iv = \sum_{j=1}^n e_{jj}v = \sum_{j=1}^n e_{j1}e_{11}e_{1j}v. \tag{4.4.8}$$

For each j, we have $e_{11}e_{1j}v \in U$, so can write $e_{11}e_{1j}v = \sum_{i=1}^m \lambda_{ij}u_i$ for some $\lambda_{ij} \in \mathbf{C}$. Then $e_{j1}e_{11}e_{1j}v = \sum_{i=1}^m \lambda_{ij}v_{ji}$. By (4.4.8) we obtain $v \in \mathrm{Span}(v_{ji})$.

Finally we show the linear independence property. Suppose that

$$\sum_{i,j} \lambda_{ji}v_{ji} = 0, \lambda_{ji} \in \mathbf{C}. \tag{4.4.9}$$

Fix $j \in \{1, \ldots, n\}$, and multiply on the left by e_{1j}. We obtain (by (4.4.6)):

$$\sum_{i=1}^m \lambda_{ji}u_i = 0. \tag{4.4.10}$$

Since the u_i are linearly independent, we get that $\lambda_{ji} = 0$ for all i. Since j was arbitrary, all the $\lambda_{ji} = 0$ as desired. □

The theorem can be interpreted as saying that all of the $\mathrm{Mat}_n(\mathbf{C})$ have the same module theory: all left modules are direct sums of copies of a single simple module. This notion can be formalised and we say that the $\mathrm{Mat}_n(\mathbf{C})$ are all *Morita equivalent* for all n.

Remark 4.4.11 (Non-examinable)**.** Two algebras A, B are called Morita equivalent if there is an equivalence of categories between their categories of left modules. This means, essentially, that there is a way to associate a left B-module (W, ρ_W) to every left A-module (V, ρ_V) and vice-versa, as well as a compatible way to associate homomorphisms of left B-modules to homomorphisms of left A-modules, such that if we apply this association twice we get something isomorphic to what we started with.

Remark 4.4.12. By Theorems 4.3.2, 4.4.1, and Exercise 4.2.13, two group algebras of finite groups are Morita equivalent if and only if they have the same number of irreducible representations (i.e., the same number of conjugacy classes).

4.5 Semisimple algebras

By Maschke's theorem, group algebras $\mathbf{C}[G]$ of finite groups G have the property that all finite-dimensional left modules are direct sums of simple modules. Theorem 4.4.1 states that the same is true for matrix algebras. By Example 4.2.13, we can take direct sums (or summands) of algebras with this property to get new ones. It is interesting to characterise such algebras. The next theorem shows how to do this and also that they all have the same behaviour as $\mathbf{C}[G]$ itself:

Theorem 4.5.1. The following are equivalent for a finite-dimensional algebra A:

(i) A is semisimple as a left module over itself;

(ii) Every finite-dimensional left A-module is semisimple.

In this case, every simple left module is isomorphic to a submodule of A.

Definition 4.5.2. A finite-dimensional algebra A is semisimple if it satisfies the equivalent conditions of the theorem.

Remark 4.5.3 (Non-examinable, omit from lectures)**.** Another equivalent condition is:

(iii) A is a direct sum of simple algebras, i.e., algebras that have no two-sided ideals other than 0 or A.

Here, A two-sided ideal is a vector subspace $I \subseteq A$ such that $I \cdot A \subseteq I$ and $A \cdot I \subseteq A$. Definition (iii) gives another explanation for the term "semisimple" that explains what it would mean to be actually "simple". The fact that (iii) is equivalent to (i) and (ii) can be deduced from the results of this subsection.

In the case that A is infinite-dimensional, definitions (iii) and (ii) are no longer equivalent. It is still true that (iii) implies (ii), but the reverse implication is false. For example, take the algebra $A = \mathbf{C}\langle x, D\rangle/(Dx - xD - 1)$, thought of as differential operators with polynomial coefficients in one variable (where $D = \frac{d}{dx}$). Then A has no proper nonzero two-sided ideals, so it satisfies (iii). It has no finite-dimensional representations at all, so also satisfies (ii). Now take any algebra B. Then $A \otimes B$ is also an algebra with no finite-dimensional modules. But it need not be a direct sum of algebras admitting no proper nonzero two-sided ideals. For instance, $B = \mathbf{C}[\varepsilon]/(\varepsilon^2)$ does the trick. So it satisfies (ii) but not iii).

This cannot be fixed by removing the word "finite-dimensional" from (ii): in this case (iii) no longer implies it. Take the algebra $A = \mathbf{C}\langle x, D\rangle/(Dx - xD - 1)$ already given. There are many infinite-dimensional, non-semisimple left A-modules, such as $V = \mathbf{C}[x, x^{-1}]$ with action $x \cdot f = x$ and $D \cdot f = \frac{df}{dx}$. The latter example has a unique simple submodule, $\mathbf{C}[x] \subseteq \mathbf{C}[x, x^{-1}] = V$, so it is indecomposable but not simple.

To prove Theorem 4.5.1, we begin with some general results, interesting in themselves.

Definition 4.5.4. If $U \subseteq V$ is a submodule, then a complementary submodule is a submodule $W \subseteq V$ such that $V = U \oplus W$.

Proposition 4.5.5. Let A be an algebra and $V = V_1 \oplus \cdots \oplus V_m$ for simple left modules V_i. Then every submodule $U \subseteq V$ has a complementary submodule of the form $V_I = \bigoplus_{i \in I} V_i$, $I \subseteq \{1, \ldots, m\}$.

Proof. Let $I \subseteq \{1, \ldots, m\}$ be maximal such that $U \cap V_I = \{0\}$. We claim that $V = U \oplus V_I$. If not, for some i, we have $V_i \not\subseteq U + V_I$. Since V_i is simple, $V_i \cap (U + V_I) = \{0\}$ and hence $U \cap (V_i + V_I) = \{0\}$, contradicting maximality. $\square$

Corollary 4.5.6. Every submodule and quotient module of a finite-dimensional semisimple left module is also semisimple. In the notation of the proposition, they are all isomorphic to V_I for some I.

Proof. Use the notation of the proposition. We need to show that U and V/U are semisimple. The composition $V_I \to V \to V/U$ is an isomorphism, since $V = V_I \oplus U$. But V_I is semisimple. Hence V/U is semisimple. By the same reasoning, $U \to V \to V/V_I$ is an isomorphism. By the first statement, V/V_I is semisimple, and isomorphic to V_J for some J (we can take $J = I^c$, the complement of I). $\square$

Lemma 4.5.7. Let V be an n-dimensional left module over an algebra A (with n finite). Then V is a quotient module of A^n.

Proof. Let $v_1, \ldots, v_n$ be a basis. Then we obtain a surjective module homomorphism $A^n \to V$ by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n \rho_V(a_i) v_i$. $\square$

Proof of Theorem 4.5.1. The implication (ii) $\Rightarrow$ (i) is obvious, so we only have to show (i) $\Rightarrow$ (ii). Assume (i). Let V be a finite-dimensional module. It is a quotient of A^n for some $n \geq 1$ by Lemma 4.5.7. Since A^n is semisimple, the result follows from Corollary 4.5.6. $\square$

We can apply this to give another proof of Theorem 4.4.1:

Lemma 4.5.8. The matrix algebra $\mathrm{Mat}_n(\mathbf{C})$, as a module over itself, is isomorphic to $(\mathbf{C}^n)^n$.

Proof. Let $1 \leq i \leq n$ and let $V_i \subseteq \mathrm{Mat}_n(\mathbf{C})$ be the subspace of matrices which are zero outside the first column. Then V_i is a submodule isomorphic to $\mathbf{C}^n$, since left-multiplication on V_i is identical to on a single column. And, $\mathrm{Mat}_n(\mathbf{C}) = \bigoplus_{i=1}^n V_i$. $\square$

Proof of Theorem 4.4.1. Let V be a finite-dimensional left module over $A = \mathrm{Mat}_n(\mathbf{C})$. By Lemma 4.5.8, $A^m \cong (\mathbf{C}^n)^{mn}$ for all $m \geq 1$. The result follows from Theorem 4.5.1. $\square$

Next we show that all finite-dimensional semisimple algebras are sums of matrix algebras:

Theorem 4.5.9 (Artin-Wedderburn)**.** The following properties are equivalent for a finite-dimensional algebra A:

(i) A is semisimple;

(ii) A is isomorphic to a direct sum of matrix algebras $A \cong \bigoplus_{i=1}^m \mathrm{Mat}_{n_i}(\mathbf{C})$;

(iii) There is a full set $V_1, \ldots, V_m$ of simple left A-modules and the map $\Phi : A \to \bigoplus_{i=1}^m \mathrm{End}(V_i)$ is an isomorphism.

Corollary 4.5.10. If A is semisimple, and $V_1, \ldots, V_m$ a full set of simple left A-modules, then $\dim A = (\dim V_1)^2 + \cdots + (\dim V_m)^2$.

To prove the theorem we need the following lemmas:

Lemma 4.5.11. There is an algebra isomorphism $\iota : \mathrm{End}_A(A) \cong A^{\mathrm{op}}, \varphi \mapsto \varphi(1)$.

Proof. Every $\varphi \in \mathrm{End}_A(A)$ satisfies $\varphi(a) = a\varphi(1)$ for all $a \in A$, and hence φ is uniquely determined by $\varphi(1)$, which can be arbitrary. This proves that ι is a linear isomorphism. Then

$$\iota(\varphi \circ \psi) = \varphi \circ \psi(1) = \varphi(\psi(1) \cdot 1) = \psi(1)\varphi(1) = m_{A^{\mathrm{op}}}(\iota(\varphi), \iota(\psi)). \tag{4.5.12}$$

$\square$

Lemma 4.5.13. For arbitrary left modules $(V_1, \rho_1), (V_2, \rho_2)$, and (W, ρ_W), we have linear isomorphisms

$$\mathrm{Hom}_A(V_1 \oplus V_2, W) \cong \mathrm{Hom}_A(V_1, W) \oplus \mathrm{Hom}_A(V_2, W), \tag{4.5.14}$$

$$\mathrm{Hom}_A(W, V_1 \oplus V_2) \cong \mathrm{Hom}_A(W, V_1) \oplus \mathrm{Hom}_A(W, V_2). \tag{4.5.15}$$

We omit the proof, as it is the same as for Lemma 2.15.3.

Proof of Theorem 4.5.9. First, (iii) implies (ii). Next, (ii) implies (i) by Lemma 4.5.8, by taking direct sums. Assume (i). Write $A \cong \bigoplus_{i=1}^m V_i^{r_i}$, a direct sum of simple submodules with $r_i \geq 1$ and the V_i pairwise *nonisomorphic*. Now let us apply $\mathrm{End}_A(-)$. By the lemmas and Schur's Lemma, we obtain

$$A^{\mathrm{op}} \cong \bigoplus_{i=1}^n \mathrm{End}_A(V_i^{r_i}) \cong \bigoplus_{i=1}^n \mathrm{Mat}_{r_i}(\mathbf{C}). \tag{4.5.16}$$

Now we obtain (ii) by taking the opposite algebras of both sides and applying Example 4.1.24.

Finally we show (ii) implies (iii). Theorem 4.5.1 and Lemma 4.5.8 (or simply Theorem 4.4.1 along with Exercise 4.2.13) shows that the simple left modules over $\bigoplus_i \mathrm{Mat}_{n_i}(\mathbf{C})$ are indeed the $\mathbf{C}^{n_i}$, which have dimension n_i. Thus $V_i := \mathbf{C}^{n_i}$ form a full set of simple left modules, and $\mathrm{End}(V_i) = \mathrm{Mat}_{n_i}(\mathbf{C})$. $\square$

4.6 Character theory

Definition 4.6.1. Let (V, ρ_V) be a finite-dimensional representation of an algebra A. The character $\chi_V : A \to \mathbf{C}$ is defined by $\chi_V(a) = \operatorname{tr} \rho_V(a)$.

Remark 4.6.2. As in the case of groups, characters are not arbitrary functions. Indeed, $\chi_V(ab) = \chi_V(ba)$, which shows that χ_V lives in the space of *Hochschild traces on* A,

$$A^*_{\mathrm{tr}} := \{f : A \to \mathbf{C} \mid f \text{ is linear and } f(ab - ba) = 0, \forall a, b \in A\}. \tag{4.6.3}$$

We will see in Exercise 4.6.5 below that, for a semisimple algebra, the characters of simple left modules form a basis for this space. In the case that $A = \mathbf{C}[G]$, the space A^*_{tr} identifies with the space of class functions (Example 4.6.4 below), so this reduces to Corollary 3.5.3.(ii) (except, there is no orthonormality statement anymore, as we have no inner product in the setting of algebras).

Example 4.6.4. If $A = \mathbf{C}[G]$ is a group algebra, then first of all $A^* \xrightarrow{\sim} \operatorname{Fun}(G, \mathbf{C})$, via the map $f \mapsto f|_G$. Next, $A^*_{\mathrm{tr}} \subseteq A^*$ identifies under this isomorphism with the subspace of class functions: indeed, $f(ab) = f(ba)$ for all $a, b \in \mathbf{C}[G]$ if and only if $f(gh) = f(hg)$ for all $g, h \in G$, which by substituting gh^{-1} for g is true if and only if $f(g) = f(hgh^{-1})$ for all $g, h \in G$.

Exercise 4.6.5. Show that, for $A = \operatorname{Mat}_n(\mathbf{C})$, then $A^*_{\mathrm{tr}} = \mathbf{C} \cdot \operatorname{Tr}$, constant multiples of the usual trace map. Hint: it is equivalent to show that, for all $f \in A^*_{\mathrm{tr}}$ and every trace-zero matrix a, then $f(a) = 0$. To see this, note that, for $i \neq j$, $e_{ij} = e_{ii}e_{ij} - e_{ij}e_{ii}$ and $e_{ii} - e_{jj} = e_{ij}e_{ji} - e_{ji}e_{ij}$. Thus, if $f \in A^*_{\mathrm{tr}}$, then $f(e_{ij}) = 0 = f(e_{ii} - e_{jj})$. Since every trace-zero matrix a is a linear combinations of e_{ij} and $e_{ii} - e_{jj}$, this implies that $f(a) = 0$.

As before, we have:

Proposition 4.6.6. If $V \cong W$, then $\chi_V = \chi_W$.

Proof. It is a consequence of the fact that trace is invariant under conjugation (by an invertible transformation). More precisely, if $\rho_W(a) \circ T = T \circ \rho_V(a)$ for T invertible, then

$$\operatorname{tr} \rho_W(a) = \operatorname{tr}(T^{-1} \circ \rho_V(a) \circ T) = \operatorname{tr}(T \circ T^{-1} \circ \rho_V(a)) = \operatorname{tr} \rho_V(a). \tag{4.6.7}$$

□

Remark 4.6.8 (Non-examinable)**.** The following results (as well as Exercise 4.6.5) could also be deduced from similar results for group algebras of finite groups, since every matrix algebra is a direct summand of a group algebra of a finite group (as in CW2, #5). But we will give direct proofs. One reason to do this is that, in practice, finite-dimensional semisimple algebras often arise where there is no finite group present, so it isn't natural to make use of one (even though it is true that one could always realise the semisimple algebra as a summand of some group algebra of a finite group, for a "large enough" group).

Theorem 4.6.9. If a finite-dimensional algebra A is semisimple, then $\chi_V = \chi_W$ implies $V \cong W$.

The theorem follows from the following result of independent interest:

Theorem 4.6.10. Let A be a finite-dimensional semisimple algebra and $(V_1, \rho_{V_1}), \ldots, (V_m, \rho_{V_m})$ a full set of simple left modules. Then the characters χ_{V_i} are linearly independent.

Proof. By Theorem 4.5.9, we can assume $A = \bigoplus_{i=1}^m \mathrm{Mat}_{n_i}(\mathbf{C})$ and $V_i = \mathbf{C}^{n_i}$. Computing the trace, we get $\chi_{\mathbf{C}^{n_i}}(a_1, \ldots, a_m) = \operatorname{tr} a_i$. Now if $\sum_{j=1}^m \lambda_j \chi_{\mathbf{C}^{n_j}} = 0$, then plugging in $a_i = I$ and $a_j = 0$ for $j \neq i$, we get $n_i \lambda_i = 0$. Thus $\lambda_i = 0$. $\square$

Proof of Theorem 4.6.9. Let $V_1, \ldots, V_m$ be a full set of simple left A-modules. Then $V \cong \bigoplus_{i=1}^m V_i^{r_i}$ and $W \cong \bigoplus_{i=1}^m V_i^{s_i}$ for some $r_i, s_i \geq 0$. Taking characters, we get $\chi_V = \sum_{i=1}^m r_i \chi_{\mathbf{C}^{n_i}}$ and $\chi_W = \sum_{i=1}^m s_i \chi_{\mathbf{C}^{n_i}}$. Now, $\chi_V = \chi_W$ implies, by linear independence of the $\chi_{\mathbf{C}^{n_i}}$ (Theorem 4.6.10), that $r_i = s_i$ for all i, and hence $V \cong W$. $\square$

Theorem 4.6.11. For A a finite-dimensional semisimple algebra, with full set of simple left A-modules $V_1, \ldots, V_m$, the characters χ_{V_i} form a basis for A_{tr}^*.

Again, by Example 4.6.4, this theorem specialises to the statement of Crollary 3.5.3.(ii) without orthonormality, in the case $A = \mathbf{C}[G]$.

Proof of Theorem 4.6.11. By Theorem 4.5.9, $A \cong \bigoplus_{i=1}^m \mathrm{Mat}_{n_i}(\mathbf{C})$. By Exercise 4.6.5, $A_{\mathrm{tr}}^* \cong \bigoplus_{i=1}^m \mathrm{Mat}_{n_i}(\mathbf{C})_{\mathrm{tr}}^* \cong \mathbf{C}^m$. Explicitly, every $f \in A_{\mathrm{tr}}^*$ has the form $f(a_1, \ldots, a_m) = \sum_{i=1}^m \lambda_i \operatorname{Tr}(a_i)$ for some $\lambda_i \in \mathbf{C}$. In particular, $\dim A_{\mathrm{tr}}^* = m$. But m also equals the number of simple left modules, by Theorem 4.5.9. By Theorem 4.6.10, the characters $\chi_{\mathbf{C}^{n_i}}$ are linearly independent, and since there are m of them, they form a basis of A_{tr}^*. The result then follows from Theorem 4.5.9. $\square$

Remark 4.6.12. For a non-semisimple algebra, it is no longer true that the necessarily linearly independent characters of simple left A-modules form a basis for A_{tr}^*. Indeed, suppose that A is commutative: $ab = ba$ for all $a, b \in A$. Then $A_{\mathrm{tr}}^* = A^*$. But in general there are fewer than $\dim A^*$ simple left A-modules. For example, for $A = \mathbf{C}[\varepsilon]/(\varepsilon)$ (Example 4.1.7), there is only one simple left module up to isomorphism, $\mathbf{C} \cdot \varepsilon$, but $\dim A = 2 > 1$.

Remark 4.6.13. As noted before, we cannot form a character table for a general algebra (even if it is finite-dimensional and semisimple algebra) since we don't have a collection of elements to take traces of. Indeed, for many finite groups G, H, the character tables can be different (so $G \ncong H$, but nonetheless there exists an isomorphism $\varphi : \mathbf{C}[G] \cong \mathbf{C}[H]$ that does not send G to H. Via φ, the characters $\mathbf{C}[G] \to \mathbf{C}, \mathbf{C}[H] \to \mathbf{C}$ of irreducible representations of G are the same, but since φ does not send G to H, this is no contradiction with the character tables being different.

Remark 4.6.14. As in Proposition 3.3.4, with the same proof, we have, for V, W finite-dimensional left modules over a general algebra A,

$$\chi_{V\oplus W} = \chi_V + \chi_W. \tag{4.6.15}$$

We can't literally do the same thing for V^* and $V \otimes W$, however, since these are *not* in general left modules over A when V and W are.

Rather, V^* is a right module over A, given by dualising endomorphisms: the map $\rho_{V^*} : A^{\mathrm{op}} \to \mathrm{End}(V^*)$ is defined by $\rho_{V^*}(a) = \rho_V(a)^*$. This is an algebra homomorphism since, as in Example 4.1.25, the dualisation map $\mathrm{End}(V) \to \mathrm{End}(V^*)$ is anti-multiplicative, i.e., $(S \circ T)^* = T^* \circ S^*$. In other words, ev have an algebra isomorphism $\mathrm{End}(V)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}(V^*)$, and therefore we get an algebra homomorphism $A^{\mathrm{op}} \to \mathrm{End}(V)^{\mathrm{op}} \to \mathrm{End}(V^*)$. In any case, $\mathrm{tr}(S^*) = \mathrm{tr}(S)$, so if we think of V^* as a right module over A, then $\chi_{V^*} = \chi_V$, rather than getting complex conjugates as we did in the group case. The reason why we had complex conjugates is because we also used the inversion map $G^{\mathrm{op}} \xrightarrow{\sim} G$, inducing $\mathbf{C}[G]^{\mathrm{op}} \xrightarrow{\sim} \mathbf{C}[G]$, to turn V^* back into a left module over $\mathbf{C}[G]$.

As for $V \otimes W$, there is no way to make this into any sort of module over A in general. However, if (V, ρ_V) is a module over A and (W, ρ_W) a module over of B, then $V \boxtimes W$ is a module over $A \otimes W$ with $\rho_{V\boxtimes W}(a \otimes b) = \widetilde{\rho}_V(a) \otimes \widetilde{\rho}_W(b)$. Then as before, $\chi_{V\boxtimes W}(a \otimes b) = \chi_V(a)\chi_W(b)$.

Remark 4.6.16. If V is a finite-dimensional left module and W a submodule, then we can take a basis $\mathcal{C}$ of W and extend it to a basis $\mathcal{B}$ of V. In this basis $\rho_V^{\mathcal{B}}(a)$ is block upper-triangular of the form:

$$\begin{pmatrix} \rho_W^{\mathcal{C}}(a) & * \\ 0 & \rho_{V/W}^{\mathcal{B}\setminus\mathcal{C}}(a) \end{pmatrix},$$

where we view $\mathcal{B} \setminus \mathcal{C}$ as a basis for V/W by adding W. As a consequence we deduce the more general identity:

$$\chi_V = \chi_W + \chi_{V/W}. \tag{4.6.17}$$

The same identity is valid for group representations (by the same argument, or simply by letting the algebra be the group algebra $\mathbf{C}[G]$).

4.7 The center

Related to the characters of an algebra is its center:

Definition 4.7.1. The center $Z(A)$ of an algebra A is the collection of elements $z \in A$ such that $za = az$ for all $a \in A$.

It is immediate that this is preserved by isomorphisms:

Proposition 4.7.2. If $\varphi : A \to B$ is an isomorphism of algebras, then it restricts to an isomorphism $\varphi_{Z(A)} : Z(A) \to Z(B)$ of the centers.

Now we turn to our main examples: the group algebra and the matrix algebra.

Proposition 4.7.3. Let G be a finite group. The center of the group algebra, $Z(\mathbf{C}[G])$, consists of the elements $\sum_{g \in G} a_g g$ such that the function $g \mapsto a_g$ is a class function ($a_{hgh^{-1}} = a_g$ for all h). The dimension equals the number of conjugacy classes of G. A basis is given by (2.18.18).

Remark 4.7.4. Under the isomorphism $\mathbf{C}[G] \xrightarrow{\sim} \operatorname{Fun}(G, \mathbf{C})$ assigning to $v = \sum_{g \in G} a_g g$ the element $\varphi_v, \varphi_v(g) = a_g$, the center maps to the class functions. So we can identify the center with the class functions in this way.

Proof of Proposition 4.7.3. Note that $z \in Z(\mathbf{C}[G])$ if and only if $z = gzg^{-1}$ for all g. Thus $Z(\mathbf{C}[G]) = (\mathbf{C}[G], \rho_{\mathrm{ad}})^G$, the invariants under the adjoint action. The result follows from Lemma 2.18.17. □

Proposition 4.7.5. Let V be a finite-dimensional vector space. Then $Z(\operatorname{End}(V)) = \mathbf{C} \cdot I$, the scalar multiples of the identity endomorphism.

Proof. Since $\operatorname{End}(V) \cong \operatorname{Mat}_{\dim V}(\mathbf{C})$, this can be done explicitly with matrices (left as an exercise). We give a proof with representation theory. By Example 4.2.8, V is a simple left module over $\operatorname{End}(V)$. Let $A := \operatorname{End}(V)$. If $z \in Z(A)$, then $\rho_V(z) : V \to V$ is A-linear: $\rho_V(z) \circ \rho_V(a) = \rho_V(a) \circ \rho_V(z)$. By Schur's Lemma (Lemma 4.2.5), $\rho_V(z)$ is a multiple of the identity. But the map $\rho_V : A = \operatorname{End}(V) \to \operatorname{End}(V)$ is the identity. So z is also a multiple of the identity. □

Since $\operatorname{End}(V) \cong \operatorname{Mat}_{\dim V}(\mathbf{C})$, we immediately conclude:

Corollary 4.7.6. The center of the $n \times n$ matrix algebra, $Z(\operatorname{Mat}_n(\mathbf{C}))$, consists of all scalar matrices, $\mathbf{C} \cdot I$.

Putting these results together, we obtain:

Corollary 4.7.7. The isomorphism $\Phi : \mathbf{C}[G] \to \bigoplus_{i=1}^m \operatorname{End}(V_i)$ restricts to an isomorphism $\Phi|_{Z(\mathbf{C}[G])} : Z(\mathbf{C}[G]) \xrightarrow{\sim} \bigoplus_{i=1}^m \mathbf{C} \cdot I_{V_i}$.

Proof. This is a direct consequence of Theorem 4.3.2 and Propositions 4.7.2 and 4.7.5. □

Applying Proposition 4.7.3, we obtain another (although similar) proof of Theorem 2.18.2: the dimension of source, $Z(\mathbf{C}[G])$ equals the number of conjugacy classes, whereas the dimension of the target is the number of irreducible representations.

Corollary 4.7.8. The following are equivalent:

(i) $a \in \mathbf{C}[G]$ is central;

(ii) $a = \sum_{g \in G} \lambda_g g$ where $\lambda_g = \lambda_{hgh^{-1}}$ for all $g, h \in G$;

(iii) For every irreducible representation (V, ρ_V), the transformation $\rho_V(a)$ is a scalar multiple of the identity.

Proof. The equivalence of (i) and (ii) is Proposition 4.7.3. The equivalence between (i) and (iii) is a consequence of Corollary 4.7.7. □

Remark 4.7.9. Just like our proof of Theorem 2.18.2 before, the equivalence between parts (ii) and (iii) of Corollary 4.7.8 can be seen without using algebras at all. Indeed instead of taking center, as in the proof of Theorem 2.18.2, we can take G-invariants. We get that an element of $\mathbf{C}[G]$ is G-invariant if and only if its image under $\Phi : \mathbf{C}[G] \xrightarrow{\sim} \bigoplus_{i=1}^m \mathrm{End}(V_i)$ is G-invariant, i.e., in $\bigoplus_{i=1}^m \mathrm{End}(V_i)^G = \bigoplus_{i=1}^m \mathrm{End}_G(V_i) = \bigoplus_{i=1}^m \mathbf{C} \cdot I_{V_i}$.

Remark 4.7.10. Notice that, for a semisimple finite-dimensional algebra A, there is a close resemblance between $Z(A)$ and A^*_{tr}. Indeed, they are both m-dimensional vector spaces, where m is the number of nonisomorphic simple left A-modules. In fact we can write an isomorphism:

$$A^*_{\mathrm{tr}} \to Z(A)^*, f \mapsto f|_{Z(A)}. \tag{4.7.11}$$

This property of the center being dual to the Hochschild traces is a general property one sees for what are called *Frobenius* or *Calabi–Yau* algebras, which are active subjects of mathematical research.

4.8 Projection operators

Given a representation $\rho_W : G \to \mathrm{GL}(W)$ of a group G, we continue to let $\widetilde{\rho}_W : \mathbf{C}[G] \to \mathrm{End}(W)$ be the extension to $\mathbf{C}[G]$.

Theorem 4.8.1. Let (V, ρ_V) be an irreducible representation of a finite group G. Consider the element

$$P_V := \frac{\dim V}{|G|} \sum_{g \in G} \overline{\chi_V}(g) g \in \mathbf{C}[G]. \tag{4.8.2}$$

Then for every finite-dimensional representation (W, ρ_W) of G, the transformation $\widetilde{\rho}_W(P_V) : W \to W$ is a G-linear projection with image the sum of all subrepresentations of W isomorphic to V.

Example 4.8.3. If $(V, \rho_V) = (\mathbf{C}, 1)$ is the trivial representation, we obtain the element $P_{\mathbf{C}} = |G|^{-1} \sum_{g \in G} g$, which is often called the *symmetriser* element. Note that $\widetilde{\rho}_W(P_{\mathbf{C}}) = |G|^{-1} \sum_{g \in G} \rho_W(g)$ is the projection we found long ago in (2.14.11).

Example 4.8.4. For a nontrivial example, let $(V, \rho_V) = (\mathbf{C}_-, \rho_-)$ be the sign representation of S_n (so $\mathbf{C}_- = \mathbf{C}$ and $\rho_-(\sigma) = \mathrm{sign}(\sigma)$. Then $P_{\mathbf{C}_-} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)$. This element is also called the antisymmetriser element.

Exercise 4.8.5. Verify explicitly that the operator $\widetilde{\rho}_W(P_{\mathbf{C}_-}) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \rho_W(\sigma)$ is a G-linear projection $W \to W$ with image the collection of all vectors $w \in W$ such that $\rho_W(\sigma)(w) = \mathrm{sign}(\sigma) w$.

Proof of Theorem 4.8.1. The element P_V is invariant under the adjoint action, i.e., $\overline{\chi_V}(g) = \overline{\chi_V}(hgh^{-1})$ for all $h \in G$. Therefore $P_V \in Z(\mathbf{C}[G])$ by Proposition 4.7.3.

First assume that W is irreducible. By Corollary 4.7.7, $\rho_W(P_V) = \lambda_{V,W} I$ for some $\lambda_{V,W} \in \mathbf{C}$. Let us compute it. Taking traces,

$$\lambda_{V,W} \dim W = \frac{\dim V}{|G|} \sum_{g \in G} \overline{\chi_V}(g) \chi_W(g) = \dim V \langle \chi_W, \chi_V \rangle. \tag{4.8.6}$$

By orthonormality of irreducible characters,

$$\lambda_{V,W} = \begin{cases} 1, & \text{if } V \cong W \\ 0, & \text{otherwise.} \end{cases} \tag{4.8.7}$$

Now, let W be a general finite-dimensional representation. By Maschke's theorem $W = \bigoplus_{i=1}^m W_i$ for some irreducible subrepresentations W_i. For each W_i, we get that $\rho_{W_i}(P_V)$ is the identity or zero, depending on whether $W_i \cong V$ or not. Hence $\rho_W(P_V)$ is the projection onto the sum of those summands isomorphic to V. This contains all subrepresentations isomorphic to V, since $\rho_W(P_V)(w) = w$ if w is contained in a subrepresentation isomorphic to V. The opposite containment is clear. Therefore $\rho_W(P_V)$ is the projection onto the sum of all subrepresentations isomorphic to V. $\square$

Remark 4.8.8. By Remark 4.7.9, we don't need to use algebras to prove Theorem 4.8.1. Simply observe that P_V is G-invariant for the adjoint action and apply Remark 4.7.9. We could have included this result therefore in Section 3.

4.9 General algebras: linear independence of characters

Finally, we demonstrate how the results on finite-dimensional semisimple algebras actually imply results for general algebras, not even finite-dimensional. Actually it is surprising how much one can deduce in the general case from the semisimple case.

Theorem 4.9.1. Let A be an algebra and $(V_1, \rho_1), \ldots, (V_m, \rho_m)$ nonisomorphic simple finite-dimensional left A-modules. Then the map $\Phi : A \to \bigoplus_{i=1}^m \rho_{V_i}$ given by $\Phi(a) = (\rho_1(a), \ldots, \rho_m(a))$ is surjective and the characters $\chi_{V_1}, \ldots, \chi_{V_m} : A \to \mathbf{C}$ are linearly independent.

Proof. Let $B := \operatorname{im} \Phi$. Then $V_1, \ldots, V_m$ are all left modules over B, via $\rho'_{V_i} : B \to \operatorname{End}(V_i)$, given by the composition of the inclusion $B \hookrightarrow \bigoplus_{j=1}^m \operatorname{End}(V_j)$ with the projection to the factor $\operatorname{End}(V_i)$. By definition, for all $a \in A$, $\rho'_{V_i}(\Phi(a)) = \rho_{V_i}(a)$. Hence, the condition for $U \subseteq V_i$ to be an A-submodule, i.e., preserved by $\rho_{V_i}(a)$ for all $a \in A$, is equivalent to being a B-submodule, i.e., preserved by $\rho'_{V_i}(\Phi(a))$ for all $a \in A$. Similarly, the condition for a linear map $V_i \to V_j$ to be A-linear is equivalent to the condition that it is B-linear. Thus, since the V_i are simple and mutually nonisomorphic as A-modules, the same holds as B-modules. As left B-modules, $\operatorname{End}(V_i) \cong V_i^{\dim V_i}$, which is semisimple. Therefore $\bigoplus_{i=1}^m \operatorname{End}(V_i)$ is itself a semisimple B-module. By Corollary 4.5.6, this implies that $B \subseteq \bigoplus_{i=1}^m \operatorname{End}(V_i)$ is semisimple as a left B-module. By Theorem 4.5.9, this implies that B is semisimple. As the $V_1, \ldots, V_m$ are mutually nonisomorphic simple left B-modules, they can be extended to a full set of simple left B-modules (by Theorem 4.5.9). But $\dim B \leq (\dim V_1)^2 + \cdots + (\dim V_m)^2$, which by Corollary 4.5.10 implies that we have equality and the $V_1, \ldots, V_m$ are already a full set of simple left B-modules. Thus, $B = \bigoplus_{i=1}^m \operatorname{End}(V_i)$ and Φ is surjective, as desired.

For the statement on linear independence of characters, note that if $\sum_{i=1}^m \lambda_i \chi_{V_i} = 0$ for $\chi_{V_i} = \operatorname{tr} \circ \rho_{V_i} : A \to \mathbf{C}$, then the same relation holds for the characters $\chi'_{V_i} = \operatorname{tr} \circ \rho'_{V_i} : B \to \mathbf{C}$, as $\rho_{V_i}(a) = \rho'_{V_i}(\Phi(a))$ for all $a \in A$. But the characters χ'_{V_i} are linearly independent by Theorem 4.6.10. So $\lambda_i = 0$ for all i. $\square$